Bernoulli **21**(1), 2015, 335–359
DOI: 10.3150/13-BEJ569

On the error bound in a combinatorial central limit theorem

LOUIS H.Y. CHEN[1] and XIAO FANG[2]

[1]*Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Republic of Singapore. E-mail: matchyl@nus.edu.sg*
[2]*Department of Statistics and Applied Probability, National University of Singapore, 6 Science Drive 2, Singapore 117546, Republic of Singapore. E-mail: stafx@nus.edu.sg*

Let $\mathbb{X} = \{X_{ij}\colon 1 \le i, j \le n\}$ be an $n \times n$ array of independent random variables where $n \ge 2$. Let π be a uniform random permutation of $\{1, 2, \dots, n\}$, independent of $\mathbb{X}$, and let $W = \sum_{i=1}^{n} X_{i\pi(i)}$. Suppose $\mathbb{X}$ is standardized so that $\mathbb{E}W = 0$, $\operatorname{Var}(W) = 1$. We prove that the Kolmogorov distance between the distribution of W and the standard normal distribution is bounded by $451 \sum_{i,j=1}^{n} \mathbb{E}|X_{ij}|^3/n$. Our approach is by Stein's method of exchangeable pairs and the use of a concentration inequality.

Keywords: combinatorial central limit theorem; concentration inequality; exchangeable pairs; Stein's method

1. Introduction and statement of the main result

Motivated by permutation tests in non-parametric statistics, Wald and Wolfowitz [23] proved a central limit theorem for the combinatorial statistics $\sum_{i=1}^{n} a_i b_{\pi(i)}$ where $\{a_i, b_j\colon i, j \in [n] := \{1, 2, \dots, n\}\}$ are real numbers and π is a uniform random permutation of $[n]$. Their result was generalized to real arrays $\{c_{ij}\colon i, j \in [n]\}$ by Hoeffding [14]. Extension to random arrays $\{X_{ij}\colon i, j \in [n]\}$ where the X_{ij} are independent random variables was considered by Ho and Chen [13]. Using the concentration inequality approach in Stein's method, they proved a bound on the Kolmogorov distance between the distribution of $\sum_{i=1}^{n} X_{i\pi(i)}$ and the normal distribution with the same mean and variance. The bound in Ho and Chen [13] is optimal only when $|X_{ij}| \le C$ for some $C > 0$. A third-moment bound for a combinatorial central limit theorem for real arrays $\{c_{ij}\colon i, j \in [n]\}$ was obtained by Bolthausen [2], who used Stein's method and induction. However, the absolute constant in the bound in Bolthausen [2] is not explicit. A bound with an explicit constant for real arrays with $|c_{ij}| \le C$ was obtained by Goldstein [11] using Stein's method and zero-bias coupling (see also Chen, Goldstein and Shao [6]). Under the same setting as Ho and Chen [13], Neammanee and Suntornchost [18] stated a third-moment bound. They used the same Stein identity in Ho and Chen [13], which dates back to Chen [3], and the concentration inequality approach. However, there is an error in the proof in Neammanee and Suntornchost [18], where the first equality and the second inequality on page 576 are incorrect because of the dependence among $S(\tau)$, ΔS and $M(t)$. Although the bound obtained by Neammanee and Rerkruthairat [17] (see Theorem 1.1 on page 1591) simplifies to a third-moment bound plus a term of a smaller order, the latter contains an undetermined constant. Besides, the proof of Theorem 1.1 uses a result of Neammanee and Rattanawong [16] (see (23) on page 21), whose correctness is in question.

1350-7265

In this paper, we give a different proof of the combinatorial central limit theorem. Our result gives a third-moment bound with an explicit constant under the setting of Ho and Chen [13]. Our approach is by Stein's method of exchangeable pairs and the use of a concentration inequality. The use of an exchangeable pair simplifies the construction of a Stein identity as compared to the construction in Ho and Chen [13], Neammanee and Suntornchost [18] and Neammanee and Rerkruthairat [17].

Stein's method was introduced by Stein [21,22] and has become a popular tool in proving distributional approximation results because of its power in handling dependence among random variables. We refer to Barbour and Chen [1] and Chen, Goldstein and Shao [6] for an introduction to Stein's method. The notion of exchangeable pair was introduced by Stein [22], and first used in Diaconis [10]. The concentration inequality approach was also introduced by Stein (see Ho and Chen [13]) and was developed by Chen [4,5] and Chen and Shao [7,8]. This approach provides a smoothing technique and a way of obtaining third-moment bounds on the Kolmogorov distance.

The following is our main result.

Theorem 1.1. *Let* $\mathbb{X} = \{X_{ij}: i, j \in [n]\}$ *be an* $n \times n$ *array of independent random variables where* $n \geq 2$, $\mathbb{E}X_{ij} = c_{ij}$, $\operatorname{Var} X_{ij} = \sigma_{ij}^2 \geq 0$ *and* $\mathbb{E}|X_{ij}|^3 < \infty$. *Assume*

$$c_{i\cdot} = c_{\cdot j} = 0, \tag{1.1}$$

where $c_{i\cdot} = \sum_{j=1}^n c_{ij}/n$, $c_{\cdot j} = \sum_{i=1}^n c_{ij}/n$. *Let* π *be a uniform random permutation of* $[n]$, *independent of* $\mathbb{X}$, *and let* $W = \sum_{i=1}^n X_{i\pi(i)}$. *Then*

(1)

$$\operatorname{Var}(W) = \frac{1}{n}\sum_{i,j=1}^n \sigma_{ij}^2 + \frac{1}{n-1}\sum_{i,j=1}^n c_{ij}^2, \tag{1.2}$$

(2) *assuming* $\operatorname{Var}(W) = 1$, *we have*

$$\sup_{z\in\mathbb{R}} \bigl|P(W \leq z) - \Phi(z)\bigr| \leq 451\gamma, \tag{1.3}$$

where Φ *is the standard normal distribution function and*

$$\gamma = \frac{1}{n}\sum_{i,j=1}^n \mathbb{E}|X_{ij}|^3. \tag{1.4}$$

Specializing Theorem 1.1 to the case where $\sigma_{ij} = 0$, we have

$$\sup_{z\in\mathbb{R}} \bigl|P(W \leq z) - \Phi(z)\bigr| \leq \frac{451}{n}\sum_{i,j=1}^n |c_{ij}|^3. \tag{1.5}$$

Moreover, if $|c_{ij}| \le C$ for all $i, j \in [n]$, then

$$\sup_{z\in\mathbb{R}} \bigl| P(W \le z) - \Phi(z) \bigr| \le 451C. \tag{1.6}$$

Remark 1.2. The constant 451 in (1.5), and therefore that in (1.6), can be reduced by a modification of the proof of Theorem 1.1 for the special case where $\sigma_{ij} = 0$. The error bound in (1.5) was obtained by Bolthausen [2] except that the constant in his bound is not explicit. In the case where $|c_{ij}| \le C$, the error bound in (1.6) is of the same order as that in Goldstein [11] and in Theorem 6.1 of Chen, Goldstein and Shao [6], although the constant in Theorem 6.1 of Chen, Goldstein and Shao [6] is 16.3, which is smaller. The constant 16.3 is not directly comparable to the constant 451 in (1.3) and (1.5) as the bounds in (1.3) and (1.5) are of third moment type, which may be smaller.

Remark 1.3. Any $n \times n$ array of independent random variables $\{Y_{ij}\colon i, j \in [n]\}$ with $\mathbb{E}Y_{ij} = \mu_{ij}$, $\operatorname{Var} Y_{ij} = \sigma_{ij}^2 \ge 0$ and $\mathbb{E}|Y_{ij}|^3 < \infty$ can be reduced to the array in Theorem 1.1 satisfying (1.1) by defining $c_{ij} = \mu_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}$ and $X_{ij} = Y_{ij} - \mu_{i\cdot} - \mu_{\cdot j} + \mu_{\cdot\cdot}$.

Theorem 1.1 has the following corollary for simple random sampling from independent random variables.

Corollary 1.4. *Let $\{Y_1, \dots, Y_n\}$ be independent random variables with $\mathbb{E}Y_i = \mu_i$, $\operatorname{Var}(Y_i) = \sigma_i^2 \ge 0$. For a positive integer $k \le n$, let $\{Y_{\xi_1}, \dots, Y_{\xi_k}\}$ be uniformly chosen from $\{Y_1, \dots, Y_n\}$ without replacement, and let $V = \sum_{i=1}^k Y_{\xi_i}$. Then we have*

$$\begin{aligned} &\sup_{z\in\mathbb{R}} \left| P\left(\frac{V - k\bar{\mu}}{\sigma} \le z\right) - \Phi(z) \right| \\ &\quad \le \frac{451}{n\sigma^3} \left[k \sum_{i=1}^n \mathbb{E} \left| \frac{k}{n}(Y_i - \mu_i) + \frac{n-k}{n}(Y_i - \bar{\mu}) \right|^3 + (n-k) \sum_{i=1}^n \left| \frac{k}{n}(\mu_i - \bar{\mu}) \right|^3 \right], \end{aligned} \tag{1.7}$$

where

$$\bar{\mu} = \sum_{i=1}^n \mu_i / n, \qquad \sigma^2 = \operatorname{Var}(V) = \frac{k}{n} \sum_{i=1}^n \sigma_i^2 + \frac{k(n-k)}{n(n-1)} \sum_{i=1}^n (\mu_i - \bar{\mu})^2,$$

and Φ is the standard normal distribution function.

