RESEARCH STATEMENT

JOHANNA N.Y. FRANKLIN

My primary research interest is recursion theory, which is a branch of mathematical logic, and my primary subfield is algorithmic randomness. Most of my research to date has focused on identifying differences between various randomness notions. I am continuing this work and hope to use technical results in this area to prove theorems in classical recursion theory as well as other areas of mathematical logic, such as reverse mathematics and recursive model theory.

1. Basic Concepts

The most fundamental concept in recursion theory is that of Turing reducibility. We say that a set A is *Turing reducible* to another set B if there is a recursive algorithm that, given B as an oracle, can answer the question "Is n in A?" for any natural number n. If this is the case, we write $A \leq_T B$. This relation can be used to generate an equivalence relation, $\equiv_T$, in the obvious way. The equivalence classes are called *Turing degrees*, and they form an upper semilattice.

There are several different ways to approach algorithmic randomness. I mainly use the notion of *initial-segment complexity*. A real A is said to be random if for some subset $\mathcal{M}$ of Turing machines, its initial-segment complexity, $K_M(A\upharpoonright n)$, is at least $n - c$ for some constant c for every $M \in \mathcal{M}$; i.e., if the shortest way to identify the first n bits of A is simply to write them down instead of using a program to generate them. Different collections of Turing machines result in different types of randomness. In the most commonly studied notion of randomness, Martin-Löf randomness, all Turing machines with prefix-free domains are considered. In Schnorr randomness, the type I have focused on, we consider only the *computable* Turing machines: those with prefix-free domains with effectively approximable measures. Clearly, every Martin-Löf random real is Schnorr random, but not vice versa. I will also refer to another notion of randomness, recursive randomness, which is strictly intermediate between Martin-Löf and Schnorr randomness.

2. Previous work

2.1. **Algorithmic randomness.** The topic of my thesis is a dual notion to randomness, triviality. A real is said to be *Schnorr trivial* if it is no more complicated to identify its first n bits than it is to write down a string of n 0s using a computable Turing machine (up to a constant). Technically, A is Schnorr trivial if for every computable machine M, there is another computable machine M' and a constant c such that

$$K_{M'}(A\upharpoonright n) \leq K_M(0^n) + c$$

for all n [5]. Clearly, every recursive real is Schnorr trivial.

The reals that are trivial in the sense of Martin-Löf, or *K-trivial*, form an ideal that is properly contained in the superlow Turing degrees [21]. I showed that this is far from the case for Schnorr trivial reals.

Theorem 2.1. [10, 7] *A Turing degree* **a** *is high* ($\mathbf{a}' \geq_T \mathbf{0}''$) *if and only if every Turing degree above* **a** *contains a Schnorr trivial real.*

One direction of this theorem is proven by constructing a uniform homogeneous perfect tree of Schnorr trivial reals recursively in an arbitrary high degree. Every real is recursively coded by a branch in this tree, so each Turing degree above the high degree contains a real that is Turing equivalent to a branch in this tree. The other direction is proven by taking a nonhigh set A and another set G that is 2-generic relative to A and showing that $A \oplus G$ is not Turing equivalent to a Schnorr trivial real.

Theorem 2.1 suggests that in the case of Schnorr randomness, initial-segment complexity and computational strength do not go together, since reals with arbitrarily high computational strength can have minimal initial-segment complexity.

I also determined the relationship between the Schnorr trivial reals and those that are low for Schnorr in my thesis. A real A is said to be *low* for a particular randomness notion $\mathcal{R}$ if using that real as an oracle does not make a random real appear to be nonrandom (that is, if $\mathcal{R}^A \subseteq \mathcal{R}$). These notions coincide for Martin-Löf randomness, but there is a Schnorr trivial real that is not low for Schnorr randomness [4, 18, 24]. However, I proved that the converse is true.

Theorem 2.2. [9] *Every real that is low for Schnorr randomness is Schnorr trivial.*

It is natural to ask if there is a degree structure in which Schnorr trivial reals do behave somewhat as expected. For instance, it seems natural that a triviality notion should be closed downward. Downey, Griffiths, and Laforte showed that this was not the case for Schnorr trivial reals in the Turing degrees, but that it did hold in the truth-table (*tt*-)degrees (the degree structure formed when only algorithms that converge on all inputs for all oracles are considered) [4]. Frank Stephan and I have shown that the Schnorr trivial reals actually form an ideal in this structure, but not in the weak truth-table (*wtt*-)degrees, which are intermediate between the Turing and *tt*-degrees.

