

# ON EQUIVARIANT MAPS AND IMMERSIONS OF REAL PROJECTIVE SPACES

by

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Let  $V_k(\mathbf{R}^m)$  denote the Stiefel manifold of orthonormal  $k$ -frames in Euclidean  $m$ -space. The antipodal involution  $\sigma$  on  $V_k(\mathbf{R}^m)$  given by  $\sigma(v_1, \dots, v_k) = (-v_1, \dots, -v_k)$  generalizes the antipodal map on spheres. A classical construction [3] establishes a bijection between the set of maps  $f: S^n \rightarrow V_k(\mathbf{R}^m)$  equivariant with respect to  $\sigma$  and the set of sections in the bundle of  $k$ -frames associated to  $m\xi_n$  where  $\xi_n$  denotes the Hopf line bundle over  $n$ -dimensional real projective space  $P^n$ . In particular,  $P^n$  immerses differentiably in  $\mathbf{R}^{n+k}$  for  $k > 0$  if and only if there exists a map  $f: S^n \rightarrow V_{n+1}(\mathbf{R}^{n+k+1})$  equivariant with respect to  $\sigma$ . The homotopy class of an equivariant map  $f: S^n \rightarrow V_k(\mathbf{R}^m)$  yields an obstruction in a certain subquotient  $\pi_{n,(m,k)}$  of  $\pi_n(V_k(\mathbf{R}^m))$  to the existence of an equivariant map  $g: S^{n+1} \rightarrow V_k(\mathbf{R}^m)$  which coincides with  $f$  on  $S^{n-1}$ . The first author determined  $\pi_{n,(m,k)}$  explicitly in [1]. In this note we interpret  $\pi_{n,(m,k)}$  in terms of indices of vector bundle monomorphisms with finite singularities and use it to derive some immersions of projective spaces in Euclidean space. We also consider the representability of the stable homotopy of spheres by immersed projective spaces.

The singularity method developed extensively by Koschorke relates the indices of vector bundle monomorphisms with finite singularities over manifolds to the normal bordism obstruction to the existence of a vector bundle *monomorphism*. (See Chapter 13 of [7].) Propositions 1.2 and 1.5 of this paper present not merely alternative proofs for results in [1], but also make important identifications with equivariant maps for applications of the singularity method to the immersion problem for projective spaces.

## 1. Equivariant Maps and Indices

Let  $\mathcal{R}^k$  denote the trivial  $k$ -dimensional vector bundle. We shall first show how to make an identification between (i) indices of vector bundle monomorphisms  $\mathcal{R}^k \rightarrow m\xi_n$  with finite singularities and (ii) homotopy classes containing equivariant maps  $S^{n-1} \rightarrow V_k(\mathbf{R}^m)$ .

Let  $W_k(m\xi_n)$  denote the bundle of nonsingular  $k$ -frames associated to  $m\xi_n$  with fiber  $W_k(\mathbf{R}^m)$ , the Stiefel manifold of nonsingular  $k$ -frames in  $\mathbf{R}^m$ . Let  $x_0$  denote the image of  $\pm e_{n+1}$  under the double covering  $S^n \rightarrow P^n$ . Given an equivariant map  $f: S^{n-1} \rightarrow V_k(\mathbf{R}^m)$ , we shall define a section  $s_f$  of the bundle of nonsingular  $k$ -frames associated to  $m\xi_n$  over  $P^n - \{x_0\}$  such that the index of

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1980 Mathematics Subject Classification: Primary 57R42, 55R25.

The second author was supported by *FINEP* and *CNPq* of Brazil.

$s_f$ , and  $s_f$ , in  $\pi_{n-1}(V_k(\mathbf{R}^m))$  is  $[f]$ . We first define  $g: S^n - \{\pm e_{n+1}\} \rightarrow W_k(\mathbf{R}^m)$  as follows. Given  $(x, t)$  in  $S^n \subset \mathbf{R}^n \times \mathbf{R}^1$ , set

$$(1.1) \quad g(x, t) = \begin{cases} (1-t)f\left(\frac{x}{\|x\|}\right) & \text{for } 0 \leq t < 1 \\ (1+t)f\left(\frac{x}{\|x\|}\right) & \text{for } -1 < t \leq 0 \end{cases}$$

