# Intertwining matrices for ideal class groups and $K$-theory

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- M. Karoubi & T. Lambre: Quelques classes caractéristiques en théorie des nombres, *J. reine angew. Math.* **543** (2002), 169-186.
- A. J. Berrick: Intertwiners and the $K$-theory of commutative rings, *J. reine angew. Math.* **569** (2004), 55–101.
- A. J. Berrick & M.-F. Lim: Intertwining matrices for number fields: supplement to "Intertwiners and the $K$-theory of commutative rings", *J. reine angew. Math.*, to appear.

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Number field = finite field extension of $\mathbb{Q}$.
Its subring of algebraic integers is a Dedekind domain $A$:
for any two nonzero ideals $\mathfrak{a} \subseteq \mathfrak{b}$ of $A$, $\exists!$ ideal $\mathfrak{c}$ of $A$ such that

$$\mathfrak{a} = \mathfrak{b}\mathfrak{c}.$$

Then the $A$-module isomorphism classes $[\mathfrak{a}]$ of nonzero ideals are called *ideal classes*.

The ideal classes form an (additive) group under multiplication of ideals, the *ideal class group* $\mathrm{C}\ell(A)$ of $A$, with zero element the class $[xA]$ of any principal ideal $xA$.

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$A$ number ring $\Rightarrow \mathrm{C}\ell(A)$ finite abelian group.
Key topic in algebraic number theory – Kummer (1850, for *e.g.* Fermat conjecture), Dedekind (1871).

Open questions... *e.g.* For real quadratic fields, how many class groups are zero?

**Example.** $\mathrm{C}\ell(A_1 = \mathbb{Z}[\sqrt{-5}\,]) = \mathbb{Z}/2$ generated by *e.g.* ideal $A_1(2+\sqrt{-5}, 3)$, with $(A_1(2+\sqrt{-5}, 3))^2 = A_1(2+\sqrt{-5})$.
Easy to check:
$u_1 = 2+\sqrt{-5}$ is not associate to a square in $A_1$.

## Lemma

*In any Dedekind domain $A$, any ideal $\mathfrak{a}$ with $[\mathfrak{a}]$ of order $n$ in $\mathrm{C}\ell(A)$ has $u, b \in \mathfrak{a}$ such that $\mathfrak{a} = A(u, b)$ with*
*(a)*

$$(A(u, b))^n = A(u),$$

*and*
*(b) $u$ not associate in $A$ to a proper power.*

**Generalization** to $K$-theory.
For more general ring $R$,
$K_0(R)$ = group completion of monoid of isomorphism classes of f.g. projective $R$-modules under direct sum.
For $A$ Dedekind, $K_0(A) \cong \mathbb{Z} \oplus \mathrm{C}\ell(A)$.

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Quillen (1970): $i \geq 1$ $K_i(R) = \varinjlim \pi_i((B\mathrm{GL}_n R)^+)$.

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II. Quillen's definition gives all $K$-groups except the most-studied!

**Aim.** Remedy these defects, by a definition of $K_0(A)$ ($A$ commutative) using matrices that extends to a definition of all $K$-groups.

Here, mostly discuss I.

For a matrix $S$ over $A$, write

$$A\langle S\rangle = \text{ideal of } A \text{ generated by the entries of } S.$$

**Example.**

$$\mathrm{C}\ell(A_1 = \mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}/2 = [\langle \mathfrak{a}\rangle]$$

where $\mathfrak{a} = A_1(2+\sqrt{-5},\, 3)$,with

$$(A_1(2+\sqrt{-5},\, 3))^2 = A_1(2+\sqrt{-5}).$$

Clearly

$$S_1 = \begin{bmatrix} 2+\sqrt{-5} & -3 \\ 3 & -1+\sqrt{-5} \end{bmatrix}$$

has $A_1 \langle S_1 \rangle = \mathfrak{a}$.

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Also,

$$\det S_1 = u_1 = 2 + \sqrt{-5} \notin U(A_1),$$

so $S_1 \notin \mathrm{GL}_2(A_1)$.

