$\Pi^1_1$ conservation theorems for COH

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A recent theorem of Chong, Slaman, and Yang:

**Theorem**

\[ \text{COH is } \Pi^1_1 \text{ conservative over } \text{RCA}_0 + B\Sigma^0_2. \]

Proof is complicated and nonuniform: makes explicit use of nonstandard models.

**Question**

*Is there an easier proof? Can theorems or techniques from classical recursion theory be brought to bear?*

**Question**

*Is \( \text{RCA}_0 \) the base system?*
Argue using PA degrees that $\text{RCA}_0 \notmodels \text{WKL}_0$.

Argue using Low Basis Theorem that $\text{WKL}_0 \notmodels \text{ACA}_0$.

(etc.)

(Harrington) Show that any countable $\mathcal{M} \models \text{RCA}_0$ can be extended to an $\mathcal{N} \models \text{WKL}_0$. (Some care is needed.) Deduce a conservation result.
The basics

Language of second-order arithmetic: \((0, 1, +, \times, \in)\) with two sorts for first- and second-order objects.
First-order variables \(x, y, z, \ldots\); second-order variables \(X, Y, Z, \ldots\).

\(\mathcal{M} = (M, S)\) a model consisting of a first-order part \(M\) and a second-order part \(S\).

\(P^-\): Basic axioms such as commutativity of +.
\(I\Sigma_n\): If \(A \subseteq M\) is \(\Sigma_n^0\)-definable (with parameters from \(M\) and \(S\)), and \(0 \in A\), and \((\forall x)x \in A \rightarrow x + 1 \in A\), then \(A = M\).

\(B\Sigma_n^0\): If \(A \subseteq M\) is \(\Delta_n^0\)-definable (with parameters), then as above.
( Characterization due to Slaman; requires \(I\Sigma_0^0 + \text{exp}\).)

\(\Delta_1^0\) comprehension: If \(A\) is \(\Delta_1^0\)-definable (with parameters) from \(M\) and \(S\), then \(A \in S\).

\(\text{RCA}_0\): \(P^- + \Delta_1^0\) comprehension + \(I\Sigma_1^0\).
Weak König’s Lemma: Every infinite \(\Delta_1^0\) (with parameters) binary tree has an infinite \(\Delta_1^0\) (with parameters) path.

\(\text{WKL}_0\): \(\text{RCA}_0 + \text{Weak König’s Lemma}\).
Conservation theorems

Suppose $T_0$ and $T_1$ are theories, and $\Gamma$ is a class of sentences. We say $T_1$ is $\Gamma$-conservative over $T_0$ if every $\phi \in \Gamma$ that is provable from $T_0 \cup T_1$ is already provable from $T_0$.

We know lots of examples:

(H. Friedman) $\text{RCA}_0$ is arith. conservative over $P^- + I\Sigma^0_1$.

(H. Friedman) $\text{ACA}_0$ is arith. conservative over PA.

(Simpson) $\text{WKL}_0$ is $\Pi^0_2$ conservative over PRA.

(Harrington) $\text{WKL}_0$ is $\Pi^1_1$ conservative over $\text{RCA}_0$.

(Simpson–Smith) $\text{WKL}_0^*$ is $\Pi^1_1$ conservative over $\text{RCA}_0^*$.

(Cholak–Jockusch–Slaman) $\text{COH}$ is $\Pi^1_1$ conservative over $\text{RCA}_0$ and over $\text{RCA}_0 + I\Sigma^0_2$.

(Chong–Slaman–Yang) $\text{COH}$ is $\Pi^1_1$ conservative over $\text{RCA}_0 + B\Sigma^0_2$. 

Cohesiveness

In recursion theory:
- *Cohesive sets* are a sort of naturally thin set.
- Cohesive sets played a role in the search for an intermediate r.e. degree.

In reverse math:
- A *cohesiveness principle* COH which is more strongly related to *p-cohesive sets* than to cohesive sets.
- Comes up in investigations of Ramsey’s theorem for pairs:
  \[ RT^2_2 \leftrightarrow SRT^2_2 + COH. \]

**Definition**

COH: If \( \langle R_0, R_1, \ldots \rangle \) is a uniformly \( \Delta^0_1 \) sequence of sets, there is an infinite \( \Delta^0_1 \) set \( C \) such that for each \( k \), either

\[
(\forall^\infty x) \ x \in C \rightarrow x \in R_k
\]

or

\[
(\forall^\infty x) \ x \in C \rightarrow x \notin R_k
\]
### Characterizations

#### Theorem (Friedman–Simpson–Smith)

**TFAE over RCA$_0$:**

1. **Weak König’s Lemma**
2. **$\Sigma^0_1$ separation:** If $A, B \subseteq \mathbb{N}$ are $\Sigma^0_1$ and disjoint, there is a $\Delta^0_1$ set $D$ s.t. $A \subseteq D \subseteq \overline{B}$.

