Exact sequences of braid and mapping class groups

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Artin’s pure braid group of \( n \) strands \( P_n(D^2) \); multiplication = concatenation of (isotopy classes of) braids.
versus
Artin’s pure braid group of $n$ strands $P_n(D^2)$; multiplication = concatenation of (isotopy classes of) braids. For any manifold $M$,

$$P_n(M) = \pi_1(\text{Conf}_n(M))$$

where

$$\text{Conf}_n(M) = \{(x_0, \ldots, x_{n-1}) \in M^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$
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Cap the disk, to get

$$c : P_n(D^2) = \pi_1(\text{Conf}_n(D^2)) \longrightarrow \pi_1(\text{Conf}_n(S^2)) = P_n(S^2).$$
Theorem (B-Cohen-Wong-Wu)

Whenever \( k \geq 4 \), there is an exact sequence

\[
\text{Brun}(P_{k+2}(S^2)) \twoheadrightarrow \text{Brun}(P_{k+1}(D^2)) \xrightarrow{c} \text{Brun}(P_{k+1}(S^2)) \rightarrow \pi_k(S^2)
\]

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\text{Brun}(P_n(M)) = \text{subgroup of classes of braids that become trivial whenever any strand is removed.}
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Motivating Questions

1. Does anything similar happen for more general surfaces?
Theorem (B-Cohen-Wong-Wu)

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Motivating Questions
1. Does anything similar happen for more general surfaces?
2. Does anything similar happen for mapping class groups?
**Simplicial Structures**

**Δ-set**: sets \( X = \{ X_0, X_1, \ldots \} \) with face maps \( d_i : X_k \to X_{k-1} \) \((i = 0, \ldots, k)\) such that

\[
d_jd_i = d_id_{j+1} \quad \text{whenever} \quad i \leq j.
\]
**Simplicial Structures**

**Δ-set:** sets $X = \{X_0, X_1, \ldots\}$ with *face maps* $d_i : X_k \to X_{k-1}$ ($i = 0, \ldots, k$) such that

$$d_j d_i = d_i d_{j+1} \text{ whenever } i \leq j.$$  

**Simplicial set:** Δ-set with *degeneracies* $s_j : X_k \to X_{k+1}$ ($j = 0, \ldots, k$) such that also

$$s_is_j = s_{j+1}s_i \text{ whenever } i \leq j,$$

$$d_is_j = \begin{cases}  
  s_{j-1}d_i & \text{whenever } i < j \\
  \text{id} & \text{whenever } i = j \text{ or } j + 1 \\
  s_jd_{i-1} & \text{whenever } i > j + 1.
\end{cases}$$
For category of groups, $\exists \Delta$-groups, simplicial groups.

Theorem (BCWW, Gonçalves-Guaschi, B-Hanbury)

(a) $P(M) = \{P^k+1(M)\}_{k \geq 0}$ forms a $\Delta$-group;

(b) $P(M)$ is a simplicial group iff $\partial M \neq \emptyset$ or $\chi(M) = 0$.

(s) $P_j = $ use vector field to shift slightly $j$th strand, to form $(j+1)$st, and relabel.

(b) gives semidirect product decomposition, normal form (e.g. Artin combing), solution to word problem.
For category of groups, ∃ Δ-groups, simplicial groups.

**Theorem (BCWW, Gonçalves-Guaschi, B-Hanbury)**

*For any surface $M$:*

(a) $P(M) = \{ P_{k+1}(M) \}_{k \geq 0}$ forms a Δ-group;

$P_i^P = \text{delete } i\text{th strand, and relabel.}$
For category of groups, \( \exists \ \Delta \)-groups, simplicial groups.

