Algebraic K-Theory and Algebraic Topology

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(Dedicated to H. Bass on the occasion of his 70th birthday)

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0 Preface

These are lecture notes from a course of lectures at the Algebraic K-Theory and its Applications Summer School at ICTP in July 2002, intended for an audience not necessarily specialist in either algebraic K-theory or algebraic topology. Much more detail about past interaction between these areas can be found in [128] and, more narrowly, [14]. Lectures of Friedlander and Weibel in the Proceedings of the 1997 Algebraic K-theory and its Applications Workshop (World Scientific, Singapore, 1999) are also recommended. These notes have a different emphasis which reflects on the one hand my own prejudices and limitations, and, on the other, the passage of time.

Of course, one has to give due weight to recent, often dramatic, developments, as in the last two lectures. First of all though, we analyse the role of algebraic topology in the modern foundation of the subject. In the first three lectures, we see that the three main definitions of higher K-theory — via the plus-construction, symmetric monoidal categories and the Q-construction – all have at their core a 'machine lemma' (respectively, the fibration preservation theorem, the group completion theorem, and Quillen's Theorem B) that ensures an adequate supply of fibrations, and thence long exact homotopy sequences upon which calculations may be based.

The next pair of lectures is devoted to the contribution of topological Hochschild and cyclic homology. We then give a topologically-oriented review of the current state of knowledge of K-theory of (the, and other) integers. The final lecture discusses recent work on classifying spaces for mapping class and related groups, somewhat in the spirit of the question: Is the plus-construction topology's gift to K-theory, or K-theory's gift to topology?

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1 The plus-construction and higher K

In order to understand the relevance of the plus-construction to K-theory, it helps to recall the development of K_1 from J. H. C. Whitehead's study of simple homotopy types.

1.1 Simple homotopy and elementary matrices

The study of K_1 of a ring developed from J. H. C. Whitehead's attempt [164], [165] to describe a homotopy equivalence of compact spaces as an iteration of elementary collapses and expansions (a map homotopic to such an iteration being called a *simple homotopy equivalence*). Each elementary collapse/expansion is determined by the attaching map of an attaching r-cell. (It was for this purpose that CW-complexes were invented.) The complete discussion takes a book [40], and is too technical to outline here. However, here are some informal remarks.

First, by means of mapping cylinders one manoeuvres into the situation of a finite r-dimensional complex K, with subcomplex L which is a strong deformation retract of K. Moreover, by a geometric normalization argument ('trading cells') one may assume that K is determined from a subcomplex K' by n attaching maps $f_i : S^{r-1} \to K'$, and K' in turn is formed from L by the attachment of n $(r-1)$-cells, via maps $g_i : S^{r-2} \to L$. Now the strong deformation retraction kills the first and last groups in the homotopy exact sequence of the triple (K, K', L):

$$\pi_r(K, L) \to \pi_r(K, K') \xrightarrow{\partial} \pi_{r-1}(K', L) \to \pi_{r-1}(K, L).$$

Thus the connecting homomorphism ∂ is an isomorphism. With $\pi = \pi_1(L)$, and $\mathbb{Z}\pi$ the integral group ring comprising finite formal $\mathbb{Z}$-linear combinations of group elements, this is an isomorphism of $\mathbb{Z}\pi$-modules because the sequence respects the natural action of π on the relative homotopy groups. In the present setup, both $\pi_r(K, K')$ and $\pi_{r-1}(K', L)$ are free $\mathbb{Z}\pi$-modules (isomorphic to the direct sum of n copies of $\mathbb{Z}\pi$), with generating classes represented by the f_i and g_i respectively. Hence ∂ defines an $n \times n$ change-of-basis matrix, an element of the group $\mathrm{GL}_n(\mathbb{Z}\pi)$ of invertible $n \times n$ matrices over the group ring.

What drives the whole theory is the effect of elementary collapses and expansions on this matrix. They correspond variously to products of the following operations:

(i) embedding $\mathrm{GL}_n(\mathbb{Z}\pi)$ in $\mathrm{GL}_{n+1}(\mathbb{Z}\pi)$, by adjunction of 1 in the $(n+1, n+1)$-slot, and 0 elsewhere in the last row and column

$$\alpha \longmapsto \begin{bmatrix} \alpha & O \\ O & 1 \end{bmatrix},$$

iteration of this process being known as *stabilization*;

(ii) multiplication by an elementary matrix $e_{ij}(r)$ (an identity matrix, save for a unique off-diagonal entry r in the (i, j)-slot), corresponding to the operation of adding a multiple of one row or column to another;

(iii) multiplication by an invertible diagonal matrix (diagonal entries comprising elements of π and their negatives, these necessarily being units of $\mathbb{Z}\pi$).

