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ASYMPTOTIC NORMALITY OF SCALING FUNCTIONS*

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Abstract. The Gaussian function $G(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, which has been a classical choice for multiscale representation, is the solution of the scaling equation

$$G(x) = \int_{\mathbb{R}} \alpha G(\alpha x - y) dg(y), \quad x \in \mathbb{R},$$

with scale $\alpha > 1$ and absolutely continuous measure

$$dg(y) = \frac{1}{\sqrt{2\pi(\alpha^2 - 1)}} e^{-y^2/2(\alpha^2-1)} dy.$$

It is known that the sequence of normalized B-splines (B_n), where B_n is the solution of the scaling equation

$$\phi(x) = \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} \phi(2x - j), \quad x \in \mathbb{R},$$

converges uniformly to G. The classical results on normal approximation of binomial distributions and the uniform B-splines are studied in the broader context of normal approximation of probability measures m_n, $n = 1, 2, \dots$, and the corresponding solutions ϕ_n of the scaling equations

$$\phi_n(x) = \int_{\mathbb{R}} \alpha \phi_n(\alpha x - y) dm_n(y), \quad x \in \mathbb{R}.$$

Various forms of convergence are considered and orders of convergence obtained. A class of probability densities are constructed that converge to the Gaussian function faster than the uniform B-splines.

Key words. normal approximation, probability measures, scaling functions, uniform B-splines, asymptotic normality

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1. Introduction.

The Gaussian function, $G(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and its derivatives have been widely used in scale-space representation (see [1], [11], [18]). The uniform B-spline, B_n, which is the solution of the scaling equation

$$\phi(x) = \sum_{j=0}^{n} \frac{1}{2^{n-1}} \binom{n}{j} \phi(2x - j), \quad x \in \mathbb{R}, \tag{1.1}$$

associated with the binomial distribution $\frac{1}{2^n}\binom{n}{j}$, $j = 0, 1, \dots, n$, approximates the Gaussian and provides fast computational algorithms for practical implementation of Gaussian scale-space representation (see [15], [16]). The B-spline, B_n, is the probability density function of the sum of n copies of independent identically distributed

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uniform random variables on the interval $[0,1)$. It is well known that the binomial distributions converge to the normal distribution in the sense that

$$\lim_{n\to\infty}\sum_{k=0}^{[x_n]}\frac{1}{2^n}\binom{n}{k}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt, \tag{1.2}$$

where $x_n=\sqrt{n}x/2+n/2$, and it is also known that

$$\lim_{n\to\infty}\int_{-\infty}^{x'_n}B_n(t)dt=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt, \tag{1.3}$$

where $x'_n:=\sqrt{n}x/2\sqrt{3}+n/2$. Further, the normalized B-splines converge uniformly on $\mathbb{R}$ to the Gaussian function (see [5] and [13]). In fact, Curry and Schoenberg [5] considered the more general class of Polya frequency functions as limits of nonuniform B-splines with arbitrary knots. The Gaussian function satisfies the integral scaling equation

$$G(x)=\int_{\mathbb{R}}\alpha G(\alpha x-y)dg(y),\quad x\in\mathbb{R},$$

where $\alpha>1$ is a scaling constant and g is the absolutely continuous measure given by

$$dg(y)=\frac{1}{\sqrt{2\pi(\alpha^2-1)}}e^{-y^2/2(\alpha^2-1)}dy.$$

The Gaussian function and its derivatives and the modulated Gaussian have been used extensively in many applications such as scale-space analysis and computer vision (see [1], [11], [18]). The normal approximation of the binomial distributions and the uniform B-splines enables the binomial coefficients and B-splines to replace the Gaussian function in the Gaussian scale-space representation and vice versa (see [11], [15], [16]). The Gaussian function is optimal in time-frequency localization, amenable to statistical analysis, and provides an accurate model of human vision (see [18]). While inheriting approximately many of the rich properties of the Gaussian, the binomial distributions and B-splines have the added advantage of providing fast algorithms for practical computations.

We shall consider a sequence of scaling equations

$$\phi_n(x)=\int_{\mathbb{R}}\alpha\phi_n(\alpha x-y)dm_n(y),\quad x\in\mathbb{R},\ n=1,2,\dots, \tag{1.4}$$

where $\alpha>1$ and (m_n) is a sequence of probability measures with finite first and second moments. It will be shown in the next section that for each n, (1.4) has a unique solution, which is also a probability measure. We shall call ϕ_n the m_n-*scaling function* and m_n its *filter*. If m_n is a discrete measure concentrated on the integers $\mathbb{Z}$ with mass $h_n(j)$ at $j\in\mathbb{Z}$, then (1.4) becomes the discrete scaling equation

$$\phi_n(x)=\sum_{j\in\mathbb{Z}}\alpha h_n(j)\phi_n(\alpha x-j),\quad x\in\mathbb{R}. \tag{1.5}$$

In particular, if m_n is the discrete measure concentrated on the set $\{0,1,\dots,n\}$ with mass $\frac{1}{2^n}\binom{n}{j}$ at $j=0,1,\dots,n$ and scale $\alpha=2$, then (1.5) reduces to (1.1). The object of this paper is to investigate the approximation of the Gaussian function by probability measures and the corresponding scaling functions in the same way as the normal approximation by binomial and B-spline distributions and to construct sequences of distributions that converge to the Gaussian faster than the binomial and B-spline distributions.

Suppose that (m_n) is a sequence of probability measures on $\mathbb{R}$ with mean $\mu(m_n) = \mu_n$ and standard deviation $\sigma(m_n) = \sigma_n$, and define

$$\widetilde{m}_n(S) = m_n(\sigma_n S + \mu_n) \quad \text{for measurable } S \subset \mathbb{R},$$

or, equivalently,

$$\widehat{\widetilde{m}}_n(u) = e^{iu\mu_n/\sigma_n}\widehat{m}_n(u/\sigma_n), \quad u \in \mathbb{R}. \tag{1.6}$$

We say that (m_n) is *asymptotically normal* if for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \int_{-\infty}^{x} d\widetilde{m}_n(t) = \int_{-\infty}^{x} G(t)dt. \tag{1.7}$$

If m_n is absolutely continuous, then by the Radon–Nikodym theorem, $dm_n(t) = f_n(t)dt$ for a probability density function f_n, and then $d\widetilde{m}_n(t) = \widetilde{f}_n(t)dt$, where

$$\widetilde{f}_n(t) = \sigma_n f_n(\sigma_n t + \mu_n).$$

The central limit theorem tells us that if m_n is the probability distribution for the sum of n independent, identically distributed random variables, then (m_n) is asymptotically normal. In the case that each such random variable is uniformly distributed on the interval $[0, 1)$, m_n has density function B_n, and the asymptotic normality is also implied by the convergence of the normalized B-splines discussed earlier. Now it is well known that asymptotic normality can be stated in terms of convergence of characteristic functions, i.e., Fourier transforms of the probability density functions. To be precise, (1.7) is equivalent to

$$\widehat{\widetilde{m}}_n(u) \to e^{-u^2/2} \text{ locally uniformly on } \mathbb{R}, \tag{1.8}$$

where local uniform convergence means convergence that is uniform on compact subsets. This result is given in [7, p. 249], and more modern expositions are given in [10] and [17].

In section 2, we show that if m is a probability measure on $\mathbb{R}$ with finite first moment, then the solution of the scaling equation

$$\phi(x) = \int_{\mathbb{R}} \alpha\phi(\alpha x - y)dm(y), \quad x \in \mathbb{R}, \tag{1.9}$$

is also a probability measure. In (1.9), and throughout the paper, α is a number larger than 1, which we call the *scale*. We remark that if the solution is absolutely continuous, then its probability density satisfies (1.9). If the solution ϕ is not absolutely continuous, then it satisfies (1.9) in the weak sense, i.e.,

$$\widehat{\phi}(u) = \widehat{m}(u/\alpha)\widehat{\phi}(u/\alpha), \quad u \in \mathbb{R}. \tag{1.10}$$

The following result puts in perspective the asymptotic normality exhibited by the binomial coefficients and the uniform B-splines.

THEOREM 1.1. *Let (m_n) be a sequence of probability measures on $\mathbb{R}$ with finite first and second moments and $(\widehat{\widehat{m}}_n{}')$ be uniformly bounded in a neighborhood of the origin. Then (m_n) is asymptotically normal if and only if the corresponding sequence of m_n-scaling functions is asymptotically normal.*

In order to study the asymptotic normality of scaling functions, we need only to study the asymptotic normality of their filters, because of Theorem 1.1. The binomial coefficients, which are the filters for the uniform B-splines, define a sequence of discrete probability measures that is asymptotically normal. It follows from Theorem 1.1 that the coefficients $b_{n,k}$ in the expansion

$$\left(\frac{1+z+\cdots+z^{\alpha-1}}{\alpha}\right)^n = \sum_{k=0}^{n(\alpha-1)} b_{n,k} z^k, \tag{1.11}$$

where the scale α is here an integer, also define a sequence of probability measures that is asymptotically normal. This is because the uniform B-splines are also the solution of the scaling equations with measures $m_n(k) = b_{n,k}$, $k = 0, 1, \ldots, n(\alpha-1)$, for any integer scale $\alpha > 1$. For such α, the roots of the polynomials on the left of (1.11) that generate $b_{n,k}$ are the complex αth roots of unity that are not equal to 1. The next theorem gives a general result that holds for a large class of polynomials including those with negative roots as well as those in (1.11).