**Theorem (BCWW, Gonçalves-Guaschi, B-Hanbury)**

For any surface \( M \):

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Let $G = \{G_k\}_{k \geq 0}$ be a $\Delta$-group. We define

$$K_i(G_k) := \text{Ker}[d_i^G : G_k \to G_{k-1}] .$$
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Moore chain complex $N(G)$ of $G$ has $k$th group

$$N(G_k) := \bigcap_{i=1}^k \text{Ker}[d_i^G : G_k \to G_{k-1}] = \bigcap_{i=1}^k K_i(G_k),$$

differential $= d_0^G$. 
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N(G_k) := \bigcap_{i=1}^{k} \text{Ker}[d_i^G : G_k \to G_{k-1}] = \bigcap_{i=1}^{k} K_i(G_k)
\]

differential = \( d_0^G \).
Face map identity \( \Rightarrow \)
\[
d_0^G(N(G_k)) \subseteq N(G_{k-1}) \quad \text{and} \quad (d_0^G)^2(N(G_k)) = 1.
\]
Group of cycles $\text{Brun}(G_k) := \bigcap_{i=0}^{k} K_i(G_k)$.

Homotopy sets $\pi_k(G)$ of $G$ are the homology sets $\text{Brun}(G_k)/d_0^G(N(G_{k+1}))$ of Moore chain complex $N(G)$.
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Lemma (BCWW)

For any surface $M$ and $k \geq 0$, the subgroup $d_0^P(\mathbb{N}(P_{k+2}(M))) \leq \text{Brun}(P_{k+1}(M))$ is normal, and so $\pi_k(P(M))$ is a group.
Basepoint $\mathbf{m}_k = (m_0, \ldots, m_{k-1}) \in \text{Conf}_k(M)$. 
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Existence of a fibration

\[
M - m_k \xrightarrow{\iota_k} \text{Conf}_{k+1}(M) \xrightarrow{\tilde{d}_k} \text{Conf}_k(M)
\]  

(1)

$\iota_k : x \mapsto (m_0, \ldots, m_{k-1}, x)$

$\tilde{d}_k : (x_0, \ldots, x_{k-1}, x_k) \mapsto (x_0, \ldots, x_{k-1})$. 

Basepoint $m_k = (m_0, \ldots, m_{k-1}) \in \text{Conf}_k(M)$.

There exists a fibration

$$M - m_k \xrightarrow{\iota_k} \text{Conf}_{k+1}(M) \xrightarrow{\tilde{d}_k} \text{Conf}_k(M) \quad (1)$$

$\iota_k : x \mapsto (m_0, \ldots, m_{k-1}, x)$

$\tilde{d}_k : (x_0, \ldots, x_{k-1}, x_k) \mapsto (x_0, \ldots, x_{k-1})$.

$\pi_1 \Rightarrow$ exact

$$\pi_1(M - m_k) \xrightarrow{(\iota_k)_*} P_{k+1}(M) \xrightarrow{d^P_k} P_k(M).$$
With $\mathbf{m}_k^j = (m_0, \ldots, \hat{m}_j, \ldots, m_{k-1}) \in \text{Conf}_k(M)$, 
\[
\text{inc} : M - m_k \hookrightarrow M - \mathbf{m}_k^j \text{ induces (after path-conjugation relabelling)}
\]
\[
d_j^\pi : \pi_1(M - m_k, m_k) \longrightarrow \pi_1(M - m_{k-1}, m_{k-1}).
\]
With \( m^j_k = (m_0, \ldots, \hat{m}_j, \ldots, m_{k-1}) \in \text{Conf}_k(M) \),
\[ \text{inc} : M - m_k \hookrightarrow M - m^j_k \text{ induces} \]
(after path-conjugation relabelling)
\[ d^\pi_j : \pi_1(M - m_k, m_k) \longrightarrow \pi_1(M - m_{k-1}, m_{k-1}). \]

**Proposition (B-Hanbury-Wu)**

(a) \( \pi(M) = \{ \pi_1(M - m_1), \pi_1(M - m_2), \ldots \} \) forms a \( \Delta \)-group.