Theorem 2.3. [12] *The Schnorr trivial reals form an ideal in the tt-degrees but not in the wtt-degrees.*

In this paper, we also defined a notion of *tt*-lowness for Schnorr randomness and proved that it coincides with Schnorr triviality. When combined, these results suggest that the *tt*-degrees may be a more natural structure in which to consider Schnorr randomness than the Turing degrees.

This work highlights some of the differences between Martin-Löf randomness and Schnorr randomness. It seems natural to ask if other theorems concerning Martin-Löf randomness also fail for other, weaker randomness notions, and if so, the extent to which they fail. One such theorem is van Lambalgen's Theorem.

Theorem 2.4 (van Lambalgen's Theorem, [25])**.** *Let* A_0 *and* A_1 *be reals. Then* $A_0 \oplus A_1$ *is Martin-Löf random if and only if* A_0 *is Martin-Löf random and* A_1 *is Martin-Löf random with respect to* A_0.

Essentially, this theorem says that if a Martin-Löf random real is broken into two halves, each half must be Martin-Löf random with respect to the other. It had previously been shown that this theorem failed for both recursive randomness and Schnorr randomness [19, 26, 22]. However, in each such proof, the Turing degrees in which van Lambalgen's Theorem was shown to fail were rather restricted. Kjos-Hanssen showed that van Lambalgen's Theorem fails for recursively random reals in high minimal degrees [22], and Yu showed that it fails for all recursively random reals that are not Martin-Löf random in a certain subclass of the Δ^0_2 degrees [26]. This suggested that the failure of this theorem might be due not only to the weakness of the randomness notions but also the low computational strength of the reals in question. Stephan and I have shown that this is not the case.

Theorem 2.5. [11] *Let* $\mathbf{a}$ *be a high Turing degree. Then there is* $B \in \mathbf{a}$ *such that* B *is recursively random and van Lambalgen's Theorem does not hold for* B*.*

Since the high degrees are precisely the Turing degrees that contain recursively random reals that are not Martin-Löf random [18], this is the best that we can hope for.

Another concept in the study of randomness is that of a basis. We say that a real A is a *basis* for a relativizable notion $\mathcal{R}$ (such as Martin-Löf randomness or 1-genericity) if there is some $B \geq_T A$ such that B has property $\mathcal{R}$ relative to A. While the bases for Martin-Löf randomness have been shown to be precisely the K-trivial reals [15], little was known about the bases for other randomness notions. In [13], we answer Question 5.2 from [20] for the specific case of Schnorr randomness.

Theorem 2.6. [13] *A real* A *is a basis for Schnorr randomness if and only iff* $A \not\geq_T 0'$*.*

In the same paper, we produced an improved partial characterization of the bases for recursive randomness and characterized the bases for weak 1-genericity as those A such that $A \not\geq_T 0'$. We also introduced the concept of highness for pairs of randomness notions as a dual to that of lowness for pairs of randomness notions introduced by Kjos-Hanssen, Nies, and Stephan in [18]. We say that a real A is in $\mathrm{High}(\mathcal{R}, \mathcal{S})$ if and only if every real in $\mathcal{R}^A$ is also in $\mathcal{S}$.

Theorem 2.7. [13] *The following are equivalent.*

(1) $A \geq_T K$.
(2) $A \in \mathit{High}(\mathit{Schnorr}, \mathit{Rec})$.
(3) $A \in \mathit{High}(\mathit{Schnorr}, \mathit{ML})$.
(4) $A \in \mathit{High}(\mathit{Schnorr}, W2R)$.

In this paper, we also characterized highness notions for several pairs of genericity notions.

I have also investigated the relationship between the Turing degrees of n-generic and Schnorr random reals. Each of these subsets of the reals is a natural class of the Kurtz random reals (those that are not in any Π^0_1 class of measure 0), so it seems reasonable to investigate how much their Turing degrees overlap.

Theorem 2.8. [8] *No* 1*-generic is truth-table equivalent to a Schnorr random real. However, there is a* 1*-generic real that is Turing equivalent to a Schnorr random real.*

2.2. **Other topics.** I have also worked on problems in the tt-degrees with Cenzer, Wu, and Liu. We have shown the following theorem.