Proof. Using the same idea as in the Poisson approximation for the hypergeometric distribution in Corollary 3.4 of Chen [3], we let the $n \times n$ array $\mathbb{Y}$ be such that the first k rows are independent copies of $\{Y_1, \dots, Y_n\}$ and the other rows are zeros. Then $\mathcal{L}(\sum_{i=1}^n Y_{i\pi(i)}) = \mathcal{L}(\sum_{i=1}^k Y_{\xi_i})$. Therefore, the bound (1.7) follows from Theorem 1.1 and Remark 1.3. □

Our interest in Corollary 1.4 is motivated by the work of Wolff [24] who considered sampling without replacement of independent random variables in the construction of random embeddings which are with high probability almost isometric in the sense of Johnson and Lindenstrauss [15]. In Corollary 1.4, if $\mu_i = 0$ for all $i \in [n]$, then $\sigma^2 = \frac{k}{n}\sum_{i=1}^n \sigma_i^2$ and the error bound in (1.7) reduces to $\frac{451k}{n\sigma^3}\sum_{i=1}^n \mathbb{E}|Y_i|^3$. Wolff [24] studied this special case and obtained the bound $\frac{3k}{n\sigma^3}\sum_{i=1}^n \mathbb{E}|Y_i|^3$ for the Wasserstein distance. The case where $\sigma_i = 0$ for all $i \in [n]$ in Corollary 1.4 was considered by Goldstein [12] using zero-bias coupling, where a similar bound with a smaller constant was obtained for the Wasserstein distance (see Theorem 5.1 of Goldstein [12]).

Although we will not consider Wasserstein distance in this paper, we wish to mention that a Wasserstein distance bound can be obtained for the normal approximation for $\mathcal{L}(W)$ where W is defined in Theorem 1.1. The proof for the bound will not require a concentration inequality and the bound will have a smaller constant. Indeed, a bound with a smaller constant has been obtained for the Wasserstein distance in Theorem 6.1 of Goldstein [12] in the case where $\sigma_{ij} = 0$.

In the next section, we prove a concentration inequality using exchangeable pairs (Lemma 2.1) and apply it to random variables with a combinatorial structure similar to that of W in Theorem 1.1. In Section 3, we prove our main result, Theorem 1.1, by Stein's method of exchangeable pairs and the concentration inequality approach.

2. A concentration inequality for exchangeable pairs

Assuming the existence of an exchangeable pair (S, S') satisfying an approximate linearity condition, the next lemma provides a bound on $\mathbb{P}(S \in [a, b])$.

Lemma 2.1. *Suppose (S, S') is an exchangeable pair of square integrable random variables and satisfies the following approximate linearity condition*

$$\mathbb{E}\bigl(S' - S|S\bigr) = -\lambda S + R \tag{2.1}$$

for a positive number λ and a random variable R. Then, for $a < b$,

$$\begin{aligned} &P\bigl(S \in [a, b]\bigr) \\ &\quad\le \frac{\mathbb{E}|S| + \mathbb{E}|R|/\lambda}{\mathbb{E}S^2 - \mathbb{E}|SR|/\lambda - 1/2}\left(\frac{b-a}{2} + \delta\right) \\ &\qquad + \frac{1}{\mathbb{E}S^2 - \mathbb{E}|SR|/\lambda - 1/2}\sqrt{\operatorname{Var}\left(\mathbb{E}\left(\frac{1}{2\lambda}\bigl(S' - S\bigr)^2 I\bigl(\bigl|S' - S\bigr| \le \delta\bigr)|S\right)\right)}, \end{aligned} \tag{2.2}$$

where

$$\delta = \frac{\mathbb{E}|S' - S|^3}{\lambda} \tag{2.3}$$

provided that $\mathbb{E}S^2 - \mathbb{E}|SR|/\lambda - 1/2 > 0$.

Remark 2.2. Bounding the last term on the right-hand side of (2.2) involves studying the conditional distribution of $(S' - S)^2$ given S. Truncating $|S' - S|$ at δ allows us to keep within third moments. Shao and Su [20] has also used a concentration inequality for exchangeable pairs in their study of character ratios of the symmetric group. Their concentration inequality, which is different from ours, is easier to prove than ours but may not be easy to apply to the combinatorial central limit theorem.

Proof of Lemma 2.1. Without loss of generality, assume $\delta < \infty$. From the exchangeability of S and S',

$$\mathbb{E}(S' - S)\big(f(S') + f(S)\big) = 0 \tag{2.4}$$

for all f such that the above expectation exists. Therefore,

$$\mathbb{E}(S' - S)\big(f(S') - f(S)\big) = 2\mathbb{E}(S - S')f(S).$$

Using the approximate linearity condition (2.1) for the right-hand side of the above equation, we have for absolutely continuous f,

$$\begin{aligned}\mathbb{E}Sf(S) &= \frac{1}{2\lambda}\mathbb{E}(S' - S)\big(f(S') - f(S)\big) + \frac{1}{\lambda}\mathbb{E}Rf(S) \\ &= \frac{1}{2\lambda}\mathbb{E}(S' - S)\int_0^{S'-S} f'(S+t)\,\mathrm{d}t + \frac{1}{\lambda}\mathbb{E}Rf(S).\end{aligned} \tag{2.5}$$

The identity (2.4) was introduced by Stein [22] and (2.5) was obtained by Stein [22] in the case $R = 0$ and by Rinott and Rotar [19] for $R \neq 0$.

Let f be such that $f'(w) = I(a - \delta \le w \le b + \delta)$ and $f(\frac{a+b}{2}) = 0$. Therefore, $|f| \le \frac{b-a}{2} + \delta$. Using the property that for all $w, w' \in \mathbb{R}$,

$$(w' - w)\int_0^{w'-w} f'(w+t)\,\mathrm{d}t \ge 0,$$

we have

$$\begin{aligned}&\frac{1}{2\lambda}\mathbb{E}(S' - S)\int_0^{S'-S} f'(S+t)\,\mathrm{d}t \\ &\quad\ge \frac{1}{2\lambda}\mathbb{E}(S' - S)\int_0^{S'-S} f'(S+t)\,\mathrm{d}t\, I\big(|S' - S| \le \delta\big)I\big(S \in [a,b]\big) \\ &\quad= \mathbb{E}I\big(S \in [a,b]\big)\frac{1}{2\lambda}(S' - S)^2 I\big(|S' - S| \le \delta\big) \\ &\quad= \mathbb{E}I\big(S \in [a,b]\big)\bigg[\mathbb{E}\bigg(\frac{1}{2\lambda}(S' - S)^2 I\big(|S' - S| \le \delta\big)|S\bigg) - \mathbb{E}\frac{1}{2\lambda}(S' - S)^2 I\big(|S' - S| \le \delta\big)\bigg] \\ &\qquad+ \mathbb{E}I\big(S \in [a,b]\big)\mathbb{E}\frac{1}{2\lambda}(S' - S)^2 I\big(|S' - S| \le \delta\big) \\ &\quad:= -R_1 + R_2.\end{aligned}$$

Using the Cauchy–Schwarz inequality,

$$R_1 \le \sqrt{\operatorname{Var}\Big(\mathbb{E}\Big(\frac{1}{2\lambda}\big(S'-S\big)^2 I\big(\big|S'-S\big| \le \delta\big) | S\Big)\Big)}.$$

From (2.1),

$$\mathbb{E}\big(S'-S\big)^2 = 2\mathbb{E}S(\lambda S - R) = 2\lambda \mathbb{E}S^2 - 2\mathbb{E}SR.$$

Therefore,

$$\begin{aligned} R_2 &= P\big(S\in[a,b]\big)\mathbb{E}\frac{1}{2\lambda}\big(S'-S\big)^2 - P\big(S\in[a,b]\big)\mathbb{E}\frac{1}{2\lambda}\big(S'-S\big)^2 I\big(\big|S'-S\big|>\delta\big) \\ &\ge P\big(S\in[a,b]\big)\Big(\mathbb{E}S^2 - \frac{\mathbb{E}SR}{\lambda}\Big) - P\big(S\in[a,b]\big)\frac{1}{\delta}\frac{\mathbb{E}|S'-S|^3}{2\lambda} \\ &= P\big(S\in[a,b]\big)\Big(\mathbb{E}S^2 - \frac{\mathbb{E}SR}{\lambda} - \frac{1}{2}\Big) \\ &\ge P\big(S\in[a,b]\big)\Big(\mathbb{E}S^2 - \frac{\mathbb{E}|SR|}{\lambda} - \frac{1}{2}\Big), \end{aligned}$$

where in the last equality we used the definition of δ in (2.3). Using the fact that $|f| \le \frac{b-a}{2} + \delta$, we have

$$\big|\mathbb{E}Sf(S)\big| \le \Big(\frac{b-a}{2}+\delta\Big)\mathbb{E}|S|, \qquad \Big|\frac{1}{\lambda}\mathbb{E}Rf(S)\Big| \le \Big(\frac{b-a}{2}+\delta\Big)\frac{\mathbb{E}|R|}{\lambda}.$$

The lemma is proved by applying all the above bounds to (2.5). □

Now we apply Lemma 2.1 to establish a concentration inequality for a sum S which is defined as follows. Let $\mathbb{X}$ be the $n \times n$ array defined in Theorem 1.1 satisfying (1.1) and

$$\frac{1}{n}\sum_{i,j=1}^{n} \sigma_{ij}^2 + \frac{1}{n-1}\sum_{i,j=1}^{n} c_{ij}^2 = 1. \tag{2.6}$$

For $n \ge 6$ and $m \in \{2,3,4\}$, we remove the last m rows and columns from the original array $\mathbb{X}$. Let τ be an independent uniform random permutation of $[n-m]$. Define the variable S by

$$S = \sum_{i=1}^{n-m} X_{i\tau(i)}. \tag{2.7}$$

To prove a concentration inequality for S, we need the following lemma which estimates the second moment of S.