($\Sigma^0_n$ and $\Delta^0_n$ always with parameters.)

#### Theorem (B., after Jockusch–Stephan and Fr–Si–Sm)

**TFAE over RCA$_0 + B\Sigma^0_2$:**

1. **Every infinite $\Delta^0_2$ binary tree has an infinite $\Delta^0_2$ path.**
2. **$\Sigma^0_2$ separation:** If $A, B \subseteq \mathbb{N}$ are $\Sigma^0_2$ and disjoint, there is a $\Delta^0_2$ set $D$ s.t. $A \subseteq D \subseteq \overline{B}$.
3. **COH**
Form a *Scott set* of degrees:

1. Start with a countable model $\mathcal{M}_0 = (\omega, S_0)$ with a top degree, i.e., $S_0 = \{ X : \deg(X) \leq a_0 \}$.

2. Take a PA($a_0$) degree $a_1$—i.e., one that computes a path for any infinite $a_0$-recursive binary tree—and adjoin to get

   $$\mathcal{M}_1 = \mathcal{M}_0[a_1] := (M, \{ X : \deg(X) \leq a_1 \}).$$

3. Repeat: find a PA($a_1$) degree $a_2$ and get $\mathcal{M}_2 = \mathcal{M}_1[a_2]$, etc.

4. The limit $\mathcal{M} = \mathcal{M}_0[a_1][a_2] \cdots$ is the desired model.
Theorem: \( \text{WKL}_0 \) is \( \Pi^1_1 \) conservative over \( \text{RCA}_0 \)

Proof (Harrington).

Suppose for contradiction \( \text{WKL}_0 \vdash \forall X \phi \) and \( \text{RCA}_0 \nvdash \forall X \phi \).

1. Fix a countable model \( \mathcal{M}_0 = (M, S_0) \) of \( \text{RCA}_0 + \exists X \neg \phi \) with \( \mathcal{M}_0 \models \neg \phi(A_0) \). Assume all sets in \( S_0 \) are \( \Delta^0_1(A_0) \).

2. Take the first infinite binary tree \( T \in S_0 \) and find a path \( A_1 \) such that

\[
\mathcal{M}_0[A_1] = (M, X: \{X \text{ is } \Delta^0_1(A_1)\})
\]

is a model of \( I \Sigma^0_1 \) and hence of \( \text{RCA}_0 \). Let \( \mathcal{M}_1 = \mathcal{M}_0[A_1] \).

3. Repeat.

4. In the limit, \( \mathcal{M} \models \text{WKL}_0 \).

But \( \mathcal{M} \) still contains \( A \), so \( \mathcal{M} \models \exists X \neg \phi \). Contradiction. \( \square \)
To build an \(\omega\)-model of COH, form a Jockusch–Stephan set:

1. Start with an \(\mathcal{M}_0\) as before.
2. Find a degree \(a_1\) such that \(a'_1\) is \(\text{PA}(a'_0)\), and let \(\mathcal{M}_1 = \mathcal{M}_0[a_1]\).
3. Repeat: Find degree \(a_2 \geq a_1\) such that \(a'_2\) is \(\text{PA}(a'_1)\), and let \(\mathcal{M}_2 = \mathcal{M}_1[a_2]\), etc.
4. The limit \(\mathcal{M} = \mathcal{M}_0[a_1][a_2] \cdots\) is the desired model.
Theorem: \( \text{COH} \) is \( \Pi^1_1 \) conservative over \( \text{RCA}_0 + B\Sigma^0_2 \)

This is a theorem of Chong, Slaman, and Yang. Our proof is new:

Proof outline.

Suppose for contradiction \( \text{RCA}_0 + B\Sigma^0_2 + \text{COH} \models \forall X \phi \) and \( \text{RCA}_0 + B\Sigma^0_2 \not\models \forall X \phi \).

1. Fix a countable model \( \mathcal{M}_0 = (M, S_0) \) of \( \text{RCA}_0 + B\Sigma^0_2 + \exists X \neg \phi \) with \( \mathcal{M}_0 \models \neg \phi(A_0) \). Assume all sets in \( S_0 \) are \( \Delta^0_1(A_0) \).