Clearly  $g$  extends  $f$  on  $S^{n-1}$  and  $g(-x, -t) = -g(x, t)$ . We define the section

$$s_f: P^n - \{x_0\} \rightarrow S^n \times_{\mathbf{Z}/2} W_k(\mathbf{R}^m)$$

by  $s_f([x, t]) = [(x, t), g(x, t)] = [(-x, -t), g(-x, -t)]$ . Now  $\text{ind } s_f$  in  $\pi_{n-1}(V_k(\mathbf{R}^m))$  is just the local obstruction at  $x_0$ , which is clearly  $[f]$ . (We identify the homotopy of  $V_k(\mathbf{R}^m)$  and  $W_k(\mathbf{R}^m)$  via the inclusion and an orthonormalization map  $\Gamma: W_k(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$ .)

Suppose that  $\alpha$  in  $\pi_{n-1}(V_k(\mathbf{R}^m))$  occurs as the index for some bundle monomorphism  $\mathcal{R}^k \rightarrow m\xi_n$  with finite singularities. We may choose a section with a unique singularity

$$s: P^n - \{x_0\} \rightarrow S^n \times_{\mathbf{Z}/2} W_k(\mathbf{R}^m)$$

whose index is  $\alpha$ . Now  $s$  determines an equivariant map  $g: S^n - \{\pm e_{n+1}\} \rightarrow W_k(\mathbf{R}^m)$  by  $s([x, t]) = [(x, t), g(x, t)]$ . Further, the index of  $s$  is  $[\Gamma \circ g]$  with orientations properly chosen. Thus we have proved the following.

**PROPOSITION 1.2.** *The elements of  $\pi_{n-1}(V_k(\mathbf{R}^m))$  which occur as indices of vector bundle monomorphisms  $\mathcal{R}^k \rightarrow m\xi_n$  with finite singularities are precisely those elements represented by equivariant maps.*

We shall now derive Lemma 6.5 of [1]. Given an equivariant map  $f: S^{n-1} \rightarrow V_k(\mathbf{R}^m)$ , the map  $g$  in (1.1) yields a section

$$s_g: S^n - \{\pm e_{n+1}\} \rightarrow S^n \times W_k(\mathbf{R}^m)$$

of the bundle  $W_k(\mathbf{R}^m)$  on  $S^n$ . Clearly the local obstruction at  $e_{n+1}$  is  $[f]$ . Now the local obstruction at  $-e_{n+1}$  can be expressed in terms of the index at  $e_{n+1}$  by  $[\sigma \circ g \circ A] = (-1)^{n+1} \sigma_*[f]$  where  $A$  is the antipodal map on  $S^n$ . So  $\text{ind } s_g = [f] + (-1)^{n+1} \sigma_*[f]$ . But  $\text{ind } s_g = 0$  by uniqueness of indices on  $S^n$  for the trivial bundle  $\mathcal{R}^m$ . So we conclude that  $[f]$  belongs to

$$(1.3) \quad \text{Ker}(1 - \sigma_*) \text{ for } n \text{ even and } \text{Ker}(1 + \sigma_*) \text{ for } n \text{ odd.}$$

We recall from [6] that the canonical automorphism  $\mu$  of the Stiefel manifolds comprises the involutions which change the sign of any fixed row of an  $(m \times k)$ -matrix. Now  $\sigma = \mu^m$  and  $\mu_* = 1 - \psi$  on  $\pi_{n-1}(V_k(\mathbf{R}^m))$  where  $\psi$  denotes the composite morphism  $i_* \circ \Sigma \circ \partial$

$$\pi_{n-1} V_k(\mathbf{R}^m) \xrightarrow{\partial} \pi_{n-2}(S^{m-k-1}) \xrightarrow{\Sigma} \pi_{n-1}(S^{m-k}) \xrightarrow{i_*} \pi_{n-1} V_k(\mathbf{R}^m) .$$

Here  $\partial$  denotes the boundary operator for the fibration  $V_{k+1}(\mathbf{R}^m) \rightarrow V_k(\mathbf{R}^m)$ ,  $\Sigma$  denotes the suspension morphism, and  $i$  denotes the fiber inclusion in the fibration  $S^{m-k} \subset V_k(\mathbf{R}^m) \rightarrow V_{k-1}(\mathbf{R}^m)$ .