THEOREM 1.2. *Let $\gamma \in [0, \pi/2)$, and define $D_\gamma = \{z \in \mathbb{C} : \text{ satisfies } (1.12)\}$:*

$$\left|\operatorname{Im}\left\{\frac{z}{(1+z)^2}\right\}\right| \le \tan\gamma \ \operatorname{Re}\ \left\{\frac{z}{(1+z)^2}\right\}. \tag{1.12}$$

For $n = 1, 2, \ldots$, take $r_{n,1}, \ldots, r_{n,n}$ in D_γ and define

$$\sum_{k=0}^{n} a_{n,k} z^k = \prod_{j=1}^{n} (z + r_{n,j})/(1 + r_{n,j}). \tag{1.13}$$

We also assume that the $r_{n,j}$, $n = 1, 2, \ldots$, $j = 1, \ldots, n$, are bounded away from -1, that the coefficients $a_{n,k}$, $n = 1, 2, \ldots$, $k = 0, \ldots n$, are real, and that

$$\sigma_n^2 = \sum_{j=1}^{n} r_{n,j}/(1 + r_{n,j})^2 \to \infty \text{ as } n \to \infty. \tag{1.14}$$

If m_n, $n = 1, 2, \ldots$, denote the discrete measures defined by $m_n(\{k\}) = a_{n,k}$, $k = 0, 1, \ldots, n$, it follows that $\widehat{\widehat{m}}_n(u) \to e^{-u^2/2}$ locally uniformly as $n \to \infty$. If, in addition, $a_{n,k} \ge 0$, $k = 0, 1, \ldots, n$, for all sufficiently large n, then (m_n) is asymptotically normal.

Remark 1. We remark that the first part of Theorem 1.2 does not require m_n to be a probability measure; i.e., some of the coefficients $a_{n,k}$ could be negative.

After some preliminary results in the next section, we shall prove Theorem 1.1 in section 3. A proof of Theorem 1.2 is given in section 4. We note that a special case of this result, when all $r_{n,j} > 0$, was proved earlier using probabilistic techniques [3]. The completely different analytic techniques, which we employ here, give considerably more general results. These techniques also allow us to analyze, in the remainder of section 4, the order of convergence in the frequency domain for both the measures m_n and the corresponding scaling functions. In particular we shall prove the following theorem.

THEOREM 1.3. *We assume the conditions of Theorem* 1.2 *and that* $a_{n,k} \geq 0$, $k = 0, 1, \ldots, n$.

(a) *Then*

$$\left\| \widehat{\widetilde{\phi}}_n - e^{-(\cdot)^2/2} \right\|_\infty = O(\sigma_n^{-1}).$$

(b) *If* $\sum_{k=0}^{n} a_{n,k} z^k$ *is a reciprocal polynomial, i.e.,* $a_{n,0} \neq 0$ *and* $a_{n,k} = a_{n,n-k}$, $k = 0, 1, \ldots, n$, *then*

$$\left\| \widehat{\widetilde{\phi}}_n - e^{-(\cdot)^2/2} \right\|_\infty = O(\sigma_n^{-2}).$$

(c) *If, in addition to the condition in* (b),

$$\sigma_n^{-1} \sum_{j=1}^{n} r_{n,j}(r_{n,j}^2 - 4r_{n,j} + 1)/(1 + r_{n,j})^4 \text{ is bounded,} \tag{1.15}$$

then

$$\left\| \widehat{\widetilde{\phi}}_n - e^{-(\cdot)^2/2} \right\|_\infty = O(\sigma_n^{-3}).$$

Asymptotic normality entails weak convergence in the time domain. We show in section 5 that, under mild conditions on the shape of the filters and the scaling functions, both the measures and the corresponding scaling functions converge uniformly in the time domain. The shape conditions are satisfied if $r_{n,j}$ are restricted to certain sectors of the complex plane, reminiscent of total positivity. It is noted that for a special case of the choice, when all $r_{n,j} > 0$, Chui and Wang [4] consider convergence of the scaling functions. However, their approach is different, and they do not consider the related convergence of the measures m_n. Finally, in the same section, we consider the order of convergence in the time domain and prove the following results.

THEOREM 1.4. *We assume the conditions of Theorem* 1.2 *and that all* $r_{n,j}$ *lie in the sector* $|\arg z| \leq \frac{\pi}{3}$. *Then as* $n \to \infty$,

$$\max_{k=0,\ldots,n} \left| \sigma_n a_{n,k} - G\left(\frac{k - \mu_n}{\sigma_n}\right) \right| = O(\sigma_n^{-\frac{1}{2}}),$$

and if $\sum_{k=0}^{n} a_{n,k} z^k$ *is reciprocal,*

$$\max_{k=0,\ldots,n} \left| \sigma_n a_{n,k} - G\left(\frac{k - \mu_n}{\sigma_n}\right) \right| = O(\sigma_n^{-\frac{2}{3}}).$$

We remark that in [3], this problem is considered, using probabilistic techniques, for the special case when $a_{n,0}, \ldots, a_{n,n}$ are the Eulerian numbers. In this case $\sigma_n = \sqrt{\pi(n+1)/6}$. Thus our result gives order of convergence $O(\sigma_n^{-\frac{2}{3}}) = O(n^{-\frac{1}{3}})$, while [3] shows only convergence $O(n^{-\frac{1}{4}})$.

THEOREM 1.5. *We assume the conditions of Theorem* 1.2, *that* $r_{n,j}$ *include* 1 *and all* $\mathrm{Re}(r_{n,j}) \geq 0$. *For* $n = 1, 2, \ldots$, *let* ϕ_n *denote the scaling function corresponding to the measure* $m_n(\{k\}) = a_{n,k}$, $k = 0, 1, \ldots, n$, *with scale* 2, *and define*

$$\widetilde{\phi}_n(x) = \sigma(\phi_n)\phi_n(\sigma(\phi_n)x + \mu(\phi_n)), \quad x \in \mathbb{R}.$$

Then

$$\|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-\frac{1}{2}}).$$

If $\sum_{n=0}^{n} a_{n,k} z^k$ *is reciprocal for large enough* n, *then*

$$\|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-1}).$$

If, in addition, (1.15) *is satisfied, then*

$$\|\widetilde{\phi}_n - G\|_\infty = O(\sigma_n^{-\frac{3}{2}}).$$

It is noted that certain sequences of scaling functions give a faster rate of convergence to the Gaussian than the uniform B-splines. Also on considering Theorems 1.3 and 1.5, it might be expected that the second part of Theorem 1.4 should give order of convergence $O(\sigma_n^{-1})$ instead of $O(\sigma_n^{-2/3})$ and that under the additional condition (1.15) we should obtain order $O(\sigma_n^{-3/2})$. We have been unable to prove orders better than $O(\sigma_n^{-2/3})$ due to a technical restriction in Lemma 5.4, and we do not know whether this restriction can be removed.

2. Probability measures and scaling equations. Consider the scaling equation (1.9) where m is a probability measure and, as before, α is a number (not necessarily an integer) satisfying $\alpha > 1$. We shall show that (1.9) has a unique solution, which is a probability measure. Further, if m has finite first and second moments, then the solution of (1.9) also has finite first and second moments. Equation (1.10) suggests that, when $\widehat{\phi}(0) = 1$, $\widehat{\phi}(u)$ is given by the infinite product (2.1) below but with n replaced by ∞. We remark that products of the form (2.1) occur in the study of groups of transformations in Hilbert space (see, for example, [6, section 38]). For the case when ϕ is the B-spline B_n and $\alpha = 2$, this reduces to the classical formula of Viète:

$$\sin x/x = \prod_{j=1}^{\infty} \cos(x/2^j).$$

So as a preliminary result we need to consider the convergence of (2.1) in Lemma 2.1 below.

LEMMA 2.1. *Suppose that* m *is a probability measure with finite first moment. Then the products*

$$\prod_{j=1}^{n} \widehat{m}(u/\alpha^j), \quad u \in \mathbb{R}, \tag{2.1}$$

converge locally uniformly as $n \to \infty$.

Proof. Since m is a probability measure, $|\widehat{m}(u)| \le 1$ for all $u \in \mathbb{R}$. Then for every nonnegative integer n and all u,

$$\left|\prod_{j=1}^{n} \widehat{m}(u/\alpha^j)\right| \le 1 \quad \text{for all } u \in \mathbb{R}.$$

Also, since m has finite first moment, $\widehat{m}'$ is bounded, and so

$$\left|\widehat{m}(u/\alpha^j) - 1\right| \le C|u|/\alpha^j, \quad j = 1, 2, \dots ,$$

for a constant $C > 0$. Thus for integers $n > \ell$,

$$\left|\prod_{j=1}^{\ell} \widehat{m}(u/\alpha^j) - \prod_{j=1}^{n} \widehat{m}(u/\alpha^j)\right| \leq \sum_{j=1}^{n-\ell} |1 - \widehat{m}(u/\alpha^{\ell+j})| \leq C|u|(\alpha^{-\ell} - \alpha^{-n})/(\alpha - 1),$$

which tends to zero uniformly on compact subsets of $\mathbb{R}$ as $\ell, n \to \infty$. Therefore, the product $\prod_{j=1}^{n} \widehat{m}(u/\alpha^j)$ converges uniformly on compact sets as $n \to \infty$. □

Proposition 2.2. *If m is a probability measure with finite first and second moments, then the scaling equation* (1.9) *has a unique solution ϕ, which is also a probability measure with finite first and second moments. Further,*

$$\mu(\phi) = (\alpha - 1)^{-1}\mu(m) \quad \textit{and} \quad \sigma(\phi)^2 = (\alpha^2 - 1)^{-1}\sigma(m)^2. \tag{2.2}$$

Proof. Choose a nonnegative initial function $f_0 \in C(\mathbb{R})$ with compact support and $\widehat{f}_0(0) = 1$, and for $n = 1, 2, \ldots$ define

$$f_n(x) = \int_{\mathbb{R}} \alpha f_{n-1}(\alpha x - y) dm(y), \quad x \in \mathbb{R}. \tag{2.3}$$

Then

$$\widehat{f}_n(u) = \widehat{f}_{n-1}(u/\alpha)\widehat{m}(u/\alpha) = \prod_{j=1}^{n} \widehat{m}(u/\alpha^j)\widehat{f}_0(u/\alpha^n), \quad u \in \mathbb{R}. \tag{2.4}$$

Further, f_n is nonnegative, and $\widehat{f}_n(0) = 1$ for $n = 0, 1, \ldots$. Therefore, f_n defines a sequence of probability measures $\mu_n \in C_0(\mathbb{R})^*$, where $d\mu_n(x) = f_n(x)dx$ and $C_0(\mathbb{R})^*$ is the dual of the space $C_0(\mathbb{R})$ of continuous functions that vanish at infinity. Therefore, $\widehat{\mu}_n = \widehat{f}_n$, $n = 0, 1, \ldots$. Since the unit ball in $C_0(\mathbb{R})^*$ is weak* compact, there exist a subsequence μ_{n_ℓ} and a probability measure ϕ on $\mathbb{R}$ such that $\mu_{n_\ell} \to \phi$ as $\ell \to \infty$ in the weak* topology. It follows (see [7, p. 249]) that $\widehat{\mu}_{n_\ell}$ converges locally uniformly to $\widehat{\phi}$ as $n \to \infty$. By Lemma 2.1 and (2.4),

$$\widehat{\phi}(u) = \prod_{j=1}^{\infty} \widehat{m}(u/\alpha^j), \quad u \in \mathbb{R},$$

which satisfies (1.10).