Thus,

$$(A_1 \langle S_1 \rangle)^2 = A_1(\det S_1).$$

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### Theorem

*[ Ber] For a commutative ring $A$ and non-zerodivisor $S \in M_n(A)$,*

$$(A \langle S \rangle)^n = A(\det S) \quad \iff \quad S \cdot M_n(A) = M_n(A) \cdot S.$$

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$$\begin{array}{ccc} A^n & \xrightarrow{M^S} & A^n \\ \downarrow^{S} & & \downarrow^{S} \\ A^n & \xrightarrow{M} & A^n \end{array}$$

Call such an $S$ an *intertwining matrix*; write $S \in \mathrm{Int}_n(A)$. Clearly $\mathrm{Int}_n(A)$ monoid with

$$A^\times \cdot \mathrm{GL}_n(A) \subseteq \mathrm{Int}_n(A) \subseteq M_n(A)$$

where $A^\times \cdot$ refers to multiplication by a non-zerodivisor scalar (matrix).

$$(A\langle S\rangle)^n = A(\det S) \quad \Longrightarrow \quad n[A\langle S\rangle] = [A(\det S)] = 0 \in \mathrm{C}\ell(A).$$

However, for $S \in A^\times \cdot \mathrm{GL}_n(A)$, already $[A\langle S\rangle] = 0 \in \mathrm{C}\ell(A)$.

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*(a) The quotient* $\mathrm{Int}_n(A)/\left(A^\times \cdot \mathrm{GL}_n(A)\right)$ *has a well-defined monoid structure (from matrix multiplication) that is in fact an abelian group.*

*(b) For $A$ Dedekind,*

$$\begin{aligned} \mathrm{Int}_n(A)/\left(A^\times \cdot \mathrm{GL}_n(A)\right) &\longrightarrow {}_n\mathrm{C}\ell(A) \\ S &\longmapsto [A\langle S\rangle] \end{aligned}$$

*is an isomorphism onto the $n$-torsion subgroup of* $\mathrm{C}\ell(A)$.

## Example.

$$S_1^2 = (2+\sqrt{-5})\begin{bmatrix} 2\sqrt{-5} & -4-\sqrt{-5} \\ 4+\sqrt{-5} & -4+\sqrt{-5} \end{bmatrix} \in A_1^\times \cdot \mathrm{GL}_2(A_1).$$

So

$$S_1^2 = I \in \mathrm{Int}_2(A_1)/\left(A_1^\times \cdot \mathrm{GL}_2(A_1)\right).$$

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So

$$S_1^2 = I \in \mathrm{Int}_2(A_1)/\left(A_1^\times \cdot \mathrm{GL}_2(A_1)\right).$$

However, because $u_1 = 2+\sqrt{-5} = \det S_1$ is not associate to a square in $A_1$,

$$S_1 \neq I \in \mathrm{Int}_2(A_1)/\left(A_1^\times \cdot \mathrm{GL}_2(A_1)\right).$$

Hence $S_1 \in \mathrm{Int}_2(A_1)$ represents the nontrivial element of $\mathrm{C}\ell(A_1)$.

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For a number field $F$ with ring of integers $A$, let $l = [F : \mathbb{Q}]$, and for $u \in A$ let the characteristic equation for $u$ over $F$ be

$$\sum_{j=0}^{l} (-1)^{l-j} N_j(u) u^j = 0. \qquad (*)$$

for some $N_0(u), \ldots, N_l(u) \in \mathbb{Z}$. (In fact, $N_0(u)$ is the norm of $u$.)

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for some $N_0(u), \ldots, N_l(u) \in \mathbb{Z}$. (In fact, $N_0(u)$ is the norm of $u$.) [ KarLam] showed how to use this equation to find elements that detect nontrivial elements in the class group. However, their method uses the Dennis trace map in $K$-theory mod $n$, not matrices.

Conditions (i),(ii) below are due to [ KarLam]. $A$ be a number ring.