(b) \( \exists \) isomorphism
\[
\text{Brun}(\pi_1(M - m_k)) \cong \text{Brun}(P_{k+1}(M)).
\]
Corollary

For $k \geq 1^*$

$\text{Brun}(P_{k+1}(M))$ is a free group, and of infinite rank when also $k \geq 2$.
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Theorem (Li-Wu)

For $J \subseteq \{0, 1, \ldots, k - 1\}^*$

$$\bigcap_{j \in J} K_j(\pi_1(M - m_k)) = [[K_j(\pi_1(M - m_k)) : j \in J]]$$

where $[[\cdots]]$ is the fat commutator subgroup.

This allows a description of normal generators for Brun($P_{k+1}(M)$) ...
Figure: The braid $a_{j,k}$.

\[
\text{inc}_* : \quad P_{k+1}(D^2) \quad \longrightarrow \quad P_{k+1}(M) \\
\quad a_{j,k} \quad \longmapsto \quad A_{j,k}
\]
Mapping class groups

\( \text{Diff}(M, m_k) = \text{group of diffeomorphisms } M \to M \text{ that fix each } m_i \in m_k \text{ and each point of } \partial M \) (orientation-preserving if \( M \) oriented).
Mapping class groups

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\( k \) th pure mapping class group of \( M \) is

\[ \Gamma^k(M) = \pi_0(\text{Diff}(M, m_k)); \]

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Define \( \underline{\Gamma}(M) = \{ \Gamma^1(M), \Gamma^2(M), \ldots \} \).

Form face maps \( d_i^{\Gamma} : \Gamma^{k+1}(M) \to \Gamma^k(M) \) (\( i = 0, \ldots, k \)) by forgetting the \( i \) th marked point, and applying a “correction” diffeomorphism.
Theorem (BHW, BH)

(a) \( \Gamma(M) \) forms a \( \Delta \)-group.
(b) If \( \partial M \neq \emptyset \), then \( \Gamma(M) \) forms a simplicial group.
Theorem (BHW, BH)

(a) \( \Gamma(M) \) forms a \( \Delta \)-group.
(b) If \( \partial M \neq \emptyset \), then \( \Gamma(M) \) forms a simplicial group.

For (b), the degeneracy maps arise because each diffeomorphism is the identity on a collar of the boundary. So, locate the extra marked point in the collar.
\exists \text{ fibration}

\[
(Diff(M, m_{k+1}), \text{id}) \longrightarrow (Diff(M, m_k), \text{id}) \xrightarrow{ev_{m_k}} (M - m_k, m_k),
\]

and so, a connecting homomorphism

\[
\partial^\pi : \pi_1(M - m_k, m_k) \longrightarrow \Gamma^{k+1}(M).
\]
∃ fibration

\[(\text{Diff}(M, m_{k+1}), \text{id}) \longrightarrow (\text{Diff}(M, m_k), \text{id}) \xrightarrow{\text{ev}_{m_k}} (M - m_k, m_k), \quad (2)\]

and so, a connecting homomorphism

\[\partial^{\pi} : \pi_1(M - m_k, m_k) \longrightarrow \Gamma^{k+1}(M).\]

Thus, we can compare \(\pi(M)\) and \(\Gamma(M)\).
**Theorem (BHW)**

Suppose that \( M \neq S^2, \mathbb{R}P^2 \).

(a) \( \pi_k(\underline{P}(M)) = 1 \) for all \( k \geq 1 \).
Theorem (BHW)

Suppose that $M \neq S^2, \mathbb{R}P^2$.

(a) $\pi_k(P(M)) = 1$ for all $k \geq 1$.

