Fixed points of homotopy idempotent selfmaps of manifolds

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First, explain

**Conjecture**

*(Bass 1976)* For any group $G$, the Hattori-Stallings trace

\[ \text{HS} : K_0(\mathbb{Z}G) \rightarrow \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s] \]

has image in $\mathbb{Z} \cdot [e]$. 
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Here, integral group ring

$$\mathbb{Z}G = \left\{ \sum_{\text{finite}} a_g g \mid a_g \in \mathbb{Z}, \ g \in G \right\}$$

and $e \in G$ is the identity element.
For a ring $R$,
$K_0(R) :=$ Grothendieck group (formally invert $\oplus$) of isomorphism classes of finitely generated projective (right) $R$-modules.

- $P$ f.g. proj. (right) $R$-module
- $\iff \exists \bar{P}, n$ with $P \oplus \bar{P} \cong R^n$
- $\iff \exists M = M^2 = (m_{ij}) \in M_n(R)$ with $P = M \cdot R^n$

$M$ not unique, e.g. $M \oplus O$ and $GMG^{-1}$ ($G \in \text{GL}_n(R)$) give same $P$. 
Define

\[ \text{Tr}(P) = \text{Tr}(M) = \sum m_{ii} \in R/[R, R], \]

where \([R, R]\) is generated by all \(rs - sr\).

Okay, since \(H = MG^{-1}\) gives

\[ \text{Tr}(M) - \text{Tr}(GMG^{-1}) = \text{Tr}(HG) - \text{Tr}(GH) \]
\[ = \sum (h_{ij}g_{jj} - g_{ji}h_{ij}) \in [R, R]. \]
Since

\[ \text{Tr}(P_1 \oplus P_2) = \text{Tr}(M_1 \oplus M_2) = \text{Tr}(M_1) + \text{Tr}(M_2) = \text{Tr}(P_1) + \text{Tr}(P_2), \]

we obtain

\[ \text{Tr} : K_0(R) \longrightarrow R/[R, R]. \]

**Example** If \( P = R^n \) is free, then \( M = I_n \), so \( \text{Tr}(R^n) = n = \text{rank}(P) \).
For $R = \mathbb{Z}G$,

$[\mathbb{Z}G, \mathbb{Z}G]$ generated by all $gh - hg = gh - h(gh)h^{-1}$ with $g, h \in G$,

so

$$\mathbb{Z}G / [\mathbb{Z}G, \mathbb{Z}G] = \bigoplus_{[G]} \mathbb{Z} \cdot [s]$$

where $[s]$ is conjugacy class of $s \in G$, and set of conjugacy classes is $[G]$. 
Exploring work of Swan on finite groups, Bass conjectured the following.

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*(Bass 1976)* For any group $G$, the Hattori-Stallings trace

\[
\text{HS}: K_0(\mathbb{Z} G) \to \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]
\]

\[
[P] \mapsto \sum_{[G]} r_P(s)[s]
\]

has image in $\mathbb{Z} \cdot [e]$. 

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Weiss (1980): \text{Bass cj for } G \text{ polycyclic-by-finite.} 
Linnell (1983): 1 < |s| < \infty \implies r_P(s) = 0. 
Hence, \text{Bass cj for } G \text{ torsion, residually finite.}
Swan (1960) \[\implies\] Bass cj for \(G\) finite.

standard commutative algebra \[\implies\] Bass cj for \(G\) abelian.

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Linnell (1983): \(1 < |s| < \infty \implies r_P(s) = 0\).

Hence, Bass cj for \(G\) torsion, residually finite.

Eckmann (1986): using Connes-Karoubi’s Chern character in cyclic homology of \(\mathbb{C}G\), Bass cj for \(G\) with \(\text{cd}_Q(G) \leq 2\) and for \(G\) solvable of finite Hirsch number.

Emmanouil (1998): using techniques of Eckmann, Burghelea, Bass cj for $G$ elementary amenable with $\text{hd}_\mathbb{Q}(G) < \infty$. 
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Eckmann (2001): using geometric group theory (Alonso & Bridson), Bass cj for $G$ hyperbolic or $\text{CAT}(0)$.

Also, Farrell-Jones cj for $G$ $\Rightarrow$ Bass cj for $G$.
Bass cj related to Baum-Connes cj (shared consequences).
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Hastings & Heller (1982): $\exists$ CW-complex $Y$ and maps $d : X \to Y$ and $u : Y \to X$ such that

\[
\begin{array}{c}
X \\
\downarrow d
\end{array}
\xrightarrow{f}
\begin{array}{c}
X \\
\downarrow d
\end{array}
\xrightarrow{u}
\begin{array}{c}
Y \\
\uparrow id
\end{array}
\xrightarrow{\text{id}}
\begin{array}{c}
Y
\end{array}
\xrightarrow{\uparrow u}
\begin{array}{c}
X \\
\uparrow id
\end{array}
\xrightarrow{f}
\begin{array}{c}
X
\end{array}
\]

commutes up to homotopy.