## Theorem

*[ BerLim] (a) Suppose that there exists $u \in A$ such that:*

*(i) the integers $N_0(u)$, $N_1(u)$ are coprime; and*

*(ii) $|N_0(u)|$ is a proper power (say, $|N_0(u)| = b^n$).*

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*Then we exhibit an $n \times n$ matrix $S$ that is an intertwining matrix of determinant $u$.*

*(b) The element represented by $S$ in the ideal class group has order $n$ if and only if $u$ is not associate in $A$ to a proper power (of exponent dividing $n$).*

The $n \times n$ matrix $S$ is given by

$$\begin{bmatrix} u & -b \\ xb & yb + zu \end{bmatrix}$$

when $n = 2$; or, for $n > 2$:

$$S = \begin{bmatrix} u & -b & 0 & \cdots & \cdots & 0 \\ 0 & u & -b & \ddots & \ddots & \vdots \\ \vdots & 0 & u & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \vdots & \ddots & u & -b \\ xb & yb & 0 & \cdots & 0 & zu \end{bmatrix}$$

(where $x, y, z \in A$ are constructible from the proof).

**Example 1.** $u_1 = 2 + \sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ has characteristic equation

$$u^2 - 4u + 9 = 0$$

so $N_0(u_1) = 3^2$ and $N_1(u_1) = 4$. So (a) holds with $b_1 = 3$, and we already know that (b) holds.

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The algorithm gives $x = 1$, $y = -1$, $z = 1$, and so

$$S_1 = \begin{bmatrix} u_1 & -b_1 \\ b_1 & -b_1 + u_1 \end{bmatrix} = \begin{bmatrix} 2 + \sqrt{-5} & -3 \\ 3 & -1 + \sqrt{-5} \end{bmatrix}.$$

## Example 2.

$$u_2 = 2 + \sqrt{-23} \in A_2 = \mathbb{Z}[1/2 + \sqrt{-23}/2]$$

has characteristic equation

$$u^2 - 4u + 27 = 0$$

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The algorithm gives $x = 4u_2 - 2$, $y = 1 - 2u_2$, $z = 4$, and so

$$S_2 = \begin{bmatrix} u_2 & -b_2 & 0 \\ 0 & u_2 & -b_2 \\ (4u_2 - 2)b_2 & (1 - 2u_2)b_2 & 4u_2 \end{bmatrix} = \begin{bmatrix} 2 + \sqrt{-23} & -3 & 0 \\ 0 & 2 + \sqrt{-23} & -3 \\ 18 + 12\sqrt{-23} & -9 - 6\sqrt{-23} & 8 + 4\sqrt{-23} \end{bmatrix}$$

Therefore $[A_2 \langle S_2 \rangle]$ has order 3 in $\mathrm{C}\ell(A_2) = \mathbb{Z}/3$.

II. Extension of Quillen's definition to encompass $K_0$ information.

Theorem

*[ Ber] Inclusion induces a natural map*

$$\pi_i(B\,\mathrm{GL}_n\,A^+) \longrightarrow \pi_i(B\,\mathrm{Int}_n\,A^+)$$

*which is an isomorphism for $i \geq 2$.*

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Theorem

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*which is an isomorphism for* $i \geq 2$.
*For* $i = 1$, *this map is a monomorphism;*
*and its abelian cokernel is given by a group extension*

$$\mathrm{Prin}\,A \rightarrowtail \mathrm{Cokernel} \twoheadrightarrow \mathrm{Int}_n\,A/\left(A^\times \cdot \mathrm{GL}_n\,A\right),$$

*where* $\mathrm{Prin}\,A =$ *group* $(A^\times)^{-1}A^\times/U(A)$ *of principal fractional ideals of* $A$,
*and the quotient group is naturally isomorphic to a subgroup of* $K_0(A)$ *of exponent* $n$.

We've already seen that when the commutative ring $A$ is a Dedekind domain,

$$\mathrm{Int}_n A / \left(A^\times \cdot \mathrm{GL}_n A\right) \cong {}_n\mathrm{C}\ell(A).$$

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Intertwining matrices give:

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Intertwining matrices give:

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- a means of defining higher $K$-theory that includes $K_0$ information