Lemma 2.3. *Let S be defined by (2.7) for some $m \in \{2,3,4\}$ and $n \geq 6$. Suppose $\gamma \leq 1/c_0$ where γ was defined in (1.4). Under the assumptions (1.1) and (2.6), we have*

$$\frac{n-1}{n-2} - \frac{2n}{(n-4)c_0^{2/3}} - \frac{24n}{(n-5)^2} \leq \mathbb{E}S^2 \leq \frac{n}{n-5} + \frac{24n}{(n-5)^2}. \tag{2.8}$$

Proof. Writing $\sum_{1\leq i\neq j\leq n-m}\sum_{1\leq k\neq l\leq n-m}$ as

$$\sum_{1\leq i,j,k,l\leq n-m} - \sum_{1\leq i=j\leq n-m}\sum_{1\leq k,l\leq n-m} - \sum_{1\leq i,j\leq n-m}\sum_{1\leq k=l\leq n-m} + \sum_{1\leq i=j\leq n-m}\sum_{1\leq k=l\leq n-m}$$

and using the assumption (1.1),

$$\begin{aligned}
\mathbb{E}S^2 &= \mathbb{E}\left(\sum_{i=1}^{n-m} X_{i\tau(i)}\right)^2 = \sum_{i=1}^{n-m}\mathbb{E}X_{i\tau(i)}^2 + \sum_{1\leq i\neq j\leq n-m}\mathbb{E}X_{i\tau(i)}X_{j\tau(j)} \\
&= \frac{1}{n-m}\sum_{i,j=1}^{n-m}\mathbb{E}X_{ij}^2 + \frac{1}{(n-m)_{(2)}}\sum_{1\leq i\neq j\leq n-m}\sum_{1\leq k\neq l\leq n-m}\mathbb{E}X_{ik}X_{jl} \\
&= \frac{1}{n-m}\sum_{i,j=1}^{n-m}\left(c_{ij}^2+\sigma_{ij}^2\right) + \frac{1}{(n-m)_{(2)}}\sum_{1\leq i\neq j\leq n-m}\sum_{1\leq k\neq l\leq n-m} c_{ik}c_{jl} \\
&= \frac{1}{n-m}\sum_{i,j=1}^{n-m}\sigma_{ij}^2 + \frac{1}{n-m}\sum_{i,j=1}^{n-m}c_{ij}^2 \qquad (2.9)\\
&\quad + \frac{1}{(n-m)_{(2)}}\sum_{i,k=1}^{n-m}c_{ik}\left(\sum_{j,l=1}^{n-m}c_{jl} - \sum_{l=1}^{n-m}c_{il} - \sum_{j=1}^{n-m}c_{jk} + c_{ik}\right) \\
&= \frac{1}{n-m}\sum_{i,j=1}^{n-m}\sigma_{ij}^2 + \frac{1}{n-m-1}\sum_{i,j=1}^{n-m}c_{ij}^2 \\
&\quad + \frac{1}{(n-m)_{(2)}}\sum_{i,j=1}^{n-m}c_{ij}\left(\sum_{k,l=n-m+1}^{n}c_{kl} + \sum_{k=n-m+1}^{n}c_{ik} + \sum_{l=n-m+1}^{n}c_{lj}\right)
\end{aligned}$$

with the falling factorial notation $(n-m)_{(2)} := (n-m)(n-m-1)$. Under the assumption (2.6), $\mathbb{E}S^2$ is close to 1 intuitively. We quantify it as follows. From (1.1) and (2.6),

$$\left|\sum_{i,j=1}^{n-m}c_{ij}\left(\sum_{k=n-m+1}^{n}c_{ik}\right)\right| = \left|\sum_{i=1}^{n-m}\left(\sum_{k=n-m+1}^{n}c_{ik}\right)^2\right| \leq \left|\sum_{i=1}^{n-m}m\sum_{k=n-m+1}^{n}c_{ik}^2\right| \leq m(n-1).$$

Similarly,

$$\left|\sum_{i,j=1}^{n-m} c_{ij} \sum_{l=n-m+1}^{n} c_{lj}\right| \le m(n-1).$$

Moreover,

$$\left|\sum_{i,j=1}^{n-m} c_{ij} \sum_{k,l=n-m+1}^{n} c_{kl}\right| = \left|\left(\sum_{k,l=n-m+1}^{n} c_{kl}\right)^2\right| \le m^2 \sum_{k,l=n-m+1}^{n} c_{kl}^2 \le m^2(n-1).$$

Therefore, from (2.9),

$$\mathbb{E}S^2 \le \frac{1}{n-m-1}\sum_{i,j=1}^{n}\left(\sigma_{ij}^2 + c_{ij}^2\right) + \frac{(2m+m^2)(n-1)}{(n-m)_{(2)}} \le \frac{n}{n-5} + \frac{24n}{(n-5)^2}.$$

Now we prove the lower bound. Since $\gamma \le 1/c_0$, using Hölder's inequality, we have

$$\sum_{\substack{1\le i,j\le n: i>n-m \\ \text{or } j>n-m}} \left(\sigma_{ij}^2 + c_{ij}^2\right) = \sum_{\substack{1\le i,j\le n: i>n-m \\ \text{or } j>n-m}} \mathbb{E}X_{ij}^2 \le (8n)^{1/3}\left(\sum_{i,j=1}^{n} \mathbb{E}|X_{ij}|^3\right)^{2/3} \le 2n/c_0^{2/3}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}S^2 &\ge \frac{1}{n-m}\sum_{i,j=1}^{n-m}\left(\sigma_{ij}^2 + c_{ij}^2\right) - \frac{24n}{(n-5)^2} \\
&= \frac{1}{n-m}\sum_{i,j=1}^{n}\left(\sigma_{ij}^2 + c_{ij}^2\right) - \frac{1}{n-m}\sum_{\substack{1\le i,j\le n: i>n-m \\ \text{or } j>n-m}}\left(\sigma_{ij}^2 + c_{ij}^2\right) - \frac{24n}{(n-5)^2} \\
&\ge \frac{n-1}{n-2} - \frac{2n}{(n-4)c_0^{2/3}} - \frac{24n}{(n-5)^2}.
\end{aligned}$$

□

In the next proposition, we provide a concentration inequality for S.

Proposition 2.4. *Let S be defined by (2.7) for some $m \in \{2, 3, 4\}$. Suppose $\gamma \le 1/c_0$ where γ was defined in (1.4), and c_0 and $n \ge 6$ are large enough to satisfy*

$$\theta := \frac{1}{2} - \frac{2n}{(n-4)c_0^{2/3}} - \frac{24n}{(n-5)^2} - \frac{4\sqrt{n}}{n-4}\sqrt{\frac{n}{n-5} + \frac{24n}{(n-5)^2}} > 0. \tag{2.10}$$

Then for all $a < b$,

$$\mathbb{P}\big(S \in [a,b]\big) \le c_1(b-a) + c_2\gamma, \tag{2.11}$$

where

$$c_1 = \left(\frac{1}{2}\sqrt{\frac{n}{n-5} + \frac{24n}{(n-5)^2}} + \frac{2\sqrt{n}}{n-4}\right)\Big/\theta \tag{2.12}$$

and

$$c_2 = \frac{64n}{n-4}c_1 + \left\{\left[\frac{8n}{(n-4)^2} + \frac{16n}{n-4} + 32\left(\frac{n}{n-4}\right)^3\right]\left[\frac{32n}{n-4}\right]\right\}^{1/2}\Big/\theta. \tag{2.13}$$

Proof. For any $m \in \{2, 3, 4\}$, we construct an exchangeable pair (S, S') by uniformly selecting two different indices $I, J \in [n-m]$ and letting $S' = S - X_{I\tau(I)} - X_{J\tau(J)} + X_{I\tau(J)} + X_{J\tau(I)}$. An approximate linearity condition with an error term can be established as

$$\begin{aligned}
&\mathbb{E}(S' - S|S) \\
&= \frac{1}{(n-m)_{(2)}} \sum_{1\le i,j\le n-m} \mathbb{E}\{[X_{i\tau(j)} + X_{j\tau(i)}] - [X_{i\tau(i)} + X_{j\tau(j)}]|S\} \\
&= \frac{1}{(n-m)_{(2)}}\mathbb{E}\left\{2\sum_{i,j=1}^{n-m} X_{ij} - 2(n-m)S\Big|S\right\} \\
&= -\lambda S + R,
\end{aligned} \tag{2.14}$$

where $\lambda = 2/(n-m-1)$ and

$$R = \frac{2}{(n-m)_{(2)}}\mathbb{E}\left(\sum_{i,j=1}^{n-m} X_{ij}\Big|S\right).$$

To apply the concentration inequality in Lemma 2.1, we need to:

1. Bound $\sqrt{\mathbb{E}R^2}/\lambda$.
2. Bound

$$\delta = \frac{\mathbb{E}|S' - S|^3}{\lambda}. \tag{2.15}$$

3. Bound

$$B_0 = \sqrt{\mathrm{Var}\left(\mathbb{E}\left(\frac{1}{2\lambda}(S' - S)^2 I(|S' - S| \le \delta)|S\right)\right)}.$$