Define

$$\Pi_n(u) := \prod_{j=1}^{n} \widehat{m}(u/\alpha^j), \quad u \in \mathbb{R}. \tag{2.5}$$

Then

$$\Pi_n(u) \to \widehat{\phi}(u) \text{ locally uniformly on } \mathbb{R}, \tag{2.6}$$

where ϕ is the solution of (1.9). We shall show that ${\Pi_n}'$ converges uniformly in a neighborhood of the origin. Since $\widehat{m}(0) = 1$, there exists a closed disc D centered at the origin such that $\widehat{m}(u) \neq 0$ for all $u \in D$. Differentiating (2.5) gives

$${\Pi_n}'(u) = \prod_{j=1}^{n} \widehat{m}(u/\alpha^j) \sum_{j=1}^{n} \frac{1}{\alpha^j} \frac{\widehat{m}\,'(u/\alpha^j)}{\widehat{m}(u/\alpha^j)}, \tag{2.7}$$

which shows that $\Pi_n{}'$ is uniformly convergent on D. It follows that $\widehat{\phi}\,'$ exists and $\Pi_n{}'$ converges uniformly to $\widehat{\phi}\,'$ on D. Hence $\widehat{\phi}\,'$ is continuous on D, and

$$\widehat{\phi}\,'(0) = (\alpha - 1)^{-1}\widehat{m}\,'(0). \tag{2.8}$$

Differentiating (2.7) gives

$$\begin{aligned}\Pi_n{}''(u) &= \prod_{j=1}^{n} \widehat{m}(u/\alpha^j) \left(\sum_{j=1}^{n} \frac{1}{\alpha^j} \frac{\widehat{m}\,'(u/\alpha^j)}{\widehat{m}(u/\alpha^j)} \right)^2 \\ &\quad + \prod_{j=1}^{n} \widehat{m}(u/\alpha^j) \sum_{j=1}^{n} \frac{1}{\alpha^{2j}} \frac{\widehat{m}\,''(u/\alpha^j)\widehat{m}(u/\alpha^j) - \widehat{m}\,'(u/\alpha^j)^2}{\widehat{m}(u/\alpha^j)^2},\end{aligned} \tag{2.9}$$

which shows that $\Pi_n{}''$ is uniformly convergent on D. Thus $\widehat{\phi}\,''$ exists and is continuous on D. A straightforward computation using (2.9) leads to

$$\widehat{\phi}\,''(0) = \frac{1}{(\alpha^2 - 1)} \left\{ \widehat{m}\,''(0) + \frac{2\widehat{m}\,'(0)^2}{(\alpha - 1)} \right\}. \tag{2.10}$$

It follows that ϕ has finite first and second moments, and the relationships (2.2) follow from (2.8) and (2.10). □

3. Proof of Theorem 1.1. We shall prove a slightly stronger result than that of Theorem 1.1. This result is contained in Theorem 3.1.

THEOREM 3.1. *Let* (m_n) *be a sequence of probability measures on* $\mathbb{R}$ *with finite first and second moments, and* $(\widehat{\widetilde{m}}_n{}')$ *is uniformly bounded in a neighborhood of* 0. *Then the following are equivalent:*

(a) $\widehat{\widetilde{m}}_n(u) \to e^{-u^2/2}$ *locally uniformly on* $\mathbb{R}$ *as* $n \to \infty$.

(b) $\widehat{\widetilde{\phi}}_n(u) \to e^{-u^2/2}$ *locally uniformly on* $\mathbb{R}$ *as* $n \to \infty$.

(c) (m_n) *is asymptotically normal.*

(d) (ϕ_n) *is asymptotically normal.*

Further, if (a) *holds locally uniformly on* $\mathbb{R}$, *then* (b) *holds uniformly on* $\mathbb{R}$.

Proof. By Proposition 2.2, for each $n = 0, 1, \ldots$, (1.4) has a unique solution ϕ_n, which is also a probability measure with finite first and second order moments, and

$$\mu(m_n) = (\alpha - 1)\mu(\phi_n) \quad \text{and} \quad \sigma(m_n)^2 = (\alpha^2 - 1)\sigma(\phi_n)^2. \tag{3.1}$$

By (1.6), (1.10), and (3.1),

$$\widehat{\widetilde{\phi}}_n(u) = \widehat{\widetilde{m}}_n(\alpha^{-1}\sqrt{\alpha^2 - 1}\, u)\widehat{\widetilde{\phi}}_n(\alpha^{-1}u), \quad u \in \mathbb{R}. \tag{3.2}$$

Iterating (3.2) leads to

$$\widehat{\widetilde{\phi}}_n(u) = \prod_{j=1}^{\infty} \widehat{\widetilde{m}}_n(\alpha^{-j}\sqrt{\alpha^2 - 1}\, u), \quad u \in \mathbb{R}, \tag{3.3}$$

where the infinite product on the right converges locally uniformly on $\mathbb{R}$ and uniformly in n, since $(\widehat{\widetilde{m}}_n{}')$ is uniformly bounded in a neighborhood of 0.

If (a) holds, then by (3.3) we have

$$\lim_{n\to\infty} \hat{\hat{\phi}}_n(u) = \lim_{n\to\infty} \prod_{j=1}^{\infty} \hat{\hat{m}}_n(\alpha^{-j}\sqrt{\alpha^2-1}\,u)$$
$$= \prod_{j=1}^{\infty} e^{-(\alpha^2-1)u^2/2\alpha^{2j}} = e^{-u^2/2}, \quad u \in \mathbb{R}.$$

Conversely, if $\lim_{n\to\infty} \hat{\hat{\phi}}_n(u) = e^{-u^2/2}$, then by (3.2)

$$\hat{\hat{m}}_n(u) = \frac{\hat{\hat{\phi}}_n(\alpha\, u/\sqrt{\alpha^2-1})}{\hat{\hat{\phi}}_n(u/\sqrt{\alpha^2-1})}, \quad u \in \mathbb{R},$$

for sufficiently large n. It follows that

$$\lim_{n\to\infty} \hat{\hat{m}}_n(u) = \frac{e^{-\alpha^2u^2/2(\alpha^2-1)}}{e^{-u^2/2(\alpha^2-1)}} = e^{-u^2/2}, \quad u \in \mathbb{R}.$$

A similar argument shows that (a) holds locally uniformly on $\mathbb{R}$ if and only if (b) holds locally uniformly on $\mathbb{R}$.

Now suppose that (a) holds uniformly on compact subsets of $\mathbb{R}$. Note that for any $u \in \mathbb{R}$ and $n \geq 1$,

$$|\hat{\hat{m}}_n(u)| = \left| \int_{-\infty}^{-\infty} e^{-iux} dm_n(x) \right| \leq \int_{-\infty}^{-\infty} dm_n(x) = 1.$$

So for any $k \geq 1$,

$$\left|\hat{\hat{\phi}}_n(u)\right| = \prod_{j=1}^{\infty} \left|\hat{\hat{m}}_n(\alpha^{-j}\sqrt{\alpha^2-1}\,u)\right|$$
$$\leq \prod_{j=k+1}^{\infty} \left|\hat{\hat{m}}_n(\alpha^{-j}\sqrt{\alpha^2-1}\,u)\right|$$
$$= \left|\hat{\hat{\phi}}_n(\alpha^{-k}\,u)\right|, \quad u \in \mathbb{R}. \tag{3.4}$$

For any $\epsilon > 0$, we choose $A > 0$ and integer N so that $e^{-A^2/2} < \epsilon$ and

$$\left|\hat{\hat{\phi}}_n(u) - e^{-u^2/2}\right| < \epsilon, \quad |u| \leq \alpha A,\ n > N.$$

Take any u with $|u| > A$. Then there is a nonnegative integer k such that $A < \alpha^{-k}|u| \leq \alpha A$, and so

$$e^{-(\alpha^{-k}u)^2/2} < e^{-A^2/2} < \epsilon.$$

Also for $n > N$, $|\hat{\hat{\phi}}_n(\alpha^{-k}u) - e^{-(\alpha^{-k}u)^2/2}| < \epsilon$, and so

$$\left|\hat{\hat{\phi}}_n(u)\right| \leq \left|\hat{\hat{\phi}}_n(\alpha^{-k}\,u)\right| < 2\epsilon.$$

Since $e^{-u^2/2} < e^{-A^2/2} < \epsilon$, it follows that $|\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}| < 3\epsilon$. Thus for all $n > N$ and $u \in \mathbb{R}$, $|\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}| < 3\epsilon$, and hence (b) holds uniformly on $\mathbb{R}$.

Recall that the asymptotic normality of a sequence of distribution functions is equivalent to the local uniform convergence of their characteristic functions (see, for instance, [7, p. 249]). $\square$

We remark that if (m_n) is a sequence of discrete probability measures on $\mathbb{Z}$ with finite first and second moments, then the condition that $(\widehat{\widetilde{m}}_n{}')$ be uniformly bounded in a neighborhood of 0 is automatically satisfied. The following lemma gives a slightly stronger result.