So, to decide whether a given homotopy equivalence is a simple homotopy equivalence one has to determine whether a matrix in $\mathrm{GL}_n(\mathbb{Z}\pi)$ is reducible to the identity matrix by a sequence of the above operations. Now define, for any ring R (associative, with identity 1), the subgroup E_nR of $\mathrm{GL}_n R$ to be that generated by all $n \times n$ elementary matrices. Combining (i) and (ii) above gives the subgroup $ER := \lim_{n\to\infty} E_nR$ of the limit group $\mathrm{GL}R := \lim_{n\to\infty} \mathrm{GL}_n R$.

A key 'elementary' result of J. H. C. Whitehead is the following.

Lemma 1.1 (Whitehead Lemma) [164] *ER is the commutator subgroup of $\mathrm{GL}\,R$.*

Therefore the group $\mathrm{GL}R/ER$ is abelian. Hence the obstruction to a homotopy equivalence being a simple homotopy equivalence (the Whitehead torsion) lies in the abelian group (the *Whitehead group*) $\mathrm{Wh}(\pi)$ obtained from $\mathrm{GL}(\mathbb{Z}\pi)/E(\mathbb{Z}\pi)$ by factoring out the subgroup of diagonal matrices occurring in (iii) above. Conversely, every element of $\mathrm{Wh}(\pi)$ is geometrically realizable as such an obstruction. The calculation of $\mathrm{Wh}(\pi)$, even for finite π, is an active area of research [112].

1.2 Definition of K_1R

One defines

$$K_1R = \mathrm{GL}\,R/ER,$$

which by the Whitehead Lemma is an abelian group for any ring R. Evidently this construction is functorial, for any ring homomorphism $R \to S$

(which we assume sends 1_R to 1_S) maps matrices entry-wise to give ring homomorphisms $M_nR \to M_nS$. Then the group of units GL_nR maps into the group of units GL_nS, and also E_nR maps into E_nS. Thereby we obtain a group homomorphism from $\mathrm{GL}R$ to $\mathrm{GL}S$ sending ER into ES, and hence a homomorphism from K_1R to K_1S. The required functoriality properties concerning preservation of composition and identity maps are clear.

The above discussion makes it clear that the group ER plays a crucial role in this subject. Here is another way of looking at it. Since for distinct i, j, k

$$e_{ij}(r) = [e_{ik}(r), e_{kj}(1)], \tag{1-1}$$

(where $[g,h] = ghg^{-1}h^{-1}$) we have that, for $n \geq 3$, each E_nR is generated by commutators, which is to say, *perfect*. Hence their union ER is also perfect. Now any perfect subgroup of $\mathrm{GL}R$, being generated by commutators, lies in the commutator subgroup $[\mathrm{GL}R, \mathrm{GL}R]$ of $\mathrm{GL}R$.

Corollary 1.2 *ER is the unique maximum perfect subgroup, the* perfect radical $\mathcal{P}\mathrm{GL}R$ *of* $\mathrm{GL}R$.

In general, any group has a perfect radical, the product of all the perfect subgroups, or equally, the intersection of the transfinite derived series of the group.

The observation that ER is perfect can be pushed further. For any ideal I of R, EI is defined to be the subgroup of ER generated by all $e_{ij}(a)$ with $a \in I$, and $E(R,I)$ is the subgroup with generators of the form $\alpha e_{ij}(a)\alpha^{-1}$ where $\alpha \in ER$ and $e_{ij}(a) \in EI$. It is readily shown to be normal in $\mathrm{GL}R$. Now suppose that I is an idempotent ideal. This means that we can write

$$a = \sum_p a'_p a''_p,$$

a finite sum with each $a'_p, a''_p \in I$. So for any k distinct from both i and j, as in Equation 1-1

$$\begin{aligned}\alpha e_{ij}(a)\alpha^{-1} &= \prod_p [\alpha e_{ik}(a'_p)\alpha^{-1}, \alpha e_{kj}(a''_p)\alpha^{-1}] \\ &\in [E(R,I), E(R,I)].\end{aligned}$$

This makes $E(R,I)$ a perfect normal subgroup of ER. With more work one can show the converse, and thereby characterize the perfect normal subgroups of GLR as precisely those $E(R, I)$ with I an idempotent ideal of R [16]. In favourable cases, such as R being a commutative domain, it is known that there are no proper nonzero idempotent ideals, making ER the sole nontrivial perfect normal subgroup of GLR.

One of the defining features of Quillen's plus-construction is that its effect on fundamental groups is to factor out the perfect radical of the original fundamental group. Thus, starting with the classifying space BGLR of the general linear group of a ring R, it provides a space with fundamental group

$$\pi_1(B\mathrm{GL}R)/\mathcal{P}\pi_1(B\mathrm{GL}R) = \mathrm{GL}R/ER = K_1R.$$

Our present aim is to describe the homotopy theory of the plus-construction, used by Quillen to define the higher K-groups of a ring.[1] It has proven to be surprisingly useful in other applications and so has considerable interest in its own right.

In outlining this construction, we work in the category of pointed spaces and basepoint-preserving maps. Whenever we can do so, we choose spaces homotopy equivalent to a connected CW-complex that contains only a finite number of cells of any given dimension. The homotopy type of such a space X is uniquely determined by its homotopy groups $\pi_*(X)$. (Basepoints are a non-variable part of the machinery and therefore dropped from the notation.)