Theorem 2.9. [2] *There are superhigh degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{0}$*,* $\mathbf{0}'$*,* $\mathbf{a}$*, and* $\mathbf{b}$ *form a diamond in the tt-degrees.*

We are currently working to prove other, similar results. In particular, we are investigating whether a diamond can be formed in the tt-degrees with both atoms Turing equivalent to $0'$ or both atoms superlow.

I am also studying a restriction of wtt-reducibility based on bounding the use with order functions with Greenberg, Stephan, and Wu. This "tiny use" has turned out to have connections to Schnorr triviality as well as r.e. traceability.

3. Future plans

3.1. **Randomness.**
I am continuing my work in the differences between randomness notions. One of the primary results that I am interested in analyzing is Demuth's Theorem.

Theorem 3.1. [3] *Suppose A is Martin-Löf random. If $B \leq_{tt} A$ and B is not recursive, there is a set $C \equiv_{wtt} B$ that is Martin-Löf random.*

There are many different approaches one can take to this theorem. Since it involves both the *wtt*- and *tt*-degrees, one may ask if it can be sharpened in that sense. For instance, can such a C be found that is *tt*-equivalent to B? One may also ask if this theorem holds for weaker notions of randomness such as recursive or Schnorr randomness.

Other approaches to distinguishing types of randomness appears in the recent work of Figueira, Miller, and Nies [6] and Lempp and Kastermans [16]. Figueira, Miller, and Nies have developed the concept of indifferent sets for a class of reals $\mathcal{C}$; i.e., those sets A such that varying a member of $\mathcal{C}$ on A will still result in a member of $\mathcal{C}$. They have several results on necessary properties of indifferent sets for the class of Martin-Löf random reals. I plan to investigate the properties of indifferent sets for weaker randomness notions.

Kastermans and Lempp, on the other hand, have investigated other randomness notions to try to clarify the relationship between Martin-Löf randomness and Kolmogorov-Loveland randomness. These notions are not known to be distinct. However, Kastermans and Lempp have shown that permutation and injective randomness, two notions weaker than Kolmogorov-Loveland randomness, differ from Martin-Löf randomness. Therefore, one cannot hope to show that Kolmogorov-Loveland and Martin-Löf randomness are identical by showing the equivalence of, for instance, permutation and Martin-Löf randomness. However, Miller and Nies have also asked if permutation or injective randomness is equivalent to Martin-Löf randomness for the restricted class of left r.e. reals [20]. This question remains open.

I also plan to investigate the use of concepts in algorithmic randomness to prove theorems in classical recursion theory. One can ask, for instance, if applying $\vee$ and $\wedge$ to the degrees containing reals that are Schnorr (or recursively) random generates all Turing degrees, or whether an automorphism of the Turing degrees can be fully determined by its behavior on the degrees containing reals that are low for Schnorr randomness. Such information could produce further results in classical degree theory.

3.2. **Other topics.**
My work with Cenzer, Liu and Wu, together with my work in Schnorr triviality, has led me to become more interested in the *tt*-degrees. Very little is known about the structure of the *tt*-degrees within the hyperimmune Turing degrees that are not recursively enumerable, for example, and Theorem 2.1 suggests that further information about this could be useful in the study of Schnorr triviality as well as a more general study of degree theory.

In Fall 2008, I gave a short course for graduate students in reverse mathematics, which is the study of the equivalence of theorems in "standard" mathematics to certain fragments of second-order arithmetic. I am particularly interested in the connections that have developed between this field and randomness. Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman have proven results in this area using techniques from algorithmic randomness [1], and Simpson has developed a degree structure directly related to the Martin-Löf random reals and reverse mathematics [23]. I would like to contribute to these programs as well as learn to apply more standard techniques in reverse mathematics.

I am also interested in recursive model theory, which is the application of recursion theory to algebraic structures [14]. The set of sentences true of a countable structure can be coded as a set of natural numbers, and the Turing degree of this set can be determined. Techniques from algorithmic randomness have already proven useful here. For instance, Khoussainov, Semukhin, and Stephan have proven results in recursive model theory by using Kolmogorov complexity to construct theories with certain desirable properties [17]. Again, I look forward to learning the standard techniques in this area as well as applying those that I already know.

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