**LEMMA 1.4.** *The indeterminacy subgroup in  $\pi_{n-1}(V_k(\mathbf{R}^m))$  for indices associated to vector bundle monomorphisms  $\mathcal{R}^k \rightarrow m\xi_n$  with finite singularities is given by*

<i>n even</i>		<i>n odd</i>
$\text{Im}(2 - \psi)$	<i>mk odd</i>	$\text{Im } \psi$
$2\pi_{n-1}(V_k(\mathbf{R}^m))$	<i>mk even</i>	$0$

*Proof.* As an obstruction cohomology class, the index of a bundle monomorphism  $\mathcal{R}^k \rightarrow m\xi_n$  with finite singularities belongs to  $H^n(P^n; \pi_{n-1}(V_k(\mathbf{R}^m))_{w_1})$ . The local coefficient system  $\pi_{n-1}(V_k(\mathbf{R}^m))_{w_1}$  is twisted if and only if  $w_1(m\xi_n)$  is nonzero, that is, if and only if  $m$  is odd. Every orientation-reversing loop in the base of a nonorientable  $m$ -plane bundle acts by  $\mu_*$  on  $\pi_*(V_k(\mathbf{R}^m))$  in the associated bundle of  $k$ -frames, since the action of the generator of  $\pi_1(BO(m))$  on the homotopy of the fiber in the fibration  $V_k(\mathbf{R}^m) \subset BO(m-k) \rightarrow BO(m)$  is induced through matrix multiplication on the left by a non-rotation [11, p.306]. Let  $\alpha$  generate  $\pi_1(P^n)$ . Thus  $\alpha_*(\beta) = \mu_*(\beta)$  for all  $\beta$  in  $\pi_{n-1}(V_k(\mathbf{R}^m))$  for  $m$  odd while  $\alpha_*(\beta) = \beta$  for  $m$  even. Let  $Z_{w_1(P^n)}$  denote the local coefficient system of integers twisted by  $w_1(P^n)$ . That is,  $\alpha_*(1) = \mu_*(1) = -1$  for  $n$  even while  $\alpha_*(1) = 1$  for  $n$  odd. By Poincaré duality and evaluation [11, p.275] we obtain the isomorphisms

$$\begin{aligned} H^n(P^n; \pi_{n-1}(V_k(\mathbf{R}^m))_{w_1}) &\cong H_0(P^n; \pi_{n-1}(V_k(\mathbf{R}^m))_{w_1} \otimes Z_{w_1(P^n)}) \\ &\cong \pi_{n-1}(V_k(\mathbf{R}^m)) \otimes \mathbf{Z}/G. \end{aligned}$$

The indeterminacy subgroup  $G$  for indices is the subgroup generated by  $\beta \otimes 1 - \alpha_*(\beta \otimes 1)$  for all  $\beta$  in  $\pi_{n-1}(V_k(\mathbf{R}^m))$ . The action of  $\alpha$  on  $\beta \otimes 1$  is given by

$$\begin{array}{ll} \mu_*(\beta) \otimes -1 & \text{for } m \text{ odd, } n \text{ even} & \beta \otimes -1 & \text{for } m \text{ even, } n \text{ even} \\ \mu_*(\beta) \otimes 1 & \text{for } m \text{ odd, } n \text{ odd} & \beta \otimes 1 & \text{for } m \text{ even, } n \text{ odd} \end{array}$$

Thus the indeterminacy subgroup  $G$  is given by  $2\pi_{n-1}(V_k(\mathbf{R}^m))$  for  $m$  and  $n$  even, and is trivial for  $m$  even and  $n$  odd. For  $m$  odd and  $k$  even, the morphism  $1 + \mu_*$  is multiplication by 2 while  $1 - \mu_*$  is trivial. For  $mk$  odd,  $1 + \mu_* = 2 - \psi$  and  $1 - \mu_* = \psi$ . The result follows.

The proof of the following proposition follows immediately from Proposition 1.2, Lemma 1.4 and (1.3). We emphasize the identification of the classical obstruction group for this problem with

the given subquotients of homotopy groups of Stiefel manifolds in our proof of this result, which is essentially Lemma 6.5 of [1].