1.3 Acyclic spaces and maps

We begin with some elementary homotopy theory, after Quillen [14], [64].

The condition that $\tilde{H}_*(W) = 0$ (homology and cohomology with trivial integer coefficients unless otherwise stated), that is, that W is *acyclic*, is weaker than contractibility of W. Indeed all it tells us about the fundamental group $\pi = \pi_1(W)$ is that it satisfies $H_1(\pi) = 0$ and $H_2(\pi) = 0$, where we recall that the former condition means that π is perfect, since $H_1(\pi) = \pi/[\pi, \pi]$. In turn, the importance for us of perfect groups comes from the

[1] Dan Quillen and George Whitehead were colleagues at MIT in the early 1970s, so the following incident just might have happened ...

As MIT's Quillen was strolling one day
Along came Kay Whitehead the opposite way.
She called: 'Dan, what's the fuss
About you and your plus?'
He just smiled, shrugged, and said: 'Hiya Kay!'

Whitehead Lemma. Its notation here uses the functor on groups $G \mapsto \mathcal{P}G$, where the *perfect radical* $\mathcal{P}G$ denotes the product of all perfect subgroups of G; thus $\mathcal{P}G$ is necessarily the largest perfect subgroup of G, and normal in G. That it is a functor follows from the observation that a homomorphism necessarily sends a commutator to a commutator.

Both of the homological trivializations above follow from the Serre spectral sequence of the fibration $\tilde{W} \to W \to B\pi$ corresponding to the universal covering space $\tilde{W}$ of W. It gives rise to an exact sequence

$$H_2(W) \to H_2(\pi) \to H_0(B\pi;\, H_1(\tilde{W})) \to H_1(W) \to H_1(\pi) \to 0,$$

in which $H_1(\tilde{W})$ is the abelianization of the trivial group $\pi_1(\tilde{W})$, making all terms zero. There is also a converse: given π with $H_1(\pi) = H_2(\pi) = 0$, we can construct an acyclic space with π as fundamental group. Moreover, the G. Whitehead theorem implies that a space W is contractible if and only if W is both acyclic and has trivial fundamental group. So one would expect the fundamental group of an ayclic space to contain a lot of information. The next result says that acyclic spaces can be characterized in this way. However, note that property (v) below is also enjoyed by $W = S^1$.

Lemma 1.3 *The following conditions on a connected space W are equivalent.*

(i) *W is acyclic.*

(ii) *Every map from W to any space whose fundamental group contains no nontrivial perfect subgroup is nullhomotopic.*

(iii) *Every map from W to any nilpotent space, or to any homotopy limit of a diagram of nilpotent spaces, is nullhomotopic.*

(iv) *Every map from W to any Eilenberg-Mac Lane space of the form $K(\mathbb{Q}, n)$ or $K(C_p, n)$, where $n \geq 1$ and C_p denotes a cyclic group of prime order, is nullhomotopic.*

Moreover, these conditions imply the following.

(v) *Any map from W that induces the trivial map on fundamental groups must be nullhomotopic.*

The key definition is that a map $f : X \to Y$ is *acyclic* if its homotopy fibre F_f is an acyclic space. Recall that $F_f = X \times_Y PY$ is given by the pull-back diagram

$$\begin{array}{ccc} F_f & \longrightarrow & PY \\ \downarrow & \cdot\lrcorner & \downarrow^{\pi_Y} \\ X & \underset{f}{\longrightarrow} & Y \end{array},$$

where PY is the contractible space of paths $\lambda : I = [0,1] \to Y$ in Y that start at the basepoint and finish at $\pi_Y(\lambda)$. This is clearly a homotopy invariant property, which conveniently allows us to regard f as a cellular map between CW-complexes, and thence even as an inclusion of a subcomplex when the occasion demands. Here are some handy tools.

Lemma 1.4 *If either f or g is a fibration in the Cartesian square*

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ \downarrow^{f'} & \cdot\lrcorner & \downarrow^{f} \\ Y' & \underset{g}{\longrightarrow} & Y \end{array}$$

then f is acyclic if and only if f' is.

Lemma 1.5 *Let*

$$\begin{array}{ccccc} E & & \xrightarrow{f} & & E' \\ & {}^{p}\searrow & & \swarrow^{p'} & \\ & & B & & \end{array}$$

commute. Then f is acyclic if and only if its induced map on the homotopy fibres of p, p' is.

Lemma 1.6 *Suppose that f_1 is a cofibration in the co-Cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ \downarrow^{f_0} & \ulcorner\cdot & \downarrow^{f_0'} \\ Y_0 & \underset{f_1'}{\longrightarrow} & Y_0 \cup_X Y_1 \end{array}.$$

If f_i is acyclic, then so is f_i' ($i = 0$ or 1).

