Since not all the matrices appearing in [1] Proposition 10.4 are intertwining matrices as claimed, we present a corrected version here. Moreover, with only a little extra work, we then generalize the result from quadratic to arbitrary number fields.

[1] Proposition 10.4. Let \( A \) be the ring of integers in the quadratic number field \( \mathbb{Q}(\sqrt{\delta}) \) of discriminant \( \delta \) \((\neq -3, -4)\). Suppose that there exist \( \alpha, b, n \in \mathbb{Z} \) \((b^2 > 1, \ n \geq 3)\) such that
\[
\delta = \alpha^2 - 4b^n \quad \text{and} \quad (\alpha, b) = 1.
\]

Thus we may define integers \( x_1, x_2, \ldots \) coprime to \( b \) by
\[
x_1 = 1, \ x_2 = \alpha, \ldots \ x_{k+1} = \alpha x_k - b^n x_{k-1}, \ldots
\]
and choose \( r_0, r_{n-1} \in \mathbb{Z} \) to satisfy
\[
r_0 x_n + r_{n-1} b^{n-1} = 1.
\]

Then, with \( u = (\alpha + \sqrt{\delta})/2 \),

(a) the matrix
\[
S_n = \begin{bmatrix}
    u & -b & 0 & \cdots & 0 & 0 \\
    0 & u & -b & \cdots & 0 & 0 \\
    0 & 0 & u & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & u & -b \\
    r_0 x_{n-1} b & r_{n-1} b & 0 & \cdots & 0 & r_0 u
\end{bmatrix}
\]
is an element of \( \text{Int}_n A \); and

(b) if further \( n \) is an odd integer, then the element of the ideal class group of \( A \) represented by \( S_n \) has order \( n \), provided that either:

(i) for every integer \( \beta \) and proper divisor \( m \) of \( n \) the number \( \delta \beta^2 \pm 4b^m \) is not a perfect square; or

(ii) \( n | \delta \), and that moreover when \( \delta > 0 \) then the fundamental unit \( (\varepsilon_1 + \varepsilon_2 \sqrt{\delta})/2 \) of \( A \) has \( n | \varepsilon_2 \).

Proof. (a) From \( u(\alpha - u) = (\alpha^2 - \delta)/4 = b^n \), we obtain inductively that
\[
u^k = x_k u - x_{k-1} b^n.
\]
That $S_n$ is an intertwiner with determinant $u$ follows readily from the proof of [1] Theorem 9.1 (where $r_1 = r_2 = \cdots = r_{n-2} = 0$).

(b) In the specified cases it is shown in [2] Propositions 1.9, 2.8 that the class obtained in (a) maps under the isomorphism of our Corollary 9.6 (b) to an element of order $n$.

For determination of the order of the element of the ideal class group represented by an intertwining matrix, the following result is useful.

**Lemma 1.** Let $A$ be a Dedekind domain, and suppose that $S \in \text{Int}_n(A)$. Let $r$ be the greatest integer such that $\det(S)$ is associate in $A$ to an $r$th power. Then the order of the element represented by $S$ in the ideal class group of $A$ is $n/r$.

**Proof.** Writing $u = \det(S)$, we deduce from [1] Theorem 3.2 that $Au = (A \langle S \rangle)^n$. By definition, the class of the ideal $A \langle S \rangle$ generated by the entries of $S$ has order dividing $s$ in the class group if and only if for some $w \in A$ we have $(A \langle S \rangle)^s = Aw$. By unique divisibility of ideals, this is equivalent to $Au = Aw^r$, where $n = rs$.

In order to generalize the proposition above, we recall the following notation from [2]. For a number field $F$ with ring of integers $A$, let $l = [F : \mathbb{Q}]$, and for $u \in A$ let the characteristic equation for $u$ over $F$ be

$$(*) \quad \sum_{j=0}^{l} (-1)^{l-j}N_j(u)u^j = 0.$$ 

for some $N_0(u), \ldots, N_l(u) \in \mathbb{Z}$. (In fact, $N_0(u)$ is the norm of $u$.)

**Theorem 2.** Let $A$ be a number ring.

(a) Suppose that there exists $u \in A$ such that:

(i) the integers $N_0(u), N_1(u)$ are coprime; and

(ii) $|N_0(u)|$ is a proper power (say, $|N_0(u)| = b^n$).

Then the $n \times n$ matrix $S$ given by

$$S = \begin{bmatrix} u & -b & 0 & \cdots & \cdots & 0 \\ 0 & u & -b & \ddots & \ddots & \vdots \\ \vdots & 0 & u & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & u & -b \\ xb & yb & 0 & \cdots & 0 & zu \end{bmatrix}$$

(when $n = 2$; or, for $n > 2$:

$$S = \begin{bmatrix} u & -b & 0 & \cdots & \cdots & \cdots \\ 0 & u & -b & \ddots & \ddots & \vdots \\ \vdots & 0 & u & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & u & -b \\ xb & yb & 0 & \cdots & 0 & zu \end{bmatrix}$$

(where $x, y, z \in A$ are constructible from the proof) is an intertwining matrix of determinant $u$.}
(b) The element represented by \( S \) in the ideal class group has order \( n \) if and only if \( u \) is not associate in \( A \) to a proper power (of exponent dividing \( n \)).

Proof. (a) From hypotheses (i), (ii), we have, as ideals in \( A \),

\[ A = Ab^n - 1 + AN_1(u), \]

while from (\( \ast \)) and (i)

\[ AN_1(u)u \subseteq Ab^n + Au. \]

Then multiplying the former equality by \( u \) and combining gives

\[ Au \subseteq Ab^n + Ab^{n-1}u + Au. \]

Iteration yields

\[ Au \subseteq Ab^n + Ab^{n-1}u + Au^k \]

for all \( k \geq 2 \); and so, there are \( x, y, z \in A \) with

\[ u = xb^n + yb^{n-1}u + zu^n. \]

Thus \( S \) has determinant \( u \), and that \( S \) is an intertwining matrix again follows from the proof of [1] Theorem 9.1.

(b) This is immediate from the lemma.

Note that the necessary and sufficient condition in (b) of the above theorem weakens that found in [2] (1.8) (namely, that no lesser power of \( b \) than \( b^n \) be the norm of an element). Thus, in \( \mathbb{Q}(\sqrt{-231}) \) for instance, the above theorem detects a nontrivial class of order 3 with \( u = (5 + \sqrt{-231})/2 \), of norm \( 4^3 \); in \( \mathbb{Q}(\sqrt{-455}) \) the theorem detects a nontrivial class of order 5 with \( u = (1 + 3\sqrt{-455})/2 \), of norm \( 4^5 \). However, in both cases, \( b = 4 = N(2) \).

References


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