**PROPOSITION 1.5** (Berrick [1]). *Let  $s: \mathcal{R}^k \rightarrow m\xi_n$  denote any vector bundle monomorphism with finite singularities on  $P^n$ . Evaluation of the associated classical cohomology obstruction on the fundamental homology class of  $P^n$  (twisted integers for  $n$  even) yields an element  $\text{Ind } s$  in the vector space  $\pi_{n-1, (m, k)}$  over  $\mathbb{Z}/2$  defined below.*

$$\begin{array}{ccc} n-1 \text{ odd} & & n-1 \text{ even} \\ \text{Ker } \psi / \text{Im}(2 - \psi) & mk \text{ odd} & \text{Ker}(2 - \psi) / \text{Im} \psi \\ \pi_{n-1}(V_k(\mathbb{R}^m)) / 2 & mk \text{ even} & \text{Ker}(\bullet 2: \pi_{n-1}(V_k(\mathbb{R}^m))) \supset). \end{array}$$

Now  $\text{Ind } s = 0$  in  $\pi_{n-1, (m, k)}$  if and only if there exists a bundle monomorphism  $\mathcal{R}^k \rightarrow m\xi_n$  which agrees with  $s$  on  $P^{n-2}$ . Equivalently, given any equivariant map  $f: S^{n-1} \rightarrow V_k(\mathbb{R}^m)$ , there exists an equivariant map  $g: S^n \rightarrow V_k(\mathbb{R}^m)$  which agrees with  $f$  on  $S^{n-2}$  if and only if  $[f]$  projects trivially into  $\pi_{n-1, (m, k)}$ .

## 2. Generalised J-Morphism and Representability

In this section we consider briefly the representability of elements of the stable homotopy of spheres through immersions of projective spaces in Euclidean space. We recall from [2] that  $\alpha$  in  $\pi_k^*$  is represented by an immersion  $P^n \subseteq \mathbb{R}^{2n-k}$  if  $\alpha = J([f])$  where  $f: S^n \rightarrow V_{n+1}(\mathbb{R}^{2n-k+1})$  denotes the Smale invariant of the immersion and  $J: \pi_n(V_{n+1}(\mathbb{R}^{2n-k+1})) \rightarrow \pi_{2n+1}(S^{2n-k+1})$  denotes the generalized  $J$ -morphism. The following lemma is an immediate generalization of Proposition 1.1. of [10] and so the proof is omitted. The condition  $a+b < 2c$ , which seems necessary in (1.1) of [10] also, insures that a self-map of degree  $-1$  on  $S^c$  will induce multiplication by  $-1$  on  $\pi_{a+b-1}(S^c)$ .

**LEMMA 2.1.** *Let  $H(f): S^{a+b-1} \rightarrow S^c$  denote the Hopf construction on a nonsingular biskew map  $f: \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^c$ . If  $a+b < 2c$  and either  $a+c$  or  $b+c$  is odd, then  $[H(f)]$  has order at most 2.*

The following result gives us the opportunity to correct a parity error in Theorem 4(b)(ii) of [2], and at the same time clarify the argument used there.

**THEOREM 2.2.** *Suppose  $\alpha$  in  $\pi_k^*$  is represented by some immersion  $P^n \subseteq \mathbb{R}^{2n-k}$ . If either  $n+k$  is odd or else the immersion is self-adjoint with  $nk$  odd, then  $2\alpha = 0$ .*

*Proof.* By (1.5) the Smale invariant of the immersion has order  $\leq 2$  in  $\pi_n(V_{n+1}(\mathbb{R}^{2n-k+1}))$  for  $n$  even and  $k$  odd. The result for  $n$  odd and  $k$  even follows by Lemma 2.1, since the Smale invariant yields a nonsingular biskew map  $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n-k+1}$ .

Next suppose  $nk$  odd and the immersion self-adjoint. Now the Smale invariant  $f: S^n \rightarrow V_{n+1}(\mathbb{R}^{2n-k+1})$  produces a biskew map  $\hat{f}: S^n \times S^n \rightarrow S^{2n-k}$  such that  $\alpha = J([f])$  is represented by the Hopf construction  $H(\hat{f})$ . Consider the following diagram.

$$\begin{array}{ccc} S^n * S^n & \xrightarrow{\bar{T}} & S^n * S^n \\ \downarrow H(\hat{f}) & & \downarrow H(\hat{f}) \\ S^{2n-k+1} & \xrightarrow{r_1} & S^{2n-k+1} \end{array}$$

For  $x$  and  $y$  in  $S^n$  and  $\theta \in [0, \pi/2]$ ,  $\bar{T}(x \cos \theta, y \sin \theta) = (y \sin \theta, x \cos \theta)$  while  $H(\hat{f})(x \cos \theta, y \sin \theta) = (\cos 2\theta, \hat{f}(x, y) \sin 2\theta)$  and  $r_1(z_1, z_2, \dots, z_{2n-k+2}) = (-z_1, z_2, \dots, z_{2n-k+2})$ . The involution  $T$  interchanging factors on  $S^n \times S^n$  is homotopic to  $\hat{f} \circ T$  by the hypothesis of self-adjointness. This makes the above diagram homotopy commutative. The result is immediate, since  $\bar{T}$  and  $r_1$  have degrees  $(-1)^{n+1}$  and  $-1$  respectively.