Say $Y$ is *dominated* by $X$, so *finitely dominated*. 
Wall (1966): The singular chain complex of the universal cover $\tilde{Y}$ of $Y$ is chain homotopy equivalent to a complex over $\mathbb{Z}\pi_1(Y)$

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to \mathbb{Z}$$

with each $P_i$ a finitely generated projective $\mathbb{Z}\pi_1(Y)$-module.
The Bass trace conjecture

Progress

Geometric version – complexes
Geometric version – manifolds

Applications

Homotopy idempotents

Obstructions

\[ w(Y) = \sum_{i=0}^{n} (-1)^i [P_i] \in K_0(\mathbb{Z}\pi_1(Y)) \]

\[ \tilde{w}(Y) \in \tilde{K}_0(\mathbb{Z}\pi_1(Y)) \]

**Wall’s finiteness obstruction**

which vanishes exactly when \( Y \) is homotopy equivalent to a finite complex.
Geoghegan (1983): Define

\[ \tilde{w}(f) = \pi_1(u)_* (\tilde{w}(Y)) \in K_0(\mathbb{Z}\pi_1(X)). \]

\( \tilde{w}(f) \) “can be interpreted as the obstruction to splitting \( f \) through a finite complex”.

\[ d \circ u \simeq \text{id} \quad \Rightarrow \quad \pi_1(u)_* \text{ split injective} \]

So

\[ \tilde{w}(f) = 0 \quad \iff \quad \tilde{w}(Y) = 0 \quad \iff \quad Y \simeq \text{finite cx.} \]

**Lemma**

\( w(f) \in K_0(\mathbb{Z}\pi_1(X)) \) *is independent of choice of* \( Y, d, u. \)
Theorem
(Wall 1966 – extended) Let $G$ be a finitely presented group, let $\alpha \in K_0(\mathbb{Z}G)$, and let $n \geq 3$. Then there is a finite $n$-dimensional CW-complex $X^n$ with fundamental group $G$ and a pointed homotopy idempotent selfmap $f$ of $X^n$ inducing the identity on $\pi_1$, with $w(f) = \alpha$.

Lemma
(Geoghegan 1983) For $f$ as above, $HS(w(f)) = R(f)$, the Reidemeister trace of $f$. 
The number of nonzero terms in $R(f) = HS(w(f))$ is the *Nielsen number* $N(f) \geq 0$.

**Theorem**

("Geoghegan 1983") Let $G$ be a finitely presented group. TFAE:

(a) The Bass conjecture holds for $G$.
(b) Every $f$ as above has $N(f) = 0$ or 1.
Proof. \((a) \Rightarrow (b)\).

\[ R(f) = HS(w(f)) \in \mathbb{Z} \cdot [e]. \]
Proof. \( (a) \Rightarrow (b). \)

\[
R(f) = HS(w(f)) \in \mathbb{Z} \cdot [e].
\]

\( (b) \Rightarrow (a). \)

\[
HS(\alpha) = HS(w(f)) = R(f) \in \begin{cases} 
\{0\} & \therefore (a) \\
\mathbb{Z} \cdot [e] & \therefore (a) \\
\mathbb{Z} \cdot [s] & s \neq e : \end{cases}
\]
Proof. \((a) \Rightarrow (b)\).

\[ R(f) = \text{HS}(w(f)) \in \mathbb{Z} \cdot [e]. \]

\((b) \Rightarrow (a)\).

\[ \text{HS}(\alpha) = \text{HS}(w(f)) = R(f) \in \begin{cases} \{0\} & \therefore (a) \\ \mathbb{Z} \cdot [e] & \therefore (a) \\ \mathbb{Z} \cdot [s] & s \neq e : \end{cases} \]

In third case, \(f' := f \lor \text{id}_{S^2}\) has \(w(f') = w(f) + \mathbb{Z}G\).

So \(R(f') = R(f) + [e]\), whence \(N(f') = 2\), so not in case (b).
Lemma

(uses Wall 1966) For finite conn’d cx $X$,

(a) There is a closed, oriented and smooth manifold $M$ of dimension at least 3 with maps $r : M \to X$ and $s : X \to M$ having $r \circ s$ pointed homotopic to $\text{id}_X$ and inducing isomorphisms of fundamental groups.

(b) For any selfmap $f : X \to X$, the selfmap $\bar{f} = s \circ f \circ r : M \to M$ has Nielsen number $N(\bar{f}) = N(f)$.

(c) If $f$ is either homotopy idempotent or pointed homotopy idempotent, then so is $\bar{f}$. 
Thus, in Geoghegan’s theorem we can assume that $X^n$ is a closed, oriented smooth manifold ($n \geq 3$). What is $N(f)$ geometrically?

**Theorem**

*(Jiang Boju 1981)* If $X^n$ as above is a connected closed smooth manifold ($n \geq 3$), then $N(f)$ is the minimum number of fixed points of any selfmap of $X^n$ homotopic to $f$.

**Theorem**

*(Jiang Boju 1993)* The above theorem fails for $n = 2$.

Nearly there!
Lemma

*(PL version: Schirmer 1976)* Any fixed-point-free selfmap of a connected closed smooth manifold $X^n$ ($n \geq 3$) $\simeq$ smooth map with unique fixed point.
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Lemma

(Bass $\leq$ 1983) The Bass conjecture is true for every group if it is true for every finitely presented group.

Theorem

The Bass conjecture is true for every group iff every homotopy idempotent selfmap of a connected closed smooth manifold ($n \geq 3$) $\simeq$ smooth map with unique fixed point.
Using Eckmann 2001,

**Theorem**

The Bass conjecture is true for the fundamental group of any connected closed smooth 3-manifold.

**Corollary**

Every homotopy idempotent selfmap of a connected closed smooth 3-manifold is homotopic to a smooth selfmap with a unique fixed point.
Surprisingly, \( \dim = 3 \) is the best-behaved for this phenomenon!
For \( F \) a closed surface of negative Euler characteristic and \( k \geq 2 \), Kelly 2005 constructs a homotopy idempotent selfmap \( f_k : F \to F \) such that every map homotopic to \( f_k \) has at least \( k \) fixed points. But fundamental groups of surfaces satisfy the Bass conjecture.
Surprisingly, dim = 3 is the best-behaved for this phenomenon!
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Using other results of Eckmann 2001,

**Corollary**

*Any homotopy idempotent selfmap of a non-positively curved, oriented closed manifold ($n \geq 3$) \( \simeq \) map with unique fixed point.*