LEMMA 3.2. *If (m_n) is a sequence of discrete probability measures on $\mathbb{Z}$ with finite first and second moments, then $(\widehat{\widetilde{m}}_n{}')$ is uniformly bounded on any compact subset of $\mathbb{R}$.*

Proof. Let $m_n(\{k\}) = b_{n,k} \geq 0$, $n = 1, 2, \ldots$, $k \in \mathbb{Z}$, where $\sum_{k=-\infty}^{\infty} b_{n,k} = 1$. As before, we write

$$\mu_n := \sum_{k=-\infty}^{\infty} k b_{n,k} \quad \text{and} \quad \sigma_n^2 := \sum_{k=-\infty}^{\infty} (k - \mu_n)^2 b_{n,k}.$$

Then

$$\widehat{\widetilde{m}}_n(u) = \sum_{k=-\infty}^{\infty} b_{n,k} e^{i(\mu_n - k)u/\sigma_n},$$

and so

$$\begin{aligned}\widehat{\widetilde{m}}_n{}'(u) &= \frac{i}{\sigma_n} \sum_{k=-\infty}^{\infty} b_{n,k}(\mu_n - k) e^{i(\mu_n - k)u/\sigma_n} \\ &= \frac{i}{\sigma_n} \sum_{k=-\infty}^{\infty} b_{n,k}(\mu_n - k)(e^{i(\mu_n - k)u/\sigma_n} - 1).\end{aligned}$$

Since $|e^{iu} - 1| \leq 2|u|$ for all $u \in \mathbb{R}$,

$$\left|\widehat{\widetilde{m}}_n{}'(u)\right| \leq \frac{2|u|}{\sigma_n^2} \sum_{k=-\infty}^{\infty} (k - \mu_n)^2 b_{n,k} = 2|u|. \qquad \square$$

COROLLARY 3.3. *Let (m_n) be a sequence of discrete probability measures on $\mathbb{Z}$ with finite first and second moments. Then (m_n) is asymptotically normal if and only if the corresponding sequence of m_n-scaling functions with scale α is asymptotically normal.*

4. Convergence in the frequency domain. In order to apply Theorem 1.1 to study the asymptotic normality of scaling functions, we need first to study the asymptotic normality of their filters. We begin with a proof of Theorem 1.2.

Proof of Theorem 1.2. Let

$$\sum_{k=0}^{n} a_{n,k} z^k = \prod_{j=1}^{n} (p_{n,j} z + q_{n,j}), \tag{4.1}$$

where $q_{n,j} = 1 - p_{n,j}$. Then

$$\widehat{m}_n(u) = \prod_{j=1}^{n} (p_{n,j} e^{-iu} + q_{n,j})$$

and

$$\widehat{\widetilde{m}}_n(u) = e^{iu\mu_n/\sigma_n} \prod_{j=1}^{n} (p_{n,j} e^{-iu/\sigma_n} + q_{n,j}),$$

where

$$\mu_n = \mu(m_n) = \sum_{j=1}^{n} p_{n,j}, \tag{4.2}$$

and

$$\sigma_n^2 = \sigma(m_n)^2 = \sum_{j=1}^{n} p_{n,j} q_{n,j}. \tag{4.3}$$

Therefore,

$$\log \widehat{\widetilde{m}}_n(u) = \frac{iu\mu_n}{\sigma_n} + \sum_{j=1}^{n} F\left(p_{n,j}, \frac{-iu}{\sigma_n}\right), \tag{4.4}$$

where

$$F(p,t) = \log(pe^t + q), \quad q = 1 - p.$$

By induction, for $n = 2, 3, \dots$,

$$F^{(n)}(p,t) := \frac{\partial^n}{\partial t^n} F(p,t) = (pe^t + q)^{-n} pq \sum_{j=0}^{n-2} (-1)^j c_n(j) p^j q^{n-2-j} e^{(j+1)t}, \tag{4.5}$$

where $c_2(j) = \delta_0(j)$, $j \in \mathbb{Z}$, and for $n = 2, 3, \dots$, c_n satisfies the recursive relation

$$c_{n+1}(j) = (j+1)c_n(j) + (n-j)c_n(j-1), \quad j \in \mathbb{Z}. \tag{4.6}$$

From (4.6) we have $\sum_{j=-\infty}^{\infty} c_{n+1}(j) = n \sum_{j=-\infty}^{\infty} c_n(j)$, and since $\sum_{j=-\infty}^{\infty} c_2(j) = 1$, we have

$$\sum_{j=-\infty}^{\infty} c_n(j) = (n-1)!, \quad n = 2, 3. \dots . \tag{4.7}$$

By (4.5) the Taylor series of $F(p,t)$ is given by

$$F(p,t) = \sum_{\nu=0}^{\infty} a_\nu(p) t^\nu, \tag{4.8}$$

where

$$a_0(p) = 0,\ a_1(p) = p,\ a_2(p) = \frac{1}{2} pq, \tag{4.9}$$

and for $\nu = 3, 4, \dots$,

$$a_\nu(p) = \frac{pq}{\nu!} \sum_{k=0}^{\nu-2} (-1)^k c_\nu(k) p^k q^{\nu-2-k}. \tag{4.10}$$

By (4.4) and (4.8),

$$\log \widehat{\widehat{m}}_n(u) = \frac{iu\mu_n}{\sigma_n} + \sum_{j=1}^{n} \sum_{\nu=0}^{\infty} a_\nu(p_{n,j}) \sigma_n^{-\nu} (-iu)^\nu. \tag{4.11}$$

By (4.2), (4.3), and (4.9),

$$\sum_{j=1}^{n} a_1(p_{n,j}) \sigma_n^{-1} (-iu) = -\frac{iu\mu_n}{\sigma_n},$$

$$\sum_{j=1}^{n} a_2(p_{n,j}) \sigma_n^{-2} (-iu)^2 = -\frac{u^2}{2},$$

so that (4.11) becomes

$$\log \widehat{\widehat{m}}_n(u) = -\frac{u^2}{2} + \sum_{\nu=3}^{\infty} \sigma_n^{-\nu} (-iu)^\nu \sum_{j=1}^{n} a_\nu(p_{n,j}). \tag{4.12}$$

Now $r_{n,j} \in D_\gamma$ if and only if

$$\left| \mathrm{Im} \left\{ \frac{r_{n,j}}{(1+r_{n,j})^2} \right\} \right| \le \tan\gamma \ \mathrm{Re} \left\{ \frac{r_{n,j}}{(1+r_{n,j})^2} \right\}$$

or

$$|\mathrm{Im}\,(p_{n,j} q_{n,j})| \le \tan\gamma \ \mathrm{Re}\,(p_{n,j} q_{n,j}) \, .$$

Therefore,

$$|p_{n,j} q_{n,j}| \le \sec\gamma \ \mathrm{Re}\,(p_{n,j} q_{n,j}) \, . \tag{4.13}$$

On the other hand, $r_{n,j}$ being bounded away from -1 is equivalent to

$$|p_{n,j}| \le A - 1, \qquad n = 1, 2, \dots , \quad j = 1, 2, \dots , n, \tag{4.14}$$

for some constant A. By (4.10), (4.13), and (4.14),

$$\begin{aligned} |a_\nu(p_{n,j})| &\le \frac{|p_{n,j} q_{n,j}|}{\nu!} \sum_{k=0}^{\nu-2} c_\nu(k) |p_{n,j}|^k |q_{n,j}|^{\nu-2-k} \\ &\le \sec\gamma \ \mathrm{Re}(p_{n,j} q_{n,j}) A^{\nu-2}/\nu. \qquad (4.15) \end{aligned}$$

By (4.12) and (4.15),

$$\begin{aligned} \left| \log \widehat{\widehat{m}}_n(u) + \frac{u^2}{2} \right| &\le \sec\gamma \ \sum_{\nu=3}^{\infty} \frac{\sigma_n^{-\nu} |u|^\nu}{\nu} \sum_{j=1}^{n} \mathrm{Re}(p_{n,j} q_{n,j}) A^{\nu-2} \\ &\le \sec\gamma \ \sum_{\nu=3}^{\infty} \frac{|u|^\nu}{\nu} \left(\frac{A}{\sigma_n} \right)^{\nu-2} \\ &\le \sec\gamma \ \frac{A|u|^3}{\sigma_n} \left(1 - \frac{A|u|}{\sigma_n} \right)^{-1} \qquad (4.16) \end{aligned}$$

whenever $A|u| < \sigma_n$. Since $\sigma_n \to \infty$ as $n \to \infty$, taking the limits as $n \to \infty$, (4.16) gives $\lim_{n\to\infty} \widehat{\widehat{m}}_n(u) = e^{-u^2/2}$ locally uniformly. □

Recall that the region D_γ in Theorem 1.2 comprises all $z \in \mathbb{C}$ satisfying

$$\left|\operatorname{Im}\left\{\frac{z}{(1+z)^2}\right\}\right| \le \tan\gamma \operatorname{Re}\left\{\frac{z}{(1+z)^2}\right\}.$$

It can be seen that D_γ contains the sector $|\arg z| \le \gamma$, and for $z = \pm re^{i\theta}$, $r > 0$, $\gamma \le \theta \le \pi$, (1.12) is equivalent to

$$\frac{\sin(\frac{\theta-\gamma}{2})}{\sin(\frac{\theta+\gamma}{2})} \le r \le \frac{\sin(\frac{\theta+\gamma}{2})}{\sin(\frac{\theta-\gamma}{2})}.$$

In particular D_γ contains the unit circle $r = 1$.

For the special case of Theorem 1.2, when all $r_{n,j} > 0$, the result was proved using probabilistic methods in [3] and [12]. Our analytic techniques allow us not only to prove asymptotic normality for a much larger class of measures but also, in the next result, to give information on the order of convergence in the frequency domain.