First,

$$\begin{aligned}
&\sqrt{\mathbb{E}R^2} \\
&= \frac{2}{(n-m)_{(2)}}\sqrt{\mathbb{E}\left(\mathbb{E}\left(\sum_{i,j=1}^{n-m} X_{ij}\Big|S\right)\right)^2}
\end{aligned}$$

$$= \frac{2}{(n-m)_{(2)}} \sqrt{\operatorname{Var}\left(\mathbb{E}\left(\sum_{i,j=1}^{n-m} X_{ij} \middle| S\right)\right) + \left(\mathbb{E}\sum_{i,j=1}^{n-m} X_{ij}\right)^2}$$

$$\leq \frac{2}{(n-m)_{(2)}} \sqrt{\operatorname{Var}\left(\sum_{i,j=1}^{n-m} X_{ij}\right) + \left(\sum_{i,j=1}^{n-m} c_{ij}\right)^2}$$

$$= \frac{2}{(n-m)_{(2)}} \sqrt{\sum_{i,j=1}^{n-m} \sigma_{ij}^2 + \left(\sum_{i,j=n-m+1}^{n} c_{ij}\right)^2}$$

$$\leq \frac{2}{(n-m)_{(2)}} \sqrt{\sum_{i,j=1}^{n-m} \sigma_{ij}^2 + m^2 \sum_{i,j=n-m+1}^{n} c_{ij}^2}$$

$$\leq \frac{2}{(n-m)_{(2)}} \sqrt{m^2\left(\sum_{i,j=1}^{n} \sigma_{ij}^2 + \sum_{i,j=1}^{n} c_{ij}^2\right)} \leq \frac{2m\sqrt{n}}{(n-m)_{(2)}},$$

where we used the assumptions (1.1) and (2.6). Therefore,

$$\frac{\sqrt{\mathbb{E}R^2}}{\lambda} \leq \frac{4\sqrt{n}}{n-4}. \tag{2.16}$$

Next, we bound δ of (2.15). From the fact that

$$\begin{aligned} &|X_{i\tau(j)} + X_{j\tau(i)} - X_{i\tau(i)} - X_{j\tau(j)}|^3 \\ &\quad \leq 16\big(|X_{i\tau(j)}|^3 + |X_{j\tau(i)}|^3 + |X_{i\tau(i)}|^3 + |X_{j\tau(j)}|^3\big), \end{aligned} \tag{2.17}$$

we have

$$\begin{aligned} &\mathbb{E}|S' - S|^3 \\ &\quad = \mathbb{E}\frac{1}{(n-m)_{(2)}} \sum_{1\leq i,j\leq n-m} \mathbb{E}\big(|X_{i\tau(i)} + X_{j\tau(j)} - X_{i\tau(j)} - X_{j\tau(i)}|^3|\mathbb{X}\big) \\ &\quad \leq \mathbb{E}\frac{16}{(n-m)_{(2)}} \sum_{1\leq i,j\leq n-m} \mathbb{E}\big(\big(|X_{i\tau(i)}|^3 + |X_{j\tau(j)}|^3 + |X_{i\tau(j)}|^3 + |X_{j\tau(i)}|^3\big)|\mathbb{X}\big) \\ &\quad \leq \frac{64n}{(n-m)_{(2)}}\gamma. \end{aligned}$$

Therefore,

$$\delta \leq \frac{32n}{n-4}\gamma. \tag{2.18}$$

Now we turn to the final step of bounding B_0. Denote

$$\alpha_{ij}^{\tau} = (X_{i\tau(i)} + X_{j\tau(j)} - X_{i\tau(j)} - X_{j\tau(i)})^2 I\big(|X_{i\tau(i)} + X_{j\tau(j)} - X_{i\tau(j)} - X_{j\tau(i)}| \le \delta\big).$$

We have

$$\frac{1}{2\lambda}\mathbb{E}\big((S'-S)^2 I\big(|S'-S| \le \delta\big)|\mathbb{X},\tau\big) = \frac{1}{4(n-m)} \sum_{1\le i\ne j\le n-m} \alpha_{ij}^{\tau}.$$

Therefore, with $|\cdot|$ denoting cardinality when applied to a subset of $[n]$,

$$\begin{aligned} B_0^2 &= \mathrm{Var}\bigg(\mathbb{E}\bigg(\frac{1}{2\lambda}(S'-S)^2 I\big(|S'-S| \le \delta\big)|S\bigg)\bigg) \\ &\le \mathrm{Var}\bigg(\frac{1}{2\lambda}\mathbb{E}\big((S'-S)^2 I\big(|S'-S| \le \delta\big)|\mathbb{X},\tau\big)\bigg) \\ &= \frac{1}{16(n-m)^2}\bigg\{2\sum_{1\le i\ne j\le n-m} \mathrm{Var}\big(\alpha_{ij}^{\tau}\big) + \sum_{\substack{1\le i,j,i',j'\le n-m,\\ i\ne j, i'\ne j', |\{i,j,i',j'\}|=3}} \mathrm{Cov}\big(\alpha_{ij}^{\tau}, \alpha_{i'j'}^{\tau}\big) \\ &\qquad + \sum_{\substack{1\le i,j,i',j'\le n-m,\\ |\{i,j,i',j'\}|=4}} \mathrm{Cov}\big(\alpha_{ij}^{\tau}, \alpha_{i'j'}^{\tau}\big)\bigg\} \\ &:= R_1 + R_2 + R_3. \end{aligned}$$

The terms R_1 and R_2 are easy to bound.

$$|R_1| \le \frac{2}{16(n-m)^2} \sum_{i,j=1}^{n-m} \delta\mathbb{E}|X_{i\tau(i)} + X_{j\tau(j)} - X_{i\tau(j)} - X_{j\tau(i)}|^3 \le \frac{8n\delta}{(n-4)^2}\gamma. \tag{2.19}$$

From $\mathrm{Cov}(X,Y) \le (\mathrm{Var}(X) + \mathrm{Var}(Y))/2$, (2.17) and the restriction that $i \ne j, i' \ne j'$, $|\{i,j,i',j'\}| = 3$, we have

$$\begin{aligned} |R_2| &\le \frac{\delta}{16(n-m)^2} \sum_{\substack{1\le i,j,i',j'\le n-m,\\ i\ne j, i'\ne j', |\{i,j,i',j'\}|=3}} \mathbb{E}|X_{i\tau(i)} + X_{j\tau(j)} - X_{i\tau(j)} - X_{j\tau(i)}|^3 \\ &\le \frac{\delta}{(n-m)^2} \sum_{\substack{1\le i,j,i',j'\le n-m,\\ i\ne j, i'\ne j', |\{i,j,i',j'\}|=3}} \big(\mathbb{E}|X_{i\tau(i)}|^3 + \mathbb{E}|X_{j\tau(j)}|^3 + \mathbb{E}|X_{i\tau(j)}|^3 + \mathbb{E}|X_{j\tau(i)}|^3\big) \\ &= \frac{8\delta(n-m-2)}{(n-m)^2}\bigg[(n-m-1)\sum_{i=1}^{n-m} \mathbb{E}|X_{i\tau(i)}|^3 + \sum_{1\le i\ne j\le n-m} \mathbb{E}|X_{i\tau(j)}|^3\bigg] \\ &\le \frac{16n\delta}{n-4}\gamma. \end{aligned} \tag{2.20}$$

Let $\alpha_{ij}^{kl} = (X_{ik} + X_{jl} - X_{il} - X_{jk})^2 I(|X_{ik} + X_{jl} - X_{il} - X_{jk}| \le \delta)$. For $|\{i, j, i', j'\}| = 4$,