Proposition 1.7 *The following properties of a map $f : X \to Y$ are equivalent.*

(i) *f is acyclic;*

(ii) *for any local coefficient system* $\{L\}$ *of abelian groups on* Y,

$$f_* : H_*(X;\, f^*\{L\}) \to H_*(Y;\, \{L\})$$

is an isomorphism;

(iii) $f_* : H_*(X;\,\mathbb{Z}) \to H_*(Y;\,\mathbb{Z})$ *is an isomorphism and the induced action of* $\pi_1(Y)$ *on* $H_*(F_f;\,\mathbb{Z})$ *is nilpotent (that is, trivial on each factor of some finite filtration of* $H_*(F_f;\,\mathbb{Z})$*).*

Lemma 1.8 *If* $f : X \to Y$ *is acyclic, then there is a perfect normal subgroup* N *of* $\pi_1(X)$ *such that* $\pi_1(f)$ *induces* $\pi_1(Y) \cong \pi_1(X)/N$.

Proof. Let F be the homotopy fibre of $f : X \to Y$. Then in the exact sequence

$$\pi_2(Y) \to \pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y) \to \pi_0(F),$$

$\pi_1(f)$ is onto because $\tilde{H}_0(F) = 0$ implies $\pi_0(F) = 1$ while $\mathrm{Ker}\pi_1(f) = \mathrm{Im}\pi_1(i)$. The fact that $H_1(F) = 0$ implies that $\pi_1(F)$ is perfect, and so its homomorphic image N is also perfect. □

Examples 1.9 (1) For $n \geq 2$, let M be a closed homology n-sphere. Since $H_n(M) \neq 0$, M is oriented. So the map $f : M \to S^n$ defined by collapsing the complement of a neighbourhood U ($\cong \mathbb{R}^n$) of a small cell ($\cong B^n$) in M induces an isomorphism

$$f_* : H_n(M) \to H_n(M,\, M - U) \cong H_n(S^n).$$

Because S^n is simply-connected, f is acyclic.

(2) The Poincaré 3-sphere is derived from the faithful smooth representation of the perfect alternating group A_5 in $SO(3)$, that is, in (1) above let $n = 3$, $M = SO(3)/A_5 = S^3/\mathrm{SL}(2,5)$.

(3) With UT_2R as the ring of 2×2 upper triangular matrices over R, and projection

$$\pi_0 : \mathrm{UT}_2R \longrightarrow R \times R$$
$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \longmapsto (a, d)\ ,$$

consider the map $B\pi$ induced on classifying spaces by $\pi = \mathrm{GL}(\pi_0) : \mathrm{GL}(\mathrm{UT}_2R) \to \mathrm{GL}(R \times R)$. It is an integral homology equivalence. However on fundamental groups it yields the original short exact sequence of groups being classified, namely

$$MR \rightarrowtail \mathrm{GL}(\mathrm{UT}_2R) \overset{\pi}{\twoheadrightarrow} \mathrm{GL}(R \times R).$$

In other words, $\mathrm{Ker}\pi_1(B\pi) = \mathrm{Ker}\pi$ is the additive group MR of all finite matrices, stabilized by

$$\alpha \longmapsto \begin{bmatrix} \alpha & O \\ O & O \end{bmatrix},$$

which is most definitely not perfect. Thus $B\pi$ cannot be acyclic.

1.4 The plus-construction

Theorem 1.10 *Let N be a perfect normal subgroup of $\pi_1(X)$. Then there exists an acyclic cofibration $q : X \to Y$ with* $\mathrm{Ker}\pi_1(q) = N$. *If $g : X \to Z$ is another acyclic cofibration with* $\mathrm{Ker}\pi_1(g) = N$, *then there exists a homotopy equivalence $h : Y \to Z$ such that $h \circ q = g$.*

Sketch **Proof.** *Existence.* We first treat the case where $N = \pi_1(X)$. For each normal generator α_λ of $\pi_1(X)$ attach a 2-cell to X with characteristic map some $a_\lambda : S^1 \to X$ whose homotopy class is α_λ. The resulting space W is simply-connected (van Kampen). Then attach 3-cells to W by $\vee b_\lambda : \bigvee_\lambda S^2 \to W$ to form another simply-connected space Y.

$$\begin{array}{ccc} \bigvee_\lambda S^2 & \overset{\vee b_\lambda}{\longrightarrow} & W \\ \downarrow & \ulcorner. & \downarrow \\ \bigvee_\lambda B^3 & \longrightarrow & Y \end{array}$$

Because Y is simply-connected it has the fundamental group we are seeking and also, in order to establish that $X \hookrightarrow Y$ is acyclic, we need only check that $H_*(Y, X) = 0$ (by Proposition 1.7(ii)). This follows by the 5-lemma and excision.

For the general case, let $X' \to X$ be a covering of X with $\pi_1(X') = N$. By the above there is an acyclic cofibration $q' : X' \to Y'$ where Y' is simply-connected. Form the cofibration $q : X \to Y$ by the push-out

$$\begin{array}{ccc} X' & \overset{q'}{\longrightarrow} & Y' \\ \downarrow & \ulcorner. & \downarrow \\ X & \underset{q}{\longrightarrow} & Y \end{array}$$

Lemma 1.6 shows that q is acyclic with $\mathrm{Ker}\pi_1(q) = N$.