PROPOSITION 4.1. *We assume the conditions of Theorem* 1.2 *(except that we do not require $a_{n,k} \ge 0$, $k = 0, 1, \dots, n$). As before,*

$$\sigma_n^2 = \sum_{j=1}^{n} \frac{r_{n,j}}{(1+r_{n,j})^2}.$$

Then there is a constant $K > 0$ so that for $S_n := \{u : |u| \le K\sigma_n\}$ the following hold.

(a) *There is a constant B such that*

$$\left|\widehat{\widehat{m}}_n(u) - e^{-u^2/2}\right| \le B\sigma_n^{-1}, \quad u \in S_n,\ n = 1, 2, \dots . \tag{4.17}$$

(b) *If $\sum_{k=0}^{n} a_{n,k}z^k$ is a reciprocal polynomial, then there is a constant C such that*

$$\left|\widehat{\widehat{m}}_n(u) - e^{-u^2/2}\right| \le C\sigma_n^{-2}, \quad u \in S_n,\ n = 1, 2, \dots . \tag{4.18}$$

(c) *Finally, if in addition to the condition in* (b), (1.15) *is satisfied, then there is a constant D such that*

$$\left|\widehat{\widehat{m}}_n(u) - e^{-u^2/2}\right| \le D\sigma_n^{-3}, \quad u \in S_n,\ n = 1, 2, \dots . \tag{4.19}$$

Proof. (a) From (4.16) we see that for $|u| \le \frac{1}{4}A^{-1}\cos\gamma\ \sigma_n$,

$$\log\widehat{\widehat{m}}_n(u) + \frac{u^2}{2} \le \sec\gamma\ \frac{4A|u|^3}{3\sigma_n} \le \frac{1}{3}u^2,$$

and so $\log\widehat{\widehat{m}}_n(u) \le -\frac{1}{6}u^2$. By the mean value theorem,

$$\begin{aligned}\left|\widehat{\widehat{m}}_n(u) - e^{-u^2/2}\right| &\le e^{-u^2/6}\left|\log\widehat{\widehat{m}}_n(u) + \frac{u^2}{2}\right| \\ &\le \sec\gamma\left(\frac{4A|u|^3}{3\sigma_n}\right)e^{-u^2/6} \\ &\le B\sigma_n^{-1}\end{aligned}$$

for a constant B, which gives (4.17).

(b) We note from (4.6) and (4.10) that

$$a_3(p) = \frac{pq}{3!}(q-p), \tag{4.20}$$

$$a_4(p) = \frac{pq}{4!}(q^2 - 4pq + p^2). \tag{4.21}$$

Suppose that $P_n(z) = \sum_{k=0}^{n} a_{n,k} z^k$ is a reciprocal polynomial. Then $P_n(z) = 0$ if and only if $P_n(z^{-1}) = 0$. Noting that if $r_{n,j} = r_{n,k}^{-1}$, then $p_{n,j} = q_{n,k}$ and $q_{n,j} = p_{n,k}$, it follows that

$$\sum_{j=1}^{n} a_3(p_{n,j}) = 0. \tag{4.22}$$

So from (4.12) and (4.15),

$$\left| \log \widehat{\widetilde{m}}_n(u) + \frac{u^2}{2} \right| \le \sec\gamma \; \frac{A^2|u|^4}{\sigma_n^2} \; \left(1 - \frac{A|u|}{\sigma_n}\right)^{-1}$$

whenever $A|u| < \sigma_n$. Then (4.18) follows in a similar manner as before.

(c) Finally, we assume (1.15). Then (4.12), (4.21), (4.22), and (4.15) give (4.19). □

We note that $r^2 - 4r + 1 = 0$ when $r = 2 \pm \sqrt{3}$, and so (1.15) requires that in some sense the roots of $P_n(z) := \sum_{k=0}^{n} a_{n,k} z^k$ are close to $-2 \pm \sqrt{3}$. In particular, (1.15) will be satisfied if

$$P_n(z) = Q_{\ell_n}(z)(z^2 + 4z + 1)^{k_n},$$

where Q_{ℓ_n} is a reciprocal polynomial of degree $\ell_n = n - 2k_n$ and $n^{-1/2}\ell_n$ is bounded over n. In this case (4.19) takes the form

$$\left| \widehat{\widetilde{m}}_n(u) - e^{-u^2/2} \right| \le C n^{-3/2}, \quad u \in S_n,\ n = 1, 2, \dots .$$

We now consider the order of convergence of the normalized m_n-scaling functions $\widetilde{\phi}_n$ as in Theorem 1.1, again in the frequency domain. From (3.3) it follows as in (4.4) that

$$\log \widehat{\widetilde{\phi}}_n(u) = \frac{iu\mu_n}{\sigma_n} + \sum_{j=1}^{\infty} \sum_{k=1}^{n} F\left(p_{n,k}, -\frac{iu}{\alpha^j \sigma_n}\right)$$

and as in (4.12) that

$$\log \widehat{\widetilde{\phi}}_n(u) = -\frac{u^2}{2} + \sum_{\nu=3}^{\infty} \frac{(-iu)^\nu}{(\alpha^2 - 1)\sigma_n^\nu} \sum_{j=1}^{n} a_\nu(p_{n,j}).$$

So as in (4.16) there is a constant A with

$$\left| \log \widehat{\widetilde{\phi}}_n(u) + \frac{u^2}{2} \right| \le \frac{A|u|^3}{\sigma_n} \left(1 - \frac{A|u|}{\sigma_n}\right)^{-1}$$

whenever $A|u| < \sigma_n$. By the mean value theorem, for $A|u| < \frac{1}{2}\sigma_n$,

$$\left|\widehat{\widehat{\phi}}_n(u) - e^{-u^2/2}\right| \le \left\{e^{-u^2/2} + \left|\widehat{\widehat{\phi}}(u) - e^{-u^2/2}\right|\right\} \frac{2A|u|^3}{\sigma_n},$$

and so

$$\begin{aligned}\left|\widehat{\widehat{\phi}}(u) - e^{-u^2/2}\right| &\le e^{-u^2/2}\ \frac{2A|u|^3}{\sigma_n}\left(1 - \frac{2A|u|^3}{\sigma_n}\right)^{-1} \\ &\le e^{-u^2/2}\frac{4A|u|^3}{\sigma_n} \\ &\le B\sigma_n^{-1}\end{aligned}$$

if $|u|^3 < \sigma_n/4A$ for some constant B.

Similarly, if P_n is a reciprocal polynomial, then as in the derivation of (4.18), there are constants $A, B > 0$ such that

$$\left|\widehat{\widehat{\phi}}_n(u) - e^{-u^2/2}\right| \le B\sigma_n^{-2}$$

whenever $|u| < A\sigma_n^{1/2}$. Finally, if (1.15) is satisfied, then there are constants $A, B > 0$ with

$$\left|\widehat{\widehat{\phi}}_n(u) - e^{-u^2/2}\right| \le B\sigma_n^{-3}$$

whenever $|u| < A\sigma_n^{3/5}$.

To extend these estimates to all of $\mathbb{R}$ we need the following result.

LEMMA 4.2. *Suppose that m_n is a probability measure, $n = 1, 2, \ldots$, and there is a sequence (β_n) with $\lim \beta_n = 0$ so that*

$$\left|\widehat{\widehat{\phi}}_n(u) - e^{-u^2/2}\right| < \beta_n$$

whenever $|u| \le A|\log \beta_n|$ for some $A > 0$. Then

$$\overline{\lim}_{n\to\infty}\ \beta_n^{-1}\|\widehat{\widehat{\phi}}_n - e^{-(\cdot)^2/2}\|_\infty \le 1.$$

Proof. Take $0 < \epsilon < 1$. Choose n large enough so that

$$2|\log(\beta_n\epsilon)| < \alpha^{-2}A^2\ |\log \beta_n|^2.$$

Take any u in $\mathbb{R}$ with $|u| > A|\log \beta_n|$. Then for some integer $k \ge 1$,

$$\alpha^{-1}A|\log \beta_n| < \alpha^{-k}|u| \le A|\log \beta_n|.$$

Putting $v = \alpha^{-k}|u|$, we have

$$v^2 > \alpha^{-2}A^2|\log \beta_n|^2 > 2|\log(\beta_n\epsilon)|,$$

and so

$$e^{-v^2/2} < \beta_n\epsilon.$$

Since $|\widehat{\widetilde{\phi}}_n(v) - e^{-v^2/2}| < \beta_n$, recalling (3.4) gives

$$\left|\widehat{\widetilde{\phi}}_n(u)\right| \le \left|\widehat{\widetilde{\phi}}(v)\right| < \beta_n(1+\epsilon).$$

Also $e^{-u^2/2} < e^{-v^2/2} < \beta_n\epsilon$, and so

$$\left|\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}\right| < \beta_n(1+2\epsilon).$$

For any u with $|u| \le A|\log\beta_n|$ we have $|\widehat{\widetilde{\phi}}_n(u) - e^{-u^2/2}| < \beta_n$, and thus $\|\widehat{\widetilde{\phi}}_n - e^{-(\cdot)^2/2}\|_\infty \le \beta_n(1+2\epsilon)$ for all $u \in \mathbb{R}$. The result follows. ∎

Proof of Theorem 1.3. Theorem 1.3 follows from Lemma 4.2 and the preceding discussions. ∎

5. Convergence in the time domain. From Theorems 1.1 and 1.2 we can deduce the convergence of $\widetilde{m}_n$ and $\widetilde{\phi}_n$ to the Gaussian function G in the time domain only in the weak sense of (1.7). In this section we shall show that under mild assumptions on $(r_{n,j})$ in Theorem 1.2, both $\widetilde{m}_n$ and $\widetilde{\phi}_n$ have a "nice" shape, which ensures that the convergence is uniform. We consider two possibilities for the shape. For a continuous function ψ, we say ψ is *bell-shaped* if $\psi \ge 0$, $\lim_{x\to\pm\infty}\psi(x) = 0$, and there are two points $\alpha < \beta$ such that ψ is convex on $(-\infty,\alpha]$ and $[\beta,\infty)$ and concave on $[\alpha,\beta]$. We say that ψ is *logconcave* if it is supported on a closed interval, $\psi > 0$, and $\log\psi$ is concave on its interior. Neither of these properties implies the other. We note that in both cases there is a point γ such that ψ is increasing on $(-\infty,\gamma]$ and decreasing on $[\gamma,\infty)$. We also note that logconcavity is equivalent to *total positivity* of order 2, which says that for any $x_1 < x_2$ and $y_1 < y_2$,

$$\begin{vmatrix} \psi(x_1-y_1) & \psi(x_1-y_2) \\ \psi(x_2-y_1) & \psi(x_2-y_2) \end{vmatrix} \ge 0.$$

The following lemma shows that for a sequence of bell-shaped or logconcave functions, asymptotic normality implies uniform convergence. The result was stated in [5] for the case of logconcave functions, but no proof was given.