### 3. Immersing Projective Spaces

**THEOREM 3.1.** *Suppose that  $P^n$  immerses in  $\mathbb{R}^{2n-12}$  for  $n \equiv 4(8)$ . Then  $P^{n+1} \subseteq \mathbb{R}^{2n-12}$ . In addition,  $P^{n+2}$  immerses in  $\mathbb{R}^{2n-11}$  for  $n \equiv 12(16)$ .*

*Proof.* By hypothesis there exists an equivariant map  $f: S^n \rightarrow V_{n+1}(\mathbb{R}^{2n-11})$  with  $n \equiv 4(8)$ . We remark that  $n \geq 60$  from known results in low dimensions. Since the 2-primary components of  $\pi_n(V_{n+1}(\mathbb{R}^{2n-11}))$  and  $\pi_n(V_{n+2}(\mathbb{R}^{2n-11}))$  are trivial by [9], there exists an immersion  $P^{n+1} \subseteq \mathbb{R}^{2n-12}$  by [1, Prop. 6.6].

Suppose now that  $n \equiv 12(16)$ . Since  $\pi_{n+1}(V_{n+2}(\mathbb{R}^{2n-11})) = 0$  by [9], the Smale invariant of any immersion  $P^{n+1} \subseteq \mathbb{R}^{2n-12}$  extends to an equivariant map  $S^{n+2} \rightarrow V_{n+2}(\mathbb{R}^{2n-11})$ . We recall from [5, Theorem 1.2] that an equivariant map  $S^i \rightarrow V_k(\mathbb{R}^m)$  produces an equivariant map  $S^{k-1} \rightarrow V_{i+1}(\mathbb{R}^m)$  provided  $k+2 \leq 2(m-i)$ . Thus we obtain an equivariant map  $g: S^{n+1} \rightarrow V_{n+3}(\mathbb{R}^{2n-11})$ . Finally, the equivariant map  $j \circ g$  where  $j: V_{n+3}(\mathbb{R}^{2n-11}) \hookrightarrow V_{n+3}(\mathbb{R}^{2n-10})$  denotes the natural inclusion extends to an equivariant map  $S^{n+2} \rightarrow V_{n+3}(\mathbb{R}^{2n-10})$  because  $\pi_{n+1}(V_{n+3}(\mathbb{R}^{2n-10})) = 0$  by [9].

Let  $\alpha(n)$  and  $\nu(n)$  denote the number of 1s in the dyadic expansion of  $n$  and the highest power of 2 dividing  $n$  respectively. Now Davis proved in [4] that  $P^n$  immerses in  $\mathbb{R}^{2n-12}$  for  $n \equiv 12(16)$  with  $\alpha(n) \geq 5$  and  $\nu(n+4) < 7$ . Thus we obtain the following.

**COROLLARY 3.2.**  *$P^n \subseteq \mathbb{R}^{2n-14}$  for  $n \equiv 13(16)$  with  $\alpha(n) \geq 6$  and  $\nu(n+3) < 7$ .  $P^n \subseteq \mathbb{R}^{2n-15}$  for  $n \equiv 14(16)$  with  $\alpha(n) \geq 6$  and  $\nu(n+2) < 7$ .*

The above results improve those of [1] by 2 and 1 dimensions respectively. For example,  $P^{189} \subseteq \mathbb{R}^{364}$  by (3.2) whereas  $P^{189} \subseteq \mathbb{R}^{366}$  by [1] and  $P^{189} \not\subseteq \mathbb{R}^{369}$  by [4]. Further  $P^{190} \subseteq \mathbb{R}^{365}$  by (3.2) while  $P^{190} \subseteq \mathbb{R}^{366}$  by [1] and  $P^{190} \not\subseteq \mathbb{R}^{369}$  from the table of [4].

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