(b) $\pi_k(\Gamma(M)) = 1$ for all $k \geq \begin{cases} 2 & \text{if } M = T, K, \\ 1 & \text{otherwise.} \end{cases}$
Theorem (BHW)

Suppose that $M \neq S^2, \mathbb{RP}^2$.
(a) $\pi_k(P(M)) = 1$ for all $k \geq 1$.
(b) $\pi_k(\Gamma(M)) = 1$ for all $k \geq \begin{cases} 2 & \text{if } M = T, K, \\ 1 & \text{otherwise.} \end{cases}$
(c) $\pi_0(\Gamma(M)) \cong \Gamma(M)$. 
Exceptions

Theorem

\((S^2)\) \(\pi_k(\Gamma(S^2)) = 1\) for \(k = 0, 1, 2\) and
\(\pi_3(\Gamma(S^2)) \cong F_2 / [[F_2, F_2], F_2].\)

For \(k \geq 4\), \(\pi_k(\Gamma(S^2)) \cong \pi_k(S^2).\)
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For \(k \geq 4\), \(\pi_k(\Gamma(S^2)) \cong \pi_k(S^2)\).

(\(\mathbb{RP}^2\)) \(\pi_k(\Gamma(\mathbb{RP}^2)) = 1\) for \(k = 0, 1\) and 
\(\pi_2(\Gamma(\mathbb{RP}^2)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}\).
For \(k \geq 3\), \(\pi_k(\Gamma(\mathbb{RP}^2)) \cong \pi_k(S^2)\).
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Theorem

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\( \pi_3(\Gamma(S^2)) \cong F_2 / [[F_2, F_2], F_2] \).
\[ \text{For } k \geq 4, \pi_k(\Gamma(S^2)) \cong \pi_k(S^2). \]

(\( \mathbb{R}P^2 \)) \( \pi_k(\Gamma(\mathbb{R}P^2)) = 1 \) for \( k = 0, 1 \) and 
\( \pi_2(\Gamma(\mathbb{R}P^2)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \)
\[ \text{For } k \geq 3, \pi_k(\Gamma(\mathbb{R}P^2)) \cong \pi_k(S^2). \]

(T) \( \pi_1(\Gamma(T)) \cong \mathbb{Z} \oplus \mathbb{Z}. \)
Exceptions

Theorem

\((S^2)\) \(\pi_k(\Gamma(S^2)) = 1\) for \(k = 0, 1, 2\) and 
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For \(k \geq 4\), \(\pi_k(\Gamma(S^2)) \cong \pi_k(S^2)\).

\((\mathbb{R}P^2)\) \(\pi_k(\Gamma(\mathbb{R}P^2)) = 1\) for \(k = 0, 1\) and 
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For \(k \geq 3\), \(\pi_k(\Gamma(\mathbb{R}P^2)) \cong \pi_k(S^2)\).

\((T)\) \(\pi_1(\Gamma(T)) \cong \mathbb{Z} \oplus \mathbb{Z}\).

\((K)\) \(\pi_1(\Gamma(K)) \cong \mathbb{Z}\).
Comparison of braids and mapping class groups \exists

evaluation fibration

\[
\text{Diff}(M, m_k) \longrightarrow \text{Diff}(M) \xrightarrow{\text{ev}_{m_k}} \text{Conf}_k(M). \quad (3)
\]
Comparison of braids and mapping class groups

Evaluation fibration

$$\text{Diff}(M, m_k) \longrightarrow \text{Diff}(M) \xrightarrow{\text{ev}_{m_k}} \text{Conf}_k(M). \quad (3)$$

The three fibrations fit together to give the commuting diagram
Comparison of disk mapping class groups

Proposition (B-Duzhin-Wu)

We have the following commutative diagram of groups and group homomorphisms:

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathbb{Z}^n & \rightarrow & \text{DF}_n & \rightarrow & \text{PF}_n & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \mathbb{Z}^n & \rightarrow & \text{DP}_n & \rightarrow & \text{PP}_n & \rightarrow & 1,
\end{array}
\]

whose rows and columns are exact sequences.
Moreover, the first row is a trivial group extension; that is,

$$DF_n \cong PF_n \times \mathbb{Z}^n \cong P_n \times \mathbb{Z}^n \cong \text{framed braids on } n \text{ strands}.$$ 

For the second row, we have only a semidirect product

$$DP_n \cong PP_n \rtimes \mathbb{Z}^n.$$