LEMMA 5.1. *Suppose that (g_n) is a sequence of continuous functions with $\int_{-\infty}^{\infty} g_n = 1$, which are either bell-shaped or logconcave, and for each $x \in \mathbb{R}$,*

$$\lim_{n\to\infty}\int_{-\infty}^{x} g_n = \int_{-\infty}^{x} G. \tag{5.1}$$

Then g_n converges to G uniformly on $\mathbb{R}$.

Proof. By (5.1), for any interval $I \subset \mathbb{R}$,

$$\lim_{n\to\infty}\int_I g_n = \int_I G. \tag{5.2}$$

Take $\epsilon > 0$. Then

$$\lim_{n\to\infty}\int_{-3\epsilon}^{-\epsilon} g_n = \int_{-3\epsilon}^{-\epsilon} G, \qquad \lim_{n\to\infty}\int_{-\epsilon}^{\epsilon} g_n = \int_{-\epsilon}^{\epsilon} G.$$

Since $\int_{-3\epsilon}^{-\epsilon} G < \int_{-\epsilon}^{\epsilon} G$, we have $\int_{-3\epsilon}^{-\epsilon} g_n < \int_{-\epsilon}^{\epsilon} G$ for large enough n. Similarly, for large enough n, $\int_{\epsilon}^{3\epsilon} g_n < \int_{-\epsilon}^{\epsilon} G$. So for large enough n, there are points $-3\epsilon < a_n < -\epsilon <$

$b_n < \epsilon < c_n < 3\epsilon$ with $g_n(a_n) < g_n(b_n) > g_n(c_n)$. For any such n, $\max_{x\in\mathbb{R}} g_n(x)$ occurs only for $x \in (-3\epsilon, 3\epsilon)$. For if $\max_{x\in\mathbb{R}} g_n(x) = g_n(\alpha)$ for $\alpha \le -3\epsilon$, then $g_n(\alpha) > g_n(a_n) < g_n(b_n) > g_n(c_n)$, which contradicts the shape of g_n. Similarly, $\max_{x\in\mathbb{R}} g_n(x)$ cannot occur for $x \ge 3\epsilon$.

Again take $\epsilon > 0$. Choose $\delta > 0$ such that $|G(x) - G(y)| < \epsilon$ whenever $|x - y| < \delta$. Take a function $B \ge 0$ with support in $[0, \delta]$, $\int_0^\delta B = 1$, and $||\widehat{B}||_1 < \infty$. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} B(x-a) g_n(x)dx = \int_{-\infty}^{\infty} B(x-a)G(x)dx$$

uniformly in $a \in \mathbb{R}$. To see this, choose $A > 0$ so that $\int_{|u|>A} |\widehat{B}(u)|du < \epsilon$, and choose N so that

$$|\widehat{g}_n(u) - \widehat{G}(u)| < \epsilon \quad \text{for all } n > N,\ u \in [-A, A].$$

Then for all $n > N$,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} B(x-a)g_n(x)dx - \int_{-\infty}^{\infty} B(x-a)G(x)dx\right| \\
&= \left|\int_{-\infty}^{\infty} e^{-iau}\widehat{B}(u)\widehat{g}_n(u)du - \int_{-\infty}^{\infty} e^{-iau}\widehat{B}(u)\widehat{G}(u)du\right| \\
&\le \int_{|u|>A} |\widehat{B}(u)|\ |\widehat{g}_n(u)|du + \int_{-A}^{A} |\widehat{g}_n(u) - \widehat{G}(u)|\ |\widehat{B}(u)|du \\
&+ \int_{|x|>A} |\widehat{B}(u)|\ |\widehat{G}(u)|du < \epsilon(2 + ||\widehat{B}||_1),
\end{aligned}$$

on noting that $|\widehat{g}_n(u)| \le \int_{-\infty}^{\infty} g_n(u)du = 1$.

Take $z < 0$. Choose N so that for all $n > N$, g_n is increasing on $(-\infty, z]$ and

$$\left|\int_{-\infty}^{\infty} B(x-a)g_n(x)dx - \int_{-\infty}^{\infty} B(x-a)G(x)dx\right| < \epsilon$$

for all $a \in \mathbb{R}$. For $y \le z$, $n > N$,

$$\begin{aligned}
\int_{-\infty}^{\infty} B(x-y+\delta)g_n(x)dx &= \int_{y-\delta}^{y} B(x-y+\delta)g_n(x)dx \\
&\le \int_{y-\delta}^{y} B(x-y+\delta)g_n(y)dx \\
&= g_n(y)\int_{-\infty}^{\infty} B = g_n(y).
\end{aligned}$$

Also for $n > N$,

$$\begin{aligned}
\int_{-\infty}^{\infty} B(x-y+\delta)g_n(x)dx &> \int_{-\infty}^{\infty} B(x-y+\delta)G(x)dx - \epsilon \\
&> \int_{-\infty}^{\infty} B(x-y+\delta)G(y)dx - 2\epsilon \\
&= G(y) - 2\epsilon.
\end{aligned}$$

Thus $g_n(y) > G(y) - 2\epsilon$ for all $n > N$. Similarly, for $y + \delta \le z$, $g_n(y) < G(y) + 2\epsilon$ for all $n > N$. Thus g_n converges to G uniformly on $(-\infty, z - \delta]$. A similar argument holds for $z > 0$, and so g_n converges to G uniformly outside any open interval containing 0.

Once again take $\epsilon > 0$ and choose $\delta > 0$ so that $|G(x) - G(y)| < \frac{\epsilon}{2}$ for $|x - y| \le 2\delta$. Choose N so that for $n > N$, $|g_n(x) - G(x)| < \frac{\epsilon}{2}$ for all $|x| \ge \delta$, and $\max g_n(x)$ occurs only for x in $(-\delta, \delta)$. Take any $n > N$ and x in $(-\delta, \delta)$. Then either $g_n(x) \ge g_n(-\delta)$ or $g_n(x) \ge g_n(\delta)$. Now $g_n(\delta) > G(\delta) - \frac{\epsilon}{2} > G(x) - \epsilon$ and similarly $g_n(-\delta) > G(x) - \epsilon$. Thus $g_n(x) > G(x) - \epsilon$. So we have shown that for any $\epsilon > 0$, there exists an integer N such that for all $n > N$ and all $x \in \mathbb{R}$, $g_n(x) > G(x) - \epsilon$.

Now suppose that g_n does not converge uniformly to G on $\mathbb{R}$. Then there is a number $k > 0$ and a sequence (x_n) with $\lim x_n = 0$ so that for arbitrarily large n,

$$g_n(x_n) > G(x_n) + k \quad \text{and} \quad \log g_n(x_n) > \log G(x_n) + k. \tag{5.3}$$

Choose points $0 < a < a + h < a + 2h < 1$. Then $2G(a + h) > G(a) + G(a + 2h)$ and $2G(-a - h) > G(-a) + G(-a - 2h)$. So for large enough n,

$$2g_n(a + h) > g_n(a) + g_n(a + 2h), \tag{5.4}$$

$$2g_n(-a - h) > g_n(-a) + g_n(-a - 2h). \tag{5.5}$$

Next choose $0 < 2\delta < a$ so that $|G(x) - G(y)| < k/3$ whenever $|x - y| \le \delta$. For large enough n, $x_n + 2\delta < a$ and $x_n + \delta/2 > 0$. Since $g_n \to G$ uniformly on $[\delta/2, \infty)$ and $|G(x_n + \delta) - G(x_n + 2\delta)| < k/3$, we have for large enough n,

$$|g_n(x_n + \delta) - g_n(x_n + 2\delta)| < \frac{k}{2}. \tag{5.6}$$

Also we have $G(x_n + \delta) < G(x_n) + k/3$, and so for large enough n,

$$g_n(x_n + \delta) < G(x_n) + \frac{k}{2}. \tag{5.7}$$

Hence for large enough n, by (5.6) and (5.7),

$$2g_n(x_n + \delta) < g_n(x_n + 2\delta) + G(x_n) + k.$$

Therefore, by (5.3), we see that for arbitrarily large n,

$$2g_n(x_n + \delta) < g_n(x_n) + g_n(x_n + 2\delta). \tag{5.8}$$

Now suppose g_n is bell-shaped. Choose n so that $x_n > -a$, $x_n + 2\delta < a$, and (5.4), (5.5), and (5.8) are satisfied. Let α, β be such that g_n is convex on $(-\infty, \alpha]$ and $[\beta, \infty)$ and concave on $[\alpha, \beta]$. By (5.4) and (5.5), $\beta > a$ and $\alpha < -a$. So g_n is concave on $[-a, a]$, which contradicts (5.8).

Next suppose that g_n is logconcave. A similar (but simpler) argument to that above shows that (5.8) can be replaced by

$$2 \log g_n(x + \delta) < \log g_n(x_n) + \log g_n(x_n + 2\delta),$$

which again gives a contradiction. □

We remark that the uniform convergence of g_n to G on $\mathbb{R}$ and the condition $\int_{-\infty}^{\infty} g_n = 1 = \int_{-\infty}^{\infty} G$ imply that $g_n \to G$ in $L^p(\mathbb{R})$ as $n \to \infty$ for all p, $1 \le p \le \infty$. Since convergence in $L^1(\mathbb{R})$ implies (5.1), the converse of Lemma 5.1 also holds.