In view of the uniqueness result below, the space Y formed in this way is often written as X_N^+ and referred to as the *plus-construction* of X with respect to perfect normal subgroup N of $\pi_1(X)$, although the new space depends on the attaching of cells, involving choice of normal generating set for N.

Uniqueness. This is now an immediate consequence of the following useful lemma, proved by standard techniques, such as van Kampen's theorem.

Lemma 1.11 *Let X be any connected space. If N is a perfect normal subgroup of $\pi_1(X)$, then any map $g : X \to Z$ such that $\pi_1(g)(N)$ is trivial factors through the map q: $X \to X_N^+$, uniquely up to homotopy under X.*

The lattice of perfect normal subgroups has the trivial group as minimal member. Then $X_1^+ = X$. The other extreme is more interesting. After the Whitehead lemma we are interested in the perfect radical $\mathcal{P}\pi_1(X)$ of $\pi_1(X)$. With this choice of N we write $q : X \to X^+_{\mathcal{P}\pi_1(X)}$ as $q_X : X \to X^+$, and refer to it as the *plus-construction.* Thus q_X may be described as the terminal object, up to homotopy, in the category of acyclic cofibrations under X. The next properties follow from the fact that for any group G, $\mathcal{P}(G/\mathcal{P}G) = 1$.

Corollary 1.12 (a) *Acyclic $f : X \to Y$ is equivalent to q_X if and only if $\mathcal{P}\pi_1(Y) = 1$.*

(b) *In particular, $q_{X^+} = \mathrm{id}_{X^+}$.*

Corollary 1.13 *Given $f : X \to Y$, there is a unique homotopy class of maps $f^+ : X^+ \to Y^+$ making the following square commute.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X^+ & \xrightarrow{f^+} & Y^+ \end{array}$$

Thus the plus-construction can be thought of as a functor in an appropriate homotopy category. Since the fundamental group and perfect radical functors commute with finite direct products, and $I^+ = I$, we have ...

Corollary 1.14 *If $f_0 \simeq f_1 : X \to Y$, then $f_0^+ \simeq f_1^+ : X^+ \to Y^+$.*

Examples 1.15 (1) X^+ is contractible if and only if X is acyclic.

(2) In the case of a homology sphere M discussed previously, $f : M \to S^n$ is just $q_M : M \to M^+$.

(3) This continues Example 1.9(3). In the commuting square

$$\begin{array}{ccc} B\mathrm{GL}(\mathrm{UT}_2R) & \xrightarrow{q_1} & B\mathrm{GL}(\mathrm{UT}_2R)^+ \\ \downarrow^{B\pi} & & \downarrow^{B\pi^+} \\ B\mathrm{GL}(R\times R) & \xrightarrow[q_2]{} & B\mathrm{GL}(R\times R)^+ \end{array}$$

(where $q_1 = q_{B\mathrm{GL}(\mathrm{UT}_2R)}$, etc.), $B\pi^+$ is an homology equivalence, between nilpotent spaces. Therefore $B\pi^+$ is a homotopy equivalence. Note that q_2 and $q_2 \circ B\pi$ are acyclic although $B\pi$ is not.

1.4.1 Functoriality

On occasion, some people are concerned by the fact that the factorization in Lemma 1.11 is not strictly unique. For instance, the map $X \to X_N^+$ may depend on the choice of normal generating set for the perfect normal subgroup N of $\pi_1(X)$. Moreover, even after the generating set has been chosen there is the matter of choosing the map to represent that element of the fundamental group. Various angst-minimization techniques are available.

First, as here, we can insist that the acyclic map be a cofibration too (as always happens when cells are attached). This at least makes triangles commute up to homotopy under X. It's also a good idea in view of Lemma 1.6 above.

If one is applying the construction only to a certain class of spaces, such as $X = B\mathrm{GL}R$, then one may be able to finesse the problem. Thus [62] observes that, since there is always a canonical ring homomorphism $\mathbb{Z} \to R$, it is enough to fix the plus-construction $B\mathrm{GL}\mathbb{Z} \to B\mathrm{GL}\mathbb{Z}^+$, and then obtain the plus-construction for every other ring R by applying Lemma 1.6, so as to form $B\mathrm{GL}R^+$ as the pushout

$$\begin{array}{ccc} B\mathrm{GL}\mathbb{Z} & \longrightarrow & B\mathrm{GL}\mathbb{Z}^+ \\ \downarrow & \ulcorner. & \downarrow \\ B\mathrm{GL}R & \longrightarrow & B\mathrm{GL}R^+ \end{array}$$

Moreover, since $E\mathbb{Z}$ is normally generated in $\mathrm{GL}\mathbb{Z}$ by (the stabilization of) the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it suffices to form $B\mathrm{GL}\mathbb{Z}^+$ (and thereby any $B\mathrm{GL}R^+$) by adjunction of a single corresponding 2- and 3-cell.