$$\begin{aligned}
\operatorname{Cov}\big(\alpha_{ij}^{\tau}, \alpha_{i'j'}^{\tau}\big) &= \mathbb{E}\alpha_{ij}^{\tau}\alpha_{i'j'}^{\tau} - \mathbb{E}\alpha_{ij}^{\tau}\mathbb{E}\alpha_{i'j'}^{\tau} \\
&= \frac{1}{(n-m)_{(4)}} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ |\{k,l,k',l'\}|=4}} \mathbb{E}\alpha_{ij}^{kl}\alpha_{i'j'}^{k'l'} \\
&\quad - \Bigg[\frac{1}{(n-m)_{(2)}} \sum_{1\le k\ne l\le n-m} \mathbb{E}\alpha_{ij}^{kl}\Bigg]\Bigg[\frac{1}{(n-m)_{(2)}} \sum_{1\le k'\ne l'\le n-m} \mathbb{E}\alpha_{i'j'}^{k'l'}\Bigg] \\
&= \frac{1}{(n-m)_{(4)}} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ |\{k,l,k',l'\}|=4}} \mathbb{E}\alpha_{ij}^{kl}\mathbb{E}\alpha_{i'j'}^{k'l'} - \frac{1}{[(n-m)_{(2)}]^2} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ |\{k,l,k',l'\}|=4}} \mathbb{E}\alpha_{ij}^{kl}\mathbb{E}\alpha_{i'j'}^{k'l'} \\
&\quad - \frac{1}{[(n-m)_{(2)}]^2} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ k\ne l, k'\ne l', |\{k,l,k',l'\}|\le 3}} \mathbb{E}\alpha_{ij}^{kl}\mathbb{E}\alpha_{i'j'}^{k'l'} \\
&= \frac{4(n-m)-6}{(n-m)_{(2)}(n-m)_{(4)}} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ |\{k,l,k',l'\}|=4}} \mathbb{E}\alpha_{ij}^{kl}\mathbb{E}\alpha_{i'j'}^{k'l'} \\
&\quad - \frac{1}{[(n-m)_{(2)}]^2} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ k\ne l, k'\ne l', |\{k,l,k',l'\}|\le 3}} \mathbb{E}\alpha_{ij}^{kl}\mathbb{E}\alpha_{i'j'}^{k'l'}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|R_3| &\le \frac{1}{16(n-m)^2} \sum_{\substack{1\le i,j,i',j'\le n-m,\\ |\{i,j,i',j'\}|=4}} \Bigg[\frac{4(n-m)-6}{(n-m)_{(2)}(n-m)_{(4)}} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ |\{k,l,k',l'\}|=4}} \frac{\mathbb{E}(\alpha_{ij}^{kl})^2 + \mathbb{E}(\alpha_{i'j'}^{k'l'})^2}{2} \\
&\qquad + \frac{1}{[(n-m)_{(2)}]^2} \sum_{\substack{1\le k,l,k',l'\le n-m,\\ k\ne l, k'\ne l', |\{k,l,k',l'\}|\le 3}} \frac{\mathbb{E}(\alpha_{ij}^{kl})^2 + \mathbb{E}(\alpha_{i'j'}^{k'l'})^2}{2}\Bigg] \\
&\le \frac{1}{16(n-m)^2} \\
&\quad \times \sum_{\substack{1\le i,j,i',j'\le n-m,\\ |\{i,j,i',j'\}|=4}} \Bigg[\frac{4}{(n-m)(n-m-1)^2} \\
&\qquad \times \Bigg(\sum_{1\le k\ne l\le n-m} \mathbb{E}\big(\alpha_{ij}^{kl}\big)^2/2 + \sum_{1\le k'\ne l'\le n-m} \mathbb{E}\big(\alpha_{i'j'}^{k'l'}\big)/2\Bigg)
\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&\quad + \frac{4}{(n-m)(n-m-1)^2} \\
&\quad \times \Bigg(\sum_{1\le k\neq l\le n-m} \mathbb{E}\big(\alpha_{ij}^{kl}\big)^2/2 + \sum_{1\le k'\neq l'\le n-m} \mathbb{E}\big(\alpha_{i'j'}^{k'l'}\big)^2/2 \Bigg)\Bigg] \\
&\le \frac{1}{2(n-m)^3(n-m-1)^2} \sum_{\substack{1\le i,j,i',j'\le n-m,\\ |\{i,j,i',j'\}|=4}} \sum_{1\le k\neq l\le n-m} \mathbb{E}\big(\alpha_{ij}^{kl}\big)^2 \\
&\le \frac{\delta}{2(n-m)^3} \sum_{i,j,k,l=1}^{n} \mathbb{E}|X_{ik}+X_{jl}-X_{il}-X_{jk}|^3 \le 32\left(\frac{n}{n-4}\right)^3 \delta\gamma.
\end{aligned}
$$

From (2.19), (2.20), (2.21), and then applying (2.18), we obtain

$$
\begin{aligned}
B_0^2 &\le \left[\frac{8n}{(n-4)^2} + \frac{16n}{n-4} + 32\left(\frac{n}{n-4}\right)^3\right]\delta\gamma \\
&\le \left[\frac{8n}{(n-4)^2} + \frac{16n}{n-4} + 32\left(\frac{n}{n-4}\right)^3\right]\frac{32n\gamma^2}{n-4}.
\end{aligned}
\tag{2.22}
$$

Now we are ready to obtain a concentration inequality for S using Lemma 2.1. From (2.2), and applying the bounds (2.16), (2.8), (2.18), (2.22), we obtain

$$
\begin{aligned}
&P\big(S\in[a,b]\big) \\
&\le \Bigg(\Bigg(\sqrt{\frac{n}{n-5}+\frac{24n}{(n-5)^2}} + \frac{4\sqrt{n}}{n-4}\Bigg) \\
&\qquad \Bigg/\Bigg(\frac{n-1}{n-2} - \frac{2n}{(n-4)c_0^{2/3}} - \frac{24n}{(n-5)^2} \\
&\qquad\quad - \frac{4\sqrt{n}}{n-4}\sqrt{\frac{n}{n-5}+\frac{24n}{(n-5)^2}} - \frac{1}{2}\Bigg)\Bigg)\left(\frac{b-a}{2}+\delta\right) \\
&\quad + \Bigg(\Bigg(\sqrt{\frac{8n}{(n-4)^2}+\frac{16n}{n-4}+32\left(\frac{n}{n-4}\right)^3}\sqrt{\frac{32n}{n-4}}\Bigg) \\
&\qquad \Bigg/\Bigg(\frac{n-1}{n-2} - \frac{2n}{(n-4)c_0^{2/3}} - \frac{24n}{(n-5)^2} - \frac{4\sqrt{n}}{n-4}\sqrt{\frac{n}{n-5}+\frac{24n}{(n-5)^2}} - \frac{1}{2}\Bigg)\Bigg)\gamma \\
&\le c_1(b-a) + c_2\gamma.
\end{aligned}
$$

□

Remark 2.5. Lemma 2.3 and Proposition 2.4 still hold if S is defined similarly as in (2.7) but with any m rows and m columns removed from the array X where $m = 2, 3, 4$.

Remark 2.6. From Proposition 2.4, the error in Neammanee and Suntornchost [18] can be corrected by conditioning on (using their notation)

$$J, K, L, M, \tau(J), \tau(K), \tau(L), \tau(M),$$
$$\bigl\{\hat{X}_{ij}\colon i \in \bigl\{J, K, \tau^{-1}(L), \tau^{-1}(M)\bigr\}, j \in \bigl\{L, M, \tau(J), \tau(K)\bigr\}\bigr\},$$

and by applying our Proposition 2.4 instead of their Proposition 2.7.

3. Proof of the main result

From (1.1), $\mathbb{E}W = 0$. The variance of W can be calculated as follows. From (1.1),

$$\begin{aligned}
\operatorname{Var}(W) &= \operatorname{Var}\left(\sum_{i=1}^n X_{i\pi(i)}\right)\\
&= \sum_{i=1}^n \operatorname{Var}(X_{i\pi(i)}) + \sum_{1\le i\ne j\le n} \operatorname{Cov}(X_{i\pi(i)}, X_{j\pi(j)})\\
&= \sum_{i=1}^n \mathbb{E}(X_{i\pi(i)} - c_{i\cdot})^2 + \sum_{1\le i\ne j\le n} \mathbb{E}(X_{i\pi(i)} - c_{i\cdot})(X_{j\pi(j)} - c_{j\cdot})\\
&= \frac{1}{n}\sum_{i,j=1}^n \mathbb{E}(X_{ij} - c_{i\cdot})^2 + \frac{1}{n(n-1)} \sum_{1\le i\ne j\le n}\sum_{1\le k\ne l\le n} \mathbb{E}(X_{ik} - c_{i\cdot})(X_{jl} - c_{j\cdot})\\
&= \frac{1}{n}\sum_{i,j=1}^n \bigl(\sigma_{ij}^2 + c_{ij}^2\bigr) + \frac{1}{n(n-1)} \sum_{1\le i\ne j\le n}\sum_{1\le k\ne l\le n} c_{ik}c_{jl}\\
&= \frac{1}{n}\sum_{i,j=1}^n \bigl(\sigma_{ij}^2 + c_{ij}^2\bigr) + \frac{1}{n(n-1)} \sum_{i,j=1}^n c_{ij}^2\\
&= \frac{1}{n}\sum_{i,j=1}^n \sigma_{ij}^2 + \frac{1}{n-1} \sum_{i,j=1}^n c_{ij}^2.
\end{aligned}$$

This proves the first part of the theorem. In the following, we work under the assumption that $\operatorname{Var}(W) = 1$, that is,

$$\frac{1}{n}\sum_{i,j=1}^n \sigma_{ij}^2 + \frac{1}{n-1}\sum_{i,j=1}^n c_{ij}^2 = 1. \tag{3.1}$$

We assume $\gamma \le 1/451$, that is, $c_0 = 451$ in Proposition 2.4. Otherwise the bound (1.3) is obviously true. From (3.1) and Hölder's inequality, we have

$$n-1 \le \sum_{i,j=1}^{n} \mathbb{E}X_{ij}^2 \le n^{2/3}\left(\sum_{i,j=1}^{n} \mathbb{E}|X_{ij}|^3\right)^{2/3} = n^{4/3}\gamma^{2/3}. \tag{3.2}$$

Therefore it suffices to prove Theorem 1.1 for $n \ge 203\,000$. For $n \ge 203\,000$ and $\gamma \le 1/451$, (2.10) is satisfied, and the concentration inequality (2.11) in Proposition 2.4 is applicable.

We follow the notation in Section 1 and construct an exchangeable pair (W, W') by uniformly selecting two different indices $I, J \in [n]$ (the ranges of I and J are different from those in the proof of Proposition 2.4) and let $W' = W - X_{I\pi(I)} - X_{J\pi(J)} + X_{I\pi(J)} + X_{J\pi(I)}$. Following the argument as in (2.14), we have

$$\mathbb{E}\bigl(W' - W|W\bigr) = -\lambda W + R, \tag{3.3}$$

where $\lambda = 2/(n-1)$ and

$$R = \frac{2}{n(n-1)}\mathbb{E}\left(\sum_{i,j=1}^{n} X_{ij}\middle|W\right).$$

The following bound on $\sqrt{\mathbb{E}R^2}$

$$\sqrt{\mathbb{E}R^2} \le \frac{2}{n(n-1)}\sqrt{\operatorname{Var}\left(\sum_{i,j=1}^{n} X_{ij}\right)} = \frac{2}{n(n-1)}\sqrt{\sum_{i,j=1}^{n}\sigma_{ij}^2} \le \frac{2}{(n-1)\sqrt{n}} \tag{3.4}$$

is obtained by using the assumptions (1.1) and (3.1).