2. Take the first infinite binary tree \( T \Delta^0_2 \) in \( S_0 \) and find a path \( P \) and a second set \( A_1 \geq_T A_0 \) such that
   - (i) \( \mathcal{M}_0[A_1] \) is a model of \( B\Sigma^0_2 \);
   - (ii) \( P \) is \( \Delta^0_2(A_1) \).

3. Repeat.

4. In the limit, \( \mathcal{M} \models \text{COH} \).

Contradiction as before, but we need a way to produce \( P, A_{n+1} \).
Suppose $\mathcal{M}_n = \mathcal{M}_0[A_1] \cdots [A_n]$, and $T$ is a tree $\Delta^0_2$ definable in $A_n$. Normally, this means $T$ is $\Delta^0_1$ definable in $A'_n$.

Naïve approach:

- Expand our model to $\mathcal{M}_n[A'_n]$.
- Apply Harrington’s trick to get a path $P$ such that $\mathcal{M}_n[A'_n][P] \models I\Sigma^0_1$.
- Apply Friedberg Jump Theorem to get $A_{n+1} \geq_T A_n$ with $A_{n+1} \equiv_T P$.

Problem 1: If $\mathcal{M} \models B\Sigma^0_2$, can only guarantee $\mathcal{M}[0'] \models B\Sigma^0_1$.

Problem 2: Need a formal-enough version of the jump theorem to get $B\Sigma^0_2$ in the end.
Working in $\text{RCA}_0^*$

Problem 1: If $M \models B\Sigma^0_2$, can only guarantee $M[0'] \models B\Sigma^0_1$.

$\text{RCA}_0$: $P^- + \Delta^0_1$ comprehension + $I\Sigma^0_1$.

$\text{RCA}_0^*$: $P^- + \Delta^0_1$ comprehension + $B\Sigma^0_1$. (+ exp)

Lemma (Simpson–Smith)

If $\mathcal{N} \models \text{RCA}_0^*$ is countable with a top degree and $T$ is an infinite binary tree, there is an infinite path $P$ such that $\mathcal{N}[P] \models \text{RCA}_0^*$.

In particular, Weak König’s Lemma is $\Pi^1_1$ conservative over $\text{RCA}_0^*$.

This solves our first problem.
Problem 2: Need a formal-enough version of the jump theorem to get $B\Sigma^0_2$ back.

Lemma (B.; Friedberg Jump Theorem for $B\Sigma^0_2$)

Suppose $\mathcal{M} \models B\Sigma^0_2$ and $P \geq_T 0'$ is such that $\mathcal{M}[P] \models B\Sigma^0_1$. Then there is $A$ such that $\mathcal{M}[A] \models B\Sigma^0_2$ and $P \leq_T A'$.

Proof similar to the original, plus a requirement to preserve induction, plus Shore blocking. Lemma generalizes as one would expect.

Cf.

Theorem (Towsner 2015)

Suppose $\mathcal{M} \models I\Sigma^0_n$ and $P$ is any set. Then there is a $B$ such that $\mathcal{M}[B] \models I\Sigma^0_n$, and $P$ is $\Delta^0_{n+1}$-definable in $B$. 
Extending the method

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<td><em>If $\mathcal{N} \models RCA_0 + B\Sigma^0_n$ or $\mathcal{N} \models RCA_0 + I\Sigma^0_n, n \geq 2...</em></td>
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Get a similar conservation result for COH in each of these cases. Do not, however, get one over $RCA_0$ alone. Instead, it seems the original Mathias-forcing proof in CJS is optimal.
Questions

Question

*Is* RCA$_0$ *the* base system?

Question

If WKL$_0$ *is* $\Sigma^0_1$ separation and COH *is* $\Sigma^0_2$ separation, then what *is* $\Sigma^0_3$ separation?

Question

*Can you define principles of* $\Sigma^0_n$ separation, *prove mutual independence, and prove conservation theorems for them?*

Yes. But why would you want to do that?

Question

*Is there a use for principles such as: Every* $\Delta^0_1$ 2-colouring of triples *has an infinite* $\Delta^0_2$ homogeneous set?*
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Homework

Worry about ‘A is $\Delta^0_1$ in B’ versus $A \leq_T B$. Worry about $\Delta^0_2$ vs $\leq_T 0'$ in models of $B\Sigma^0_2$. Worry about transitivity of $\leq_T$ in models of $B\Sigma^0_1$. 