Another way to exploit Lemma 1.6 is to note that it implies that the homotopy cofibring map $Y_1 \to C_f$ of any map $f : V \to Y_1$ is acyclic when the space V is. In the situation of $Y_1 = B\mathrm{GL}R$ again, there is a good choice of V, namely the *Volodin space* [153], [142]. This is formed as the union of the classifying spaces of all subgroups of $\mathrm{GL}R$ that are conjugate to some upper triangular matrix group $\mathrm{UT}_n R$ by a permutation matrix. Its construction is functorial, and comes with a canonical map to $B\mathrm{GL}R$.

Another method, discussed below, is to form the plus-construction as W-nullification with respect to a suitable space W. Again, a preferred class of spaces under consideration may lead to a preferred choice of W.

1.4.2 Definition of higher K

Although the plus-construction has turned out to have important applications in many areas of topology, the motivation for Quillen's definition was the definition of higher algebraic K-theory, as follows.

The *higher K-groups* of a ring R, $K_i R$ ($i \geq 1$) are defined as the composition of covariant functors

$$K_i : R \longmapsto \mathrm{GL}R \longmapsto B\mathrm{GL}R \longmapsto B\mathrm{GL}R^+ \longmapsto \pi_i(B\mathrm{GL}R^+).$$

Observe that this agrees with Bass' earlier definition of K_1 as $\mathrm{GL}R/ER$ because

$$\pi_1(B\mathrm{GL}R^+) = \mathrm{GL}R/\mathcal{P}(\mathrm{GL}R) = \mathrm{GL}R/ER = K_1 R.$$

It follows from arguments presented in the next lecture that BER^+ is the universal cover, and $B\mathrm{St}R^+$ the 2-connected cover, so that we can also write:

$$\begin{aligned} K_i R &= \pi_i(BER^+) \qquad i \geq 2, \\ &= \pi_i(B\mathrm{St}R^+) \qquad i \geq 3. \end{aligned}$$

Although it is a consequence of the Kan-Thurston theorem [78] that it is possible to continue with a sequence of groups in this way, no agreeable description has yet been found. [2]

---

[2] Another excuse for verse...

It was said by the K-theorist Berrick,
Just to make his talk more atmospheric:
Higher K (R a ring) is
A homotopy-group thing, as
Described by this BER^+-ic limerick.

2 The plus-construction as a localization

The discussion below follows [16]. The idempotence property of Corollary 1.12(b) above means that the plus-construction can usefully be viewed as a *localization*, in the following sense.

Lemma 2.1 *Let $L : \mathcal{C} \to \mathcal{C}$ be a functor and $l : \mathrm{Id}_{\mathcal{C}} \to L$ be a natural transformation. Define $\mathcal{L}$ to be the full subcategory of $\mathcal{C}$ whose objects are those Y isomorphic to some LX with X an object of $\mathcal{C}$. Then the following are equivalent.*

(i) *For each object X of $\mathcal{C}$, $l_X : X \to LX$ has the universal property with respect to each object Y of $\mathcal{L}$:*

$$\begin{array}{ccccc} X & & \xrightarrow{l_X} & & LX \\ & \searrow^{f} & & \swarrow^{\exists ! f'} & \\ & & Y & & \end{array}$$

(ii) *For each object X of $\mathcal{C}$, $l_{LX} = Ll_X$ and is an isomorphism.*

(iii) *For each object X of $\mathcal{C}$, $l_{LX} = Ll_X$ and has a left inverse.*

Adams [2] called $\mathcal{L}$ a *reflective* subcategory, whose objects are the *local* objects of the theory. From this viewpoint, a localizing functor L is just the same thing as an idempotent monad functor with *coaugmentation* map l_X. (For, (ii) above is what a monad reduces to, under the further assumption of idempotence.) This is consistent with the philosophy that repetition of localization produces nothing new. In particular, the traditional localization of a ring or module is a localization in the above sense. In this light, here are some 'new' examples of localizations.

Examples 2.2 Abelianization of groups. As is well known, taking the quotient of a group G by its commutator subgroup $[G, G]$ produces a group G_{ab} which is the initial abelian group under G via the canonical homomorphism $l : G \to G/[G, G]$; moreover $(G_{\mathrm{ab}})_{\mathrm{ab}} = G_{\mathrm{ab}}$.

Hypoabelianization of groups. Here the local objects are the hypoabelian groups, namely those groups lacking a nontrivial perfect subgroup. Since every group G has a maximum perfect subgroup $\mathcal{P}G$, and always $\mathcal{P}(G/\mathcal{P}G) = 1$, the homomorphism $G \to G/\mathcal{P}G$ is initial among maps from G to hypoabelian groups. Evidently the construction is idempotent.

Plus-construction of spaces. For a connected space X, Quillen's map $q_X : X \to X^+$ is initial among all maps from X to spaces whose fundamental group is hypoabelian. Viewed as an idempotent construction, q_X induces an isomorphism of homology groups (arbitrary abelian local coefficients) and hypoabelianization of fundamental groups.