From the fact that (W, W') is an exchangeable pair and satisfies an approximate linearity condition (3.3), the following functional identity can be proved by the same argument as in (2.5).

$$\mathbb{E}Wf(W) = \frac{1}{2\lambda}\mathbb{E}\bigl(W' - W\bigr)\bigl(f\bigl(W'\bigr) - f(W)\bigr) + \frac{\mathbb{E}Rf(W)}{\lambda}. \tag{3.5}$$

Let f be the bounded solution to the Stein equation

$$f'(w) - wf(w) = I(w \le z) - \Phi(z). \tag{3.6}$$

It is known that (Chen and Shao [9])

$$\bigl|f(w)\bigr| \le \frac{\sqrt{2\pi}}{4}, \qquad \bigl|f'(w)\bigr| \le 1 \qquad \forall w \in \mathbb{R} \tag{3.7}$$

and

$$\big|(w+u)f(w+u)-(w+v)f(w+v)\big| \le \left(|w|+\frac{\sqrt{2\pi}}{4}\right)\big(|u|+|v|\big). \tag{3.8}$$

From (3.6) and (3.5), what we need to bound is

$$\begin{aligned}
&\mathbb{P}(W \le z)-\Phi(z)\\
&\quad=\mathbb{E}f'(W)-\mathbb{E}Wf(W)\\
&\quad=\mathbb{E}f'(W)\left(1-\frac{(W'-W)^2}{2\lambda}\right)+\frac{1}{2\lambda}\mathbb{E}\big(W'-W\big)\int_0^{W'-W}\big(f'(W)-f'(W+t)\big)\,\mathrm{d}t\\
&\qquad-\frac{\mathbb{E}Rf(W)}{\lambda}\\
&\quad:=R_1+R_2-R_3.
\end{aligned}$$

From (3.4) and (3.7), and recalling $\lambda=2/(n-1)$, we have

$$|R_3| \le 1/\sqrt{n}. \tag{3.9}$$

To bound R_1 and R_2, we need the concentration inequality obtained in the last section.

From (3.3) and (3.4),

$$\mathbb{E}\big(W'-W\big)^2=2\lambda-2\mathbb{E}WR\begin{cases}\le 2\lambda+2\sqrt{\mathbb{E}R^2}\le \dfrac{4}{n-1}\left(1+\dfrac{1}{\sqrt{n}}\right),\\[2ex] \ge 2\lambda-2\sqrt{\mathbb{E}R^2}\ge \dfrac{4}{n-1}\left(1-\dfrac{1}{\sqrt{n}}\right).\end{cases} \tag{3.10}$$

We bound R_2 first. From (3.6),

$$\begin{aligned}
R_2 ={}& \frac{1}{2\lambda}\mathbb{E}\big(W'-W\big)\int_0^{W'-W}\big(Wf(W)-(W+t)f(W+t)\big)\,\mathrm{d}t\\
&+\frac{1}{4n}\sum_{1\le i\ne j\le n}\mathbb{E}(X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)})\\
&\qquad\times\int_0^{X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}}\big[I(U\le z-X_{i\pi(i)}-X_{j\pi(j)})\\
&\qquad\qquad -I(U\le z-X_{i\pi(i)}-X_{j\pi(j)}-t)\big]\,\mathrm{d}t\\
:={}& R_{2,1}+R_{2,2},
\end{aligned}$$

where $U=\sum_{k\notin\{i,j\}}X_{k\pi(k)}$. Noting that U is independent of $\{X_{i\pi(i)},X_{j\pi(j)},X_{i\pi(j)},X_{j\pi(i)}\}$ given $\pi(i),\pi(j)$, and that the conditional distribution of U given $\pi(i),\pi(j)$ is the same as that of S in (2.7) for $m = 2$ under a relabeling of indices, we can apply the concentration inequality (2.11) to obtain the following upper bound on $|R_{2,2}|$.

$$
\begin{aligned}
|R_{2,2}| \le \frac{1}{4n} \sum_{1 \le i \ne j \le n} & \mathbb{E}(X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}) \\
& \times \int_0^{X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}} I\big(-(t \vee 0) \le U - (z - X_{i\pi(i)} - X_{j\pi(j)}) \\
& \qquad \le -(t \wedge 0)\big)\,\mathrm{d}t \\
\le \frac{1}{4n} \sum_{1 \le i \ne j \le n} & \mathbb{E}(X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}) \\
& \times \int_0^{X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}} \big(c_1|t| + c_2\gamma\big)\,\mathrm{d}t \\
= \frac{c_1}{8n} \sum_{1 \le i \ne j \le n} & \mathbb{E}|X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}|^3 \\
+ \frac{c_2\gamma}{4n} \sum_{1 \le i \ne j \le n} & \mathbb{E}|X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}|^2 \\
\le 8c_1\gamma + c_2\gamma & \left(1 + \frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

In the last inequality, we used (2.17) and

$$
\begin{aligned}
\sum_{1 \le i \ne j \le n} \mathbb{E}|X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}|^2 &= n(n-1)\mathbb{E}\big(W' - W\big)^2 \\
&\le 4n\left(1 + \frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

which follows from (3.10). For $R_{2,1}$, from the property (3.8) of f with $w = U$, $u = X_{i\pi(i)} + X_{j\pi(j)}$, $v = X_{i\pi(i)} + X_{j\pi(j)} + t$,

$$
\begin{aligned}
|R_{2,1}| \le \frac{1}{4n} \sum_{1 \le i \ne j \le n} & \mathbb{E}(X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)}) \\
& \times \int_0^{X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}} \left(|U| + \frac{\sqrt{2\pi}}{4}\right)\big(2|X_{i\pi(i)} + X_{j\pi(j)}| + t\big)\,\mathrm{d}t \\
= \frac{1}{4n} \sum_{1 \le i \ne j \le n} & \mathbb{E}\left(|U| + \frac{\sqrt{2\pi}}{4}\right)\Big[(X_{i\pi(j)} + X_{j\pi(i)} - X_{i\pi(i)} - X_{j\pi(j)})^2 2|X_{i\pi(i)} + X_{j\pi(j)}|
\end{aligned}
$$

$$+\frac{|X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}|^3}{2}\Bigg]$$

$$\le 24\left(\sqrt{\frac{n}{n-5}+\frac{24n}{(n-5)^2}}+\frac{\sqrt{2\pi}}{4}\right)\gamma,$$

where we used (2.8), (2.17) and

$$\begin{aligned}&|X_{i\pi(j)}+X_{j\pi(i)}-X_{i\pi(i)}-X_{j\pi(j)}|^2|X_{i\pi(i)}+X_{j\pi(j)}|\\&\quad\le \tfrac{16}{3}\big(|X_{i\pi(j)}|^3+|X_{j\pi(i)}|^3\big)+\tfrac{32}{3}\big(|X_{i\pi(i)}|^3+|X_{j\pi(j)}|^3\big).\end{aligned}$$

Therefore, with

$$c_3:=\sqrt{\frac{n}{n-5}+\frac{24n}{(n-5)^2}}, \tag{3.11}$$

$$|R_2|\le\left(8c_1+c_2\left(1+\frac{1}{\sqrt{n}}\right)+24c_3+6\sqrt{2\pi}\right)\gamma. \tag{3.12}$$

Next, we bound R_1.

$$\begin{aligned}R_1&=\mathbb{E}f'(W)\left(1-\frac{(W'-W)^2}{2\lambda}\right)\\&=\frac{1}{n^2(n-1)^2}\\&\quad\times\sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}}\mathbb{E}\left(f'(W)\left(1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\right)\right|\\&\qquad\qquad I=i,J=j,\pi(i)=k,\pi(j)=l\Bigg)\\&=\frac{1}{n^2(n-1)^2}\sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}}\mathbb{E}\left(f'(W)\left(1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\right)\Big|\pi(i)=k,\pi(j)=l\right)\end{aligned}$$

since $\mathbb{X}$, π and (I,J) are independent. For each choice of $i\ne j, k\ne l$, let $\mathbb{X}^{ijkl}:=\{X^{ijkl}_{i'j'}: i',j'\in[n]\}$ be the same as $\mathbb{X}$ except that $\{X_{ik},X_{il},X_{jk},X_{jl}\}$ has been replaced by an independent copy $\{X'_{ik},X'_{il},X'_{jk},X'_{jl}\}$. Define

$$W^{ijkl}=\sum_{i'=1}^{n}X^{ijkl}_{i'\pi(i')}.$$

Then,

$$W^{ijkl} \text{ is independent of } \{X_{ik}, X_{il}, X_{jk}, X_{jl}\} \text{ and } \mathcal{L}(W^{ijkl}) = \mathcal{L}(W). \tag{3.13}$$

Next, we define a new permutation π_{ijkl} coupled with π such that

$$\mathcal{L}(\pi_{ijkl}) = \mathcal{L}\big(\pi | \pi(i) = k, \pi(j) = l\big).$$

This coupling has been constructed by Goldstein [11]. Let τ_{ij} denote the transposition of i, j. Define

$$\pi_{ijkl} = \begin{cases} \pi & \text{if } l = \pi(j), k = \pi(i), \\ \pi \cdot \tau_{\pi^{-1}(k),i} & \text{if } l = \pi(j), k \neq \pi(i), \\ \pi \cdot \tau_{\pi^{-1}(l),j} & \text{if } l \neq \pi(j), k = \pi(i), \\ \pi \cdot \tau_{\pi^{-1}(l),i} \cdot \tau_{\pi^{-1}(k),j} \cdot \tau_{ij} & \text{if } l \neq \pi(j), k \neq \pi(i). \end{cases} \tag{3.14}$$