From Lemma 5.1 we now derive the uniform convergence of $\widetilde{m}_n$ to G under an extra condition on the numbers $(r_{n,j})$ as in Theorem 1.4.

THEOREM 5.2. *We assume the conditions of Theorem* 1.4. *Then*

$$\lim_{n\to\infty}\left\{\sigma_n a_{n,k} - G\left(\frac{k-\mu_n}{\sigma_n}\right)\right\} = 0 \tag{5.9}$$

uniformly over k in $\mathbb{Z}$.

Proof. Since all $r_{n,j}$ lie in the sector $|\arg z| \le \frac{\pi}{3}$, it follows that the matrix $(a_{n,i-j})$ is totally positive of order 2. Hence $a_{n,k} \ge 0$ and

$$a_{n,k}^2 \ge a_{n,k-1}a_{n,k+1}, \quad k = 1,\dots,n-1, \;\; n = 1,2,\dots\,. \tag{5.10}$$

For $n = 1,2,\dots$, we define ψ_n as follows. Without loss of generality we may assume $a_{n,0}a_{n,n} \ne 0$, and it follows from (5.10) that $a_{n,k} > 0$, $k = 0,\dots,n$. We define ψ_n on $[-\mu_n/\sigma_n, (n-\mu_n)/\sigma_n]$ to be the piecewise linear function with knots $(j-\mu_n)/\sigma_n$, $j = 0,\dots,n$, satisfying

$$\psi_n\left(\frac{j-\mu_n}{\sigma_n}\right) = \log(\sigma_n a_{n,j}), \quad j = 0,1,\dots,n.$$

From (5.10), ψ_n is concave on $[-\mu_n/\sigma_n, (n-\mu_n)/\sigma_n]$. We now extend ψ_n to a continuous concave function on (α,β), where $\alpha = -(\mu_n+1)/\sigma_n$, $\beta = (n-\mu_n+1)/\sigma_n$, and

$$\lim_{x\to\alpha^+}\psi_n(x) = \lim_{x\to\beta^-}\psi_n(x) = -\infty.$$

For $n = 1,2,\dots$, we define

$$g_n(x) = \begin{cases} e^{\psi_n(x)}, & \alpha < x < \beta,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, g_n is logconcave, and

$$g_n\left(\frac{j-\mu_n}{\sigma_n}\right) = \sigma_n a_{n,j}, \quad j = 0,1,\dots,n.$$

As in Theorem 1.2, we define measures m_n, $n = 1,2,\dots$, by

$$m_n(\{k\}) = a_{n,k}, \quad k = 0,1,\dots,n,$$

and it follows that (m_n) is asymptotically normal. We note that for $k \in \mathbb{Z}$,

$$\int_{-\infty}^{\frac{k-\mu_n}{\sigma_n}} d\widetilde{m}_n = \sum_{j=0}^{k} a_{n,j},$$

where we put $a_{n,j} = 0$ for $j > n$. It follows from (5.15) that as $n \to \infty$,

$$\int_{-\infty}^{x} g_n - \int_{-\infty}^{x} d\widetilde{m}_n = O(\sigma_n^{-1})$$

uniformly in x. We can then apply Lemma 5.1 to the sequence of functions $g_n/\int_{-\infty}^{\infty} g_n$ to show that this sequence converges to G on $\mathbb{R}$. Hence g_n converges uniformly to G on $\mathbb{R}$, which by (5.15) gives (5.9). ∎

We now consider the uniform convergence of the normalized m_n-scaling functions $\widetilde{\phi}_n$ to G.

THEOREM 5.3. *Assume the conditions of Theorem* 1.5. *Then* $\widetilde{\phi}_n \to G$ *as* $n \to \infty$ *uniformly on* $\mathbb{R}$.

Proof. It follows from the work of Goodman and Micchelli (see [8]) and the properties of totally positive matrices (see [2]) that the functions ϕ_n, and hence $\widetilde{\phi}_n$, are bell-shaped. The result then follows from Theorem 1.1, Theorem 1.2, and Lemma 5.1. ◻

We remark that if the set of all $r_{n,j}$ lies in Re $z \geq 0$, then the condition that it also lies in D_γ for some $\gamma \in [0, \frac{\pi}{2})$ is equivalent to requiring that for some $\beta \in [0, \frac{\pi}{2})$ the set of all $r_{n,j}$ lying outside the sector $|\arg z| \leq \beta$ is bounded and bounded away from zero. In [4], Chui and Wang consider convergence of the sequence $(\widetilde{\phi}_n)$ as in Theorem 5.3 under the assumption that the polynomial $\sum_{k=0}^{n} a_{n,k} z^k$ is reciprocal and all $r_{n,j}$ are real and positive. They also assume that for $n = 1, 2, \ldots$, $r_{n,j} = 1$ for at least Kn values of j for some fixed $K > 0$. They prove convergence in L^p, $1 \leq p < \infty$, which we have noted is weaker than uniform convergence.

We shall finish the paper by considering the order of uniform convergence for both the measures and the corresponding scaling functions. We first need to extend concepts of bell-shaped and logconcave to discrete measures. Suppose m is a probability measure on $\mathbb{Z}$ with $m(\{j\}) = a_j$, $j \in \mathbb{Z}$. We say m is *bell-shaped* if there are integers $k \leq \ell$ such that

$$\begin{aligned} 2a_j &\leq a_{j-1} + a_{j+1}, \quad j \leq k-1 \text{ and } j \geq \ell + 1, \\ 2a_j &\geq a_{j-1} + a_{j+1}, \quad k \leq j \leq \ell. \end{aligned}$$

We say m in *logconcave* if

$$a_j^2 \geq a_{j-1} a_{j+1}, \quad j \in \mathbb{Z}.$$

LEMMA 5.4. *For* $n = 1, 2, \ldots$, *let* m_n *be a probability measure on* $\{0, 1, \ldots, n\}$ *given by* $m_n(\{k\}) = a_{n,k}$, $k = 0, 1, \ldots, n$, *which is either bell-shaped or logconcave, with mean* μ_n *and standard derivation* σ_n. *Suppose that for some* $K > 0$ *and* $r \geq 1$,

$$\left| \widehat{\widetilde{m}}_n(u) - e^{-u^2/2} \right| \leq K \sigma_n^{-r} \quad \text{for } |u| \leq K\sigma_n. \tag{5.11}$$

Then as $n \to \infty$,

$$\max_{k=0,\ldots,n} \left| \sigma_n a_{n,k} - G\left(\frac{k - \mu_n}{\sigma_n} \right) \right| = O(\sigma_n^{-s}), \tag{5.12}$$

where $s = \min\{\frac{r}{2}, \frac{2}{3}\}$.

Proof. Take a nonnegative function N with support in $[-1, 1]$, $\int_{-\infty}^{\infty} N = 1$, $\|\widehat{N}\|_1 < \infty$, and for some $A > 0$,

$$|\widehat{N}(u)| \leq A(1 + |u|)^{-3r-1}, \quad u \in \mathbb{R}.$$

Take $0 < \delta < 1/2$. Let $B_1(x) := \delta^{-1} N(x/\delta)$ and $B_2(x) := \delta^{-1} N(x/\delta - 4)$. Then B_1 and B_2 have supports on $[-\delta, \delta]$ and $[3\delta, 5\delta]$, respectively, and $\int_{-\infty}^{\infty} B_1 = \int_{-\infty}^{\infty} B_2 = 1$. So

$$\int_{-\infty}^{\infty} B_1 G > G(\delta), \quad \int_{-\infty}^{\infty} B_2 G < G(3\delta).$$

and hence

$$\begin{aligned}\int_{-\infty}^{\infty} B_1 G - \int_{-\infty}^{\infty} B_2 G &> G(\delta) - G(3\delta)\\ &> |G'(\delta)| 2\delta\\ &> |G''(\delta)| 2\delta^2\\ &> |G''(1/2)| 2\delta^2.\end{aligned}$$

Also for $j = 1, 2$,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} B_j d\widetilde{m}_n - \int_{-\infty}^{\infty} B_j G\right| &= \left|\int_{-\infty}^{\infty} \widehat{B}_j(-u)(\widehat{\widetilde{m}}_n(u) - \widehat{G}(u))du\right|\\ &\leq \int_{|u|\geq K\sigma_n} (|\widehat{\widetilde{m}}_n(u)| + \widehat{G}(u))|\widehat{B}_j(-u)|du\\ &\quad + \frac{K}{\sigma_n^r}\int_{-K\sigma_n}^{K\sigma_n} |\widehat{B}_j(u)|du\\ &\leq 2\int_{|u|>K\sigma_n} |\widehat{N}(\delta u)|du + \frac{K}{\sigma_n^r}\int_{-\infty}^{\infty} |\widehat{N}(\delta u)|du\\ &= \frac{2}{\delta}\int_{|u|\geq K\delta\sigma_n} |\widehat{N}(u)|du + \frac{K}{\delta\sigma_n^r}\int_{-\infty}^{\infty} |\widehat{N}(u)|du\\ &\leq \frac{C}{\delta(K\delta\sigma_n)^{3r}} + \frac{C}{\delta\sigma_n^r}\end{aligned}$$

for some $C > 0$. Choosing $\delta = c\sigma_n^{\beta-1}$ for some $\frac{1}{3} \leq \beta < 1$, and $c > 1$, gives

$$\left|\int_{-\infty}^{\infty} B_j d\widetilde{m}_n - \int_{-\infty}^{\infty} B_j G\right| \leq \frac{D}{c\sigma_n^{r+\beta-1}} \tag{5.13}$$

for some $D > 0$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} B_1 d\widetilde{m}_n &> \int_{-\infty}^{\infty} B_1 G - \frac{D}{c\sigma_n^{r+\beta-1}}\\ &> \int_{-\infty}^{\infty} B_2 G + \left|G''\Big(\frac{1}{2}\Big)\right| 2\delta^2 - \frac{D}{c\sigma_n^{r+\beta-1}}\\ &> \int_{-\infty}^{\infty} B_2 d\widetilde{m}_n + \frac{|G''(\frac{1}{2})| 2c^2}{\sigma_n^{2-2\beta}} - \frac{2D}{c\sigma_n^{r+\beta-1}}\ . \qquad (5.14)\end{aligned}$$