Localization, rationalization and completion of spaces. As modules over $\mathbb{Z}$, abelian groups can be tensored with the rings of p-local integers $\mathbb{Z}_{(p)}$, rationals $\mathbb{Q}$, or p-adic integers $\mathbb{Z}_p$. For a nilpotent space X, there are corresponding topological functors $X \mapsto X_{(p)}$, $X \mapsto X_{\mathbb{Q}}$, $X \mapsto X_p^{\wedge}$ that effect these constructions on the homotopy and homology groups of X [70]. This means that, for example, for a simply-connected space or loop space X, the coaugmentation map $X \to X_p^{\wedge}$ induces

$$\pi_*(X) \to \pi_*(X_p^{\wedge}) = \pi_*(X) \otimes \mathbb{Z}_p$$
$$H_*(X) \to H_*(X_p^{\wedge}) = H_*(X) \otimes \mathbb{Z}_p.$$

In the setting of the based homotopy category of CW-complexes, an important advance has been the notion of localization with respect to a map $f : W \to W'$. Then a space X is said to be *f-local* (or *f-periodic*) if the induced map on based mapping spaces

$$f^* : \text{map}_*(W', X) \to \text{map}_*(W, X)$$

is a weak homotopy equivalence. This model seems to accommodate the examples of interest in homotopy theory, for instance, Anderson's version of Sullivan's P-localization of simply-connected spaces. For further discussion, see [35], [43]. Remarkably, whether or not this is really a special kind of localization, or quite general, depends on the model of set theory that one chooses to work with (see [36])!

2.1 The plus-construction as a nullification

In fact, one can simplify further, and usefully consider W' to be a point. Thus one is left considering localization with respect to a space W. This concept, known as the *W-nullification* $P_W X$ of a space X, turns out to have illuminating application to the plus-construction. A space Y is said to be *W-null* if the pointed mapping space $\text{map}_*(W, Y)$ is weakly contractible, or equivalently, if $[\Sigma^n W, Y]$ is trivial for all $n \geq 0$. Given W and X, there then exists a space $P_W X$ that is initial in the homotopy category of W-null spaces under X [43], and the coaugmentation map $l_X\colon X \to P_W X$ is called *W-nullification.*

Example 2.3 When $W = S^2$, the nullification $X \to P_W X$ is just the map $X \to K(\pi_1 X, 1)$ that induces an isomorphism on fundamental groups, and whose homotopy fibre is the universal cover of X.

Again, there is the question of how general is this kind of localization. To answer this, first let $A_L X$ be the homotopy fibre of $X \to LX$ over its given basepoint. In the list below, (i) follows the convention that two localizations are equivalent if they have the same local spaces. In the proof, in order to deduce (i) from the other conditions, we do need to assume that $L = L_f$ for some map f of connected spaces.

Theorem 2.4 [19] *For any localization $L = L_f$ with f a map of connected spaces, the following are equivalent.*

(i) *L is equivalent to a nullification.*

(ii) *If, in any fibration sequence $F \to E \xrightarrow{p} B$, both F and B are local, then so is E.*

(iii) *Every fibration sequence $F \to E \xrightarrow{p} B$ with B local is preserved by L.*

(iv) *For every space X the space $LA_L X$ is contractible.*

(v) *For every space X the space $A_L X$ is connected, and $A_L(A_L X) = A_L X$.*

The challenge is to determine for which spaces W the plus-construction $q_X : X \to X^+$ coincides with the map $X \to P_W X$. See for example [145].

We now review some of the theory of [18], by which an acyclic space W yields the plus-construction as the W-nullification functor P_W. It involves the relative form of the plus-construction $X \to X^+_N$, characterized above. In our situation, a fixed connected, based space W, not necessarily acyclic but assumed to have perfect fundamental group, determines a choice of N in the following way. For every connected, based space X, the group $S(W, X)$ *swept* by W is the subgroup of $\pi_1(X)$ generated by the images of all homomorphisms $\pi_1(W) \to \pi_1(X)$ that are induced by based maps $W \to X$. It is a perfect normal subgroup of $\pi_1(X)$ with the following characterization.

Lemma 2.5 *For each X, the space $Y(1) = X^+_{S(W,X)}$ together with the corresponding map q is initial in the homotopy category of spaces Y under X such that every composite map $W \to X \to Y$ sends $\pi_1(W)$ to 1.*

However, this is not quite the universal property that we are after. Although we seek the initial space Y under X such that $S(W,Y) = 1$, it may be that $S(W, Y(1)) \neq 1$, by virtue of the existence of a map from W to $Y(1)$ that fails to lift to X. To appreciate this point (and in the absence of a specific example!), it is simpler to consider the group-theoretic analogue.