Let

$$W_{ijkl} = \sum_{i'=1}^{n} X_{i'\pi_{ijkl}(i')}.$$

Since W_{ijkl} has the conditional distribution of W given $\pi(i) = k, \pi(j) = l$, and since $\mathbb{X}$ and π are independent, we have

$$\begin{aligned} &\mathbb{E}\bigg(f'(W)\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg)\Big|\pi(i) = k, \pi(j) = l\bigg) \\ &\quad = \mathbb{E}f'(W_{ijkl})\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg). \end{aligned}$$

Therefore,

$$\begin{aligned} R_1 &= \mathbb{E}f'(W)\bigg(1 - \frac{(W' - W)^2}{2\lambda}\bigg) \\ &= \frac{1}{(n(n-1))^2} \sum_{\substack{1\le i,j,k,l\le n,\\ i\neq j, k\neq l}} \mathbb{E}\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg)\big(f'(W_{ijkl}) - f'\big(W^{ijkl}\big)\big) \\ &\quad + \frac{1}{(n(n-1))^2} \sum_{\substack{1\le i,j,k,l\le n,\\ i\neq j, k\neq l}} \mathbb{E}\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg) f'\big(W^{ijkl}\big). \end{aligned}$$

Define index sets $\mathcal{I} = \{i, j, \pi^{-1}(k), \pi^{-1}(l)\}$ and $\mathcal{J} = \{k, l, \pi(i), \pi(j)\}$. Then $|\mathcal{I}| = |\mathcal{J}| \in \{2, 3, 4\}$. Letting $S = \sum_{i' \notin \mathcal{I}} X_{i'\pi(i')}$, we can write

$$W_{ijkl} = S + \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}$$

and

$$W^{ijkl} = S + \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')}.$$

Since S is a function depending only on the components of $\mathbb{X}$ outside the square $\mathcal{I} \times \mathcal{J}$ and $\{\pi(i)\colon i \notin \mathcal{I}\}$,

$$\begin{aligned} &S \text{ is independent of} \left\{ X_{il}, X_{jk}, X_{ik}, X_{jl}, \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}, \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')} \right\} \\ &\text{given } \pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j). \end{aligned} \tag{3.15}$$

The conditional distribution of S given $\pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)$ is the same as that of S in (2.7) under a relabeling of indices. From (2.8), $\mathbb{E}(|S||\pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)) \le c_3$ where c_3 was defined in (3.11). From (3.13), (3.7) and (3.10),

$$\begin{aligned} &\frac{1}{(n(n-1))^2} \left| \sum_{\substack{1 \le i,j,k,l \le n, \\ i \ne j, k \ne l}} \mathbb{E}\left(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\right) f'\big(W^{ijkl}\big) \right| \\ &= \frac{1}{(n(n-1))^2} \left| \mathbb{E} f'(W) \sum_{\substack{1 \le i,j,k,l \le n, \\ i \ne j, k \ne l}} \mathbb{E}\left(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\right) \right| \\ &= \left| \mathbb{E} f'(W) \mathbb{E}\left(1 - \frac{(W' - W)^2}{2\lambda}\right) \right| \le \frac{1}{\sqrt{n}}. \end{aligned}$$

Therefore,

$$\begin{aligned} |R_1| &\le \left| \frac{1}{(n(n-1))^2} \sum_{\substack{1 \le i,j,k,l \le n, \\ i \ne j, k \ne l}} \mathbb{E}\left(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\right) \big(f'(W_{ijkl}) - f'\big(W^{ijkl}\big)\big) \right| \\ &\quad + \frac{1}{\sqrt{n}} \\ &= \left| \frac{1}{(n(n-1))^2} \sum_{\substack{1 \le i,j,k,l \le n, \\ i \ne j, k \ne l}} \mathbb{E}\left(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\right) \right. \\ &\qquad \left. \times \left(f'\left(S + \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}\right) - f'\left(S + \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')}\right)\right) \right| + \frac{1}{\sqrt{n}} \\ &\le R_{1,1} + R_{1,2} + \frac{1}{\sqrt{n}}, \end{aligned}$$

where using the Stein equation (3.6),

$$R_{1,1} = \Bigg| \frac{1}{(n(n-1))^2} \sum_{\substack{1 \le i,j,k,l \le n,\\ i \ne j, k \ne l}} \mathbb{E}\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg) \times \bigg(\Big(S + \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}\Big) f\Big(S + \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}\Big) - \Big(S + \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')}\Big) f\Big(S + \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')}\Big)\bigg) \Bigg|,$$

and

$$R_{1,2} = \Bigg| \frac{1}{(n(n-1))^2} \sum_{\substack{1 \le i,j,k,l \le n,\\ i \ne j, k \ne l}} \mathbb{E}\bigg(1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg) \times \bigg(I\Big(S + \sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')} \le z\Big) - I\Big(S + \sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')} \le z\Big)\bigg) \Bigg|.$$

Applying (3.8), (3.15), (2.8) and (3.1), $R_{1,1}$ can be bounded as follows.

$$\begin{aligned} R_{1,1} &\le \frac{1}{(n(n-1))^2} \\ &\quad \times \sum_{\substack{1 \le i,j,k,l \le n,\\ i \ne j, k \ne l}} \mathbb{E}\bigg\{\mathbb{E}\big(|S| \big| \pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)\big) + \frac{\sqrt{2\pi}}{4}\bigg\} \\ &\quad \times \mathbb{E}\bigg\{\bigg(\Big|\sum_{i' \in \mathcal{I}} X_{i'\pi_{ijkl}(i')}\Big| + \Big|\sum_{i' \in \mathcal{I}} X^{ijkl}_{i'\pi(i')}\Big|\bigg) \\ &\quad \times \bigg|1 - \frac{(X_{il} + X_{jk} - X_{ik} - X_{jl})^2}{2\lambda}\bigg| \bigg| \pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)\bigg\} \\ &\le \frac{c_3 + \sqrt{2\pi}/4}{n^2(n-1)^2} \\ &\quad \times \sum_{\substack{1 \le i,j,k,l \le n,\\ i \ne j, k \ne l}} \max\bigg\{\mathbb{E}\Big(\sum_{i' \in \mathcal{I}} |X_{i'\pi_{ijkl}(i')}| + \sum_{i' \in \mathcal{I}} \big|X^{ijkl}_{i'\pi(i')}\big|\Big), \\ &\qquad \frac{4}{2\lambda} \mathbb{E}\Big(\sum_{i' \in \mathcal{I}} |X_{i'\pi_{ijkl}(i')}| + \sum_{i' \in \mathcal{I}} \big|X^{ijkl}_{i'\pi(i')}\big|\Big)\big(X_{il}^2 + X_{jk}^2 + X_{ik}^2 + X_{jl}^2\big)\bigg\} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{c_3+\sqrt{2\pi}/4}{n^2(n-1)^2}\sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}} \max\Big\{\mathbb{E}\big[|X_{ik}|+|X_{jl}|+|X_{\pi^{-1}(k)\pi(i)}|+|X_{\pi^{-1}(l)\pi(j)}| \\
&\qquad + \big|X^{ijkl}_{i\pi(i)}\big| + \big|X^{ijkl}_{j\pi(j)}\big| + \big|X^{ijkl}_{\pi^{-1}(k)k}\big| + \big|X^{ijkl}_{\pi^{-1}(l)l}\big|\big], \\
&\qquad \frac{2}{\lambda}\mathbb{E}\Big(\sum_{i'\in\mathcal{I}}\frac{4}{3}|X_{i'\pi_{ijkl}(i')}|^3 + \sum_{i'\in\mathcal{I}}\frac{4}{3}\big|X^{ijkl}_{i'\pi(i')}\big|^3 \\
&\qquad + \frac{16}{3}\big(|X_{il}|^3+|X_{jk}|^3+|X_{ik}|^3+|X_{jl}|^3\big)\Big)\Big\} \\
&\le \max\Big\{\frac{8(c_3+\sqrt{2\pi}/4)}{n^2}\sum_{i,k=1}^n \mathbb{E}|X_{ik}|, 32\Big(c_3+\frac{\sqrt{2\pi}}{4}\Big)\frac{n-1}{n}\gamma\Big\} \\
&\le \max\Big\{8\Big(c_3+\frac{\sqrt{2\pi}}{4}\Big)\frac{1}{\sqrt{n}}, 32\Big(c_3+\frac{\sqrt{2\pi}}{4}\Big)\frac{n-1}{n}\gamma\Big\},
\end{aligned}
$$

where we used

$$
\left|1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\right| \le \max\left\{1, \frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\right\} \tag{3.16}
$$

and

$$
\sum_{i,k=1}^n \mathbb{E}|X_{ik}| \le n\sqrt{\sum_{i,k=1}^n \mathbb{E}X_{ik}^2} \le n^{3/2}.
$$

Now we bound $R_{1,2}$.