Now for $n = 1, 2, \dots$, choose a continuous function g_n, which is bell-shaped or logconcave as m_n is bell-shaped or logconcave, respectively, and satisfies

$$g_n\left(\frac{j-\mu}{\sigma_n}\right) = \sigma_n a_{n,j}, \quad j = 0, 1, \dots, n. \tag{5.15}$$

If m_n is logconcave, then this can be done as in the proof of Theorem 5.2, while if m_n is bell-shaped we can take g_n to be simply the piecewise linear interpolant. Note that if, for some constant b, $g_n \geq b$ on the support of B_j, $j = 1$ or 2, then $\int_{-\infty}^{\infty} B_j d\widetilde{m}_n$ bounds the product of b and a Riemann sum for B_j over its support with interval length σ_n^{-1}. This Riemann sum equals a Riemann sum for N over $[0,1]$ with interval length $\delta^{-1}\sigma_n^{-1}$, which differs from $\int_0^1 N$ by $O(\delta^{-2}\sigma_n^{-2})$. Thus, by the uniform boundedness of g_n, we have

$$\int B_j d\widetilde{m}_n \geq b + O(\delta^{-2}\sigma_n^{-2}),$$

and similarly the result holds with $\geq$ replaced by $\leq$. Thus if $g_n(x) \leq g_n(y)$ for all $x \in [-\delta, \delta]$, $y \in [3\delta, 5\delta]$, we have

$$\int_{-\infty}^{\infty} B_1 d\widetilde{m}_n \leq \int_{-\infty}^{\infty} B_2 d\widetilde{m}_n + \frac{a}{\delta^2\sigma_n^2} = \int_{-\infty}^{\infty} B_2 d\widetilde{m}_n + \frac{a}{c^2\sigma_n^{2\beta}}$$

for a fixed constant a. Choosing $\beta = 2/3$ and c large enough, this would contradict (5.14), and so there are points $-\delta < b_n < \delta$, $3\delta < c_n < 5\delta$ with $g_n(b_n) > g_n(c_n)$. Similarly, we can choose b_n so that there is a point a_n in $(-5\delta, -3\delta)$ with $g_n(a_n) < g_n(b_n)$. As in the proof of Lemma 5.1, it follows from the shape of g_n that the maximum of $g_n(x)$ occurs only for x in $(-5\delta, 5\delta)$. So we have shown that for a constant a, maximum of $g_n(x)$ occurs for x in $(-a\sigma_n^{-1/3}, a\sigma_n^{1/3})$ for $n = 1, 2, \ldots$.

Now take $\delta = \sigma_n^{\beta-1}$ for some $1/3 \leq \beta < 1$ and $\gamma \geq a\sigma_n^{-1/3} + \delta$. Let $B(x) = \delta^{-1}N(\delta^{-1}(x-\gamma))$ so that B has support on $[\gamma-\delta, \gamma+\delta]$. As in (5.13)

$$\left|\int_{-\infty}^{\infty} B d\widetilde{m}_n - \int_{-\infty}^{\infty} BG\right| \leq \frac{D}{\sigma_n^{r+\beta-1}}$$

for some $D > 0$. Since g_n is decreasing on $[a\sigma_n^{-1/3}, \infty)$, for a constant $b > 0$,

$$\begin{aligned} g_n(\gamma-\delta) &\geq \int_{-\infty}^{\infty} B d\widetilde{m}_n - \frac{b}{\delta^2\sigma_n^2} \\ &\geq \int_{-\infty}^{\infty} BG - \frac{b}{\delta^2\sigma_n^2} - \frac{D}{\sigma_n^{r+\beta-1}} \\ &\geq G(\gamma+\delta) - \frac{b}{\sigma_n^{2\beta}} - \frac{D}{\sigma_n^{r+\beta-1}} \ . \end{aligned}$$

Since $|G'(\tau)| < 1$ for all τ in $\mathbb{R}$, $|G(x) - G(y)| \leq |x-y|$ for all $x, y \in \mathbb{R}$. So $G(\gamma+\delta) \geq G(\gamma-\delta) - 2\delta$, and so

$$g_n(\gamma-\delta) \geq G(\gamma-\delta) - \frac{b}{\sigma_n^{2\beta}} - \frac{D}{\sigma_n^{r+\beta-1}} - 2\delta.$$

Similarly,

$$g_n(\gamma+\delta) \leq G(\gamma+\delta) + \frac{b}{\sigma_n^{2\beta}} + \frac{D}{\sigma_n^{r+\beta-1}} + 2\delta.$$

Thus for all $x \geq a\sigma_n^{-1/3} + 2\delta$,

$$|g_n(x) - G(x)| \leq \frac{b}{\sigma_n^{2\beta}} + \frac{D}{\sigma_n^{r+\beta-1}} + \frac{2}{\sigma_n^{1-\beta}}.$$

For $r \geq 4/3$, put $\beta = 1/3$ to give

$$|g_n(x) - G(x)| = 0(\sigma_n^{-\frac{2}{3}}).$$

For $1 \le r \le 4/3$, put $\beta = 1 - r/2$ to give

$$|g_n(x) - G(x)| = 0(\sigma_n^{-\frac{r}{2}}).$$

Similarly, the result holds for $x \le -a\sigma_n^{-\frac{1}{3}} - 2\delta$. Thus for a constant $b > a$,

$$\sup\{|g_n(x) - G(x)| : |x| \ge b\sigma_n^{-\frac{1}{3}}\} = O(\sigma_n^{-s}) \tag{5.16}$$

for s as in the statement of Lemma 5.4. Note that for any $\delta > 0$ and $x, y \in (-\delta, \delta)$,

$$|G(x) - G(y)| \le |G''(0)|\delta|x - y| \le \delta|x - y|. \tag{5.17}$$

Take any $x \in (-b\sigma_n^{-\frac{1}{3}}, b\sigma_n^{\frac{-1}{3}})$. Then either

$$g_n(x) \ge g_n(b\sigma_n^{-\frac{1}{3}}) \quad \text{or} \quad g_n(x) \ge g_n(-b\sigma_n^{-\frac{1}{3}}).$$

Suppose the former. Then

$$\begin{aligned} g_n(x) \ge g_n(b\sigma_n^{-\frac{1}{3}}) &> G(b\sigma_n^{-\frac{1}{3}}) - O(\sigma_n^{-s}) \\ &> G(x) - O(\sigma_n^{-s}) - 2(b\sigma_n^{-\frac{1}{3}})^2. \end{aligned}$$

The same holds similarly for the latter case. Thus

$$\sup\{g_n(x) - G(x) : |x| \le b\sigma_n^{-\frac{1}{3}}\} = O(\sigma_n^{-s}). \tag{5.18}$$

Now note, as in the proof of Lemma 5.1, that if g_n is bell-shaped, then for all large enough n, g_n is concave on $[-\frac{2}{3}, \frac{2}{3}]$. Since concavity implies logconcavity, g_n is logconcave on $[-\frac{2}{3}, \frac{2}{3}]$ for all large enough n. By (5.16) and the mean value theorem,

$$\sup\left\{|\log g_n(x) - \log G(x)| : b\sigma_n^{-\frac{1}{3}} \le |x| \le \frac{2}{3}\right\} = O(\sigma_n^{-s}).$$

Take $0 \le x \le b\sigma_n^{-\frac{1}{3}}$ and n so large that $b\sigma_n^{-\frac{1}{3}} \le \frac{2}{9}$. Then

$$\begin{aligned} \log g_n(x) &\le 2\log g_n(x + b\sigma_n^{-\frac{1}{3}}) - \log g_n(x + 2b\sigma_n^{-\frac{1}{3}}) \\ &\le 2\log G(x + b\sigma_n^{-\frac{1}{3}}) - \log G(x + 2b\sigma_n^{-\frac{1}{3}}) + O(\sigma_n^{-s}) \\ &\le \log G(x) + |\log G(x + b\sigma_n^{-\frac{1}{3}}) - \log G(x)| \\ &\qquad + |\log G(x + b\sigma_n^{-\frac{1}{3}}) - \log G(x + 2b\sigma_n^{-\frac{1}{3}})| + O(\sigma_n^{-s}) \\ &\le \log G(x) + O(\sigma_n^{-\frac{2}{3}}) + O(\sigma_n^{-s}). \end{aligned}$$

A similar argument holds for $-b\sigma_n^{-\frac{1}{3}} \le x \le 0$, and applying the mean value theorem gives

$$\sup\{G(x) - g_n(x) : |x| \le b\sigma_n^{-\frac{1}{3}}\} = O(\sigma_n^{-3}). \tag{5.19}$$

Combining (5.16), (5.18), and (5.19) and recalling (5.15) then gives the result. ∎

Proof of Theorem 1.4. Theorem 1.4 follows from Proposition 4.1 and Lemma 5.4. ∎

To consider the order of uniform convergence for the scaling functions, we need the following analogue of Lemma 5.4. This can be proved in a similar manner to Lemma 5.4, but the proof is simpler, in particular because there is no restriction on the range of u as in (5.11).

LEMMA 5.5. *Suppose that* (g_n) *is a sequence of continuous functions, which are either bell-shaped or logconcave with* $\int_{-\infty}^{\infty} g_n = 1$ *and* $\|\widehat{g}_n(u) - e^{-u^2/2}\|_\infty < \alpha_n$ *for* $n = 1, 2, \ldots$, *where* $\lim_{n\to\infty} \alpha_n = 0$. *Then as* $n \to \infty$,

$$\|g_n - G\|_\infty = O(\alpha_n^{\frac{1}{2}}).$$

Proof of Theorem 1.5. Theorem 1.5 follows from Theorem 1.3 and Lemma 5.5. □

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