$$
\begin{aligned}
R_{1,2} = \Bigg| &\frac{1}{(n(n-1))^2} \\
&\times \sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}} \mathbb{E}\Big\{\mathbb{E}\Big[\Big(1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\Big) \\
&\qquad \times \Big(I\Big(S+\sum_{i'\in\mathcal{I}}X_{i'\pi_{ijkl}(i')}\le z\Big) - I\Big(S+\sum_{i'\in\mathcal{I}}X^{ijkl}_{i'\pi(i')}\le z\Big)\Big)\Big| \\
&\qquad \pi^{-1}(k),\pi^{-1}(l),\pi(i),\pi(j)\Big]\Big\}\Bigg| \\
\le &\frac{1}{(n(n-1))^2}
\end{aligned}
$$

$$\times \sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}} \mathbb{E}\Biggl\{\mathbb{E}\Biggl[\biggl|1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\biggr| \times I\biggl(z-\max\biggl\{\sum_{i'\in\mathcal{I}} X_{i'\pi_{ijkl}(i')}, \sum_{i'\in\mathcal{I}} X^{ijkl}_{i'\pi(i')}\biggr\} \le S \le z-\min\biggl\{\sum_{i'\in\mathcal{I}} X_{i'\pi_{ijkl}(i')}, \sum_{i'\in\mathcal{I}} X^{ijkl}_{i'\pi(i')}\biggr\}\biggr)\biggr| \pi^{-1}(k), \pi^{-1}(l), \pi(i), \pi(j)\Biggr]\Biggr\}.$$

Recall (3.15) and the fact that the conditional distribution of S given $\pi^{-1}(k)$, $\pi^{-1}(l)$, $\pi(i)$, $\pi(j)$ is the same as that of S in (2.7) under a relabeling of indices. We can therefore apply the concentration inequality (2.11) to obtain the following upper bound on $R_{1,2}$.

$$\begin{aligned} R_{1,2} &\le \frac{1}{(n(n-1))^2} \sum_{\substack{1\le i,j,k,l\le n,\\ i\ne j,k\ne l}} \mathbb{E}\biggl|1-\frac{(X_{il}+X_{jk}-X_{ik}-X_{jl})^2}{2\lambda}\biggr| \\ &\qquad \times \biggl\{c_1\biggl(\biggl|\sum_{i'\in\mathcal{I}} X_{i'\pi_{ijkl}(i')}\biggr| + \biggl|\sum_{i'\in\mathcal{I}} X^{ijkl}_{i'\pi(i')}\biggr|\biggr) + c_2\gamma\biggr\} \\ &\le c_1 \max\biggl\{\frac{8}{\sqrt{n}}, 32\frac{n-1}{n}\gamma\biggr\} + c_2\biggl(1+\frac{1}{\sqrt{n}}\biggr)\gamma, \end{aligned}$$

where we used (3.16) and (3.10). Therefore,

$$|R_1| \le \biggl(c_1+c_3+\frac{\sqrt{2\pi}}{4}\biggr)\max\biggl\{\frac{8}{\sqrt{n}}, 32\frac{n-1}{n}\gamma\biggr\} + c_2\biggl(1+\frac{1}{\sqrt{n}}\biggr)\gamma + \frac{1}{\sqrt{n}}. \tag{3.17}$$

Summing (3.17), (3.12), (3.9) yields an upper bound on $\sup_{z\in\mathbb{R}} |P(W\le z)-\Phi(z)|$ as

$$\begin{aligned} &\sup_{z\in\mathbb{R}}\bigl|P(W\le z)-\Phi(z)\bigr| \\ &\quad \le \biggl(40c_1 + 2\biggl(1+\frac{1}{\sqrt{n}}\biggr)c_2 + 14\sqrt{2\pi} + 56c_3 + 2\biggl(\frac{n}{n-1}\biggr)^{3/2}\biggr)\gamma, \end{aligned} \tag{3.18}$$

where we used (3.2). Recall $c_0 = 451$ and $n \ge 203\,000$. Using $c_0 = 451$ and $n = 203\,000$, the upper bound in (3.18) is calculated to be smaller than 451γ. Since c_1, c_2 and c_3 decrease as n increases, (1.3) holds for $n \ge 203\,000$. This completes the proof of Theorem 1.1.

Remark 3.1. Radoslaw Adamczak has brought to our attention an inconsistency between (4.3) and (4.6) in Ho and Chen [13]. This error can be corrected by defining ρ given (I, K, L, M) in (4.6) as π_{IKLM} in (3.14) in this paper. This correction will not affect the rest of Ho and Chen [13].

Acknowledgements

This work is based on part of the Ph.D. thesis of the second author. The second author is thankful to the first author for his guidance and helpful discussions. Both the authors would like to thank the referee for their very detailed and helpful comments. We would also like to thank Radoslaw Adamczak for bringing to our attention an error in Ho and Chen [13]. This work is partially supported by Grant C-389-000-010-101 and Grant C-389-000-012-101 at the National University of Singapore.

References

[1] Barbour, A.D. and Chen, L.H.Y. (2005). *An Introduction to Stein's Method*. *Lecture Notes Series* **4**. Singapore: Institute for Mathematical Sciences, National Univ. Singapore, Singapore Univ. Press and World Scientific.

[2] Bolthausen, E. (1984). An estimate of the remainder in a combinatorial central limit theorem. *Z. Wahrsch. Verw. Gebiete* **66** 379–386. MR0751577

[3] Chen, L.H.Y. (1975/76). An approximation theorem for sums of certain randomly selected indicators. *Z. Wahrsch. Verw. Gebiete* **33** 69–74. MR0385977

[4] Chen, L.H.Y. (1986). *The Rate of Convergence in a Central Limit Theorem for Dependent Random Variables with Arbitrary Index Set*. *IMA Preprint Series* **243**. Minneapolis: Univ. Minnesota.

[5] Chen, L.H.Y. (1998). Stein's method: Some perspectives with applications. In *Probability Towards* 2000 (L. Accardi and C.C. Heyde, eds.). *Lecture Notes in Statistics* **128**. New York: Springer. Papers from the symposium held at Columbia Univ., New York, October 2–6, 1995.

[6] Chen, L.H.Y., Goldstein, L. and Shao, Q.-M. (2011). *Normal Approximation by Stein's Method*. *Probability and Its Applications* (*New York*). Heidelberg: Springer. MR2732624

[7] Chen, L.H.Y. and Shao, Q.-M. (2001). A non-uniform Berry–Esseen bound via Stein's method. *Probab. Theory Related Fields* **120** 236–254. MR1841329

[8] Chen, L.H.Y. and Shao, Q.-M. (2004). Normal approximation under local dependence. *Ann. Probab.* **32** 1985–2028. MR2073183

[9] Chen, L.H.Y. and Shao, Q.-M. (2005). Stein's method for normal approximation. In *An Introduction to Stein's Method* (A.D. Barbour and L.H.Y. Chen, eds.). *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.* **4** 1–59. Singapore: Singapore Univ. Press. MR2235448

[10] Diaconis, P. (1977). The distribution of leading digits and uniform distribution mod 1. *Ann. Probab.* **5** 72–81. MR0422186

[11] Goldstein, L. (2005). Berry–Esseen bounds for combinatorial central limit theorems and pattern occurrences, using zero and size biasing. *J. Appl. Probab.* **42** 661–683. MR2157512

[12] Goldstein, L. (2007). L^1 bounds in normal approximation. *Ann. Probab.* **35** 1888–1930. MR2349578

[13] Ho, S.T. and Chen, L.H.Y. (1978). An L_p bound for the remainder in a combinatorial central limit theorem. *Ann. Probab.* **6** 231–249. MR0478291

[14] Hoeffding, W. (1951). A combinatorial central limit theorem. *Ann. Math. Statistics* **22** 558–566. MR0044058

[15] Johnson, W.B. and Lindenstrauss, J. (1984). Extensions of Lipschitz mappings into a Hilbert space. In *Conference in Modern Analysis and Probability* (*New Haven*, *Conn*., 1982). *Contemp. Math.* **26** 189–206. Providence, RI: Amer. Math. Soc. MR0737400

[16] Neammanee, K. and Rattanawong, P. (2008). A uniform bound on the generalization of a combinatorial central limit theorem. *Int. Math. Forum* **3** 11–27. MR2373483

[17] Neammanee, K. and Rerkruthairat, N. (2012). An improvement of a uniform bound on a combinatorial central limit theorem. *Comm. Statist. Theory Methods* **41** 1590–1602. MR3003811

[18] Neammanee, K. and Suntornchost, J. (2005). A uniform bound on a combinatorial central limit theorem. *Stoch. Anal. Appl.* **23** 559–578. MR2140977

[19] Rinott, Y. and Rotar, V. (1997). On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. *Ann. Appl. Probab.* **7** 1080–1105. MR1484798

[20] Shao, Q.-M. and Su, Z.-G. (2006). The Berry–Esseen bound for character ratios. *Proc. Amer. Math. Soc.* **134** 2153–2159. MR2215787

[21] Stein, C. (1972). A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* (*Univ. California*, *Berkeley*, *Calif.*, 1970/1971), *Vol. II*: *Probability Theory* 583–602. Berkeley, CA: Univ. California Press. MR0402873

[22] Stein, C. (1986). *Approximate Computation of Expectations*. *Institute of Mathematical Statistics Lecture Notes—Monograph Series*, **7**. Hayward, CA: IMS. MR0882007

[23] Wald, A. and Wolfowitz, J. (1944). Statistical tests based on permutations of the observations. *Ann. Math. Statistics* **15** 358–372. MR0011424

[24] Wolff, P. (2012). On randomness reduction in the Johnson–Lindenstrauss lemma. Preprint, available at http://arxiv.org/abs/1202.5500.

Received November 2011 and revised September 2013