For groups G and H, the normal subgroup $S(G,H)$ of H swept by G is that subgroup generated by the images of all homomorphisms $G \to H$. It is perfect when G is. Although $Q(H,1) = H/S(G,H)$ is initial in the category of groups Q under H such that every composite homomorphism $G \to H \to Q$ has a trivial image, there may exist a nontrivial homomorphism from G to $Q(H,1)$. The simplest example is when G, H and $Q(H,1)$ are cyclic of orders p, p^2 and p respectively. In order to construct a quotient of H that receives no nontrivial homomorphism from G, it is necessary to iterate transfinitely. This leads to $T(G,H)$ such that $H/T(G,H)$ is the initial group under H that fails to receive a nontrivial homomorphism from G.

For the topological counterpart, instead of group quotients we proceed by taking relative plus-constructions. Put $N(1) = S(W,X)$ and, given the subgroup $N(\alpha)$ of $\pi_1(X)$ and space $Y(X,\alpha) = X^+_{N(\alpha)}$, define

$$Y(X, \alpha+1) = Y(Y(X,\alpha), 1) = Y(X,\alpha)^+_{S(W,Y(X,\alpha))} = X^+_{N(\alpha+1)}.$$

For a limit ordinal α, define $N(\alpha)$ to be the union of the subgroups $N(\beta)$ with $\beta < \alpha$. Again there is suitable functoriality, and the union of all $N(\gamma)$ is a functorial subgroup $T(W,X)$ of $\pi_1(X)$, such that the space $X^+_{T(W,X)}$ is the initial space in the homotopy category of spaces Y under X with $S(W,Y)$ trivial. Moreover, when $\pi_1(W)$ is perfect, so is $T(W,X)$ and we obtain a lattice of perfect normal subgroups of $\pi_1(X)$ (with all arrows inclusions):

$$\begin{array}{ccccc} & & \mathcal{P}\pi_1(X) & & \\ & & \uparrow & & \\ & & T(\pi_1(W), \pi_1(X)) & & \\ & \nearrow & & \nwarrow & \\ S(\pi_1(W), \pi_1(X)) & & & & T(W,X) \\ & \nwarrow & & \nearrow & \\ & & S(W,X) & & \end{array}$$

Here is the result linking $T(W,X)$ to the W-nullification $P_W X$.

Theorem 2.6 [18] *Let W be any connected space with perfect fundamental group. Then the following statements are equivalent:*

(i) *W is acyclic.*

(ii) *The class of W-null spaces coincides with the class of spaces X such that $S(W,X)$ is trivial.*

(iii) *For every space X, the W-nullification map $l_X\colon X \to P_W X$ coincides, up to homotopy under X, with $X \to X^+_{T(W,X)}$.*

(iv) *For every space X, the map $l_X\colon X \to P_W X$ is an integral homology equivalence.*

It is shown in [18] that the free product $\mathcal{F}$ of a set of representatives of all isomorphism classes of countable, locally free, perfect groups has acyclic classifying space $B\mathcal{F}$ and

$$S(B\mathcal{F},X) = T(B\mathcal{F},X) = T(\mathcal{F},\pi_1(X)) = \mathcal{P}\pi_1(X)$$

for all spaces X, so that the $B\mathcal{F}$-nullification functor $P_{B\mathcal{F}}$ is naturally isomorphic to the plus-construction.

Subsequently, this procedure has been adopted in [39] in order to set up a comparable nullification for the algebra $sl(A)$ of matrices of an associative algebra A, thereby extending results of [87] in the rational case.

The matter of representing the plus-construction as P_W for suitable W becomes more delicate when we wish to restrict attention to spaces of the form $X = B\mathrm{GL}R$ – in which case we say that W *defines algebraic K-theory.* By Theorem 2.6 above, it suffices that W be acyclic and that always $T(W, B\mathrm{GL}R) = ER$. In fact, one need check this condition only for $R = \mathbb{Z}$, since ER is normally generated in $\mathrm{GL}R$ by the image of $E\mathbb{Z}$. It is shown in [16] that for a commutative domain R the only nontrivial perfect normal subgroup is ER. So, provided W is acyclic, it follows from (ii) of the theorem above that the condition that $\pi_1(W)$ have a nontrivial representation in $\mathrm{GL}\mathbb{Z}$ is both necessary and sufficient. This leaves the remaining question (to be answered in Lecture 5): in order that W define algebraic K-theory, must it be acyclic?

2.2 The plus-construction and homotopy groups

Since the plus-construction leaves homology groups unchanged, from the point of view of algebraic topology there is much interest in its effect on homotopy groups.

2.2.1 Hurewicz and K

One tool here is the *Hurewicz homomorphism* $\pi_i(Y)_{\mathrm{ab}} \to H_i(Y)$ that sends the class of $f : S^i \to Y$ to $f_*(\iota)$ where ι generates $H_i(S^i) = \mathbb{Z}$. Its key property is that it is an isomorphism when Y is $(i-1)$-connected. (Actually, this is the plain-vanilla version. Arguments below are more likely to consider homotopy and homology with other coefficients, or modulo a Serre class of abelian groups.)

For the plus-construction, we have $\pi_i(X^+) \to H_i(X^+) = H_i(X)$, and in particular

$$K_i(R) = \pi_i(B\mathrm{GL}R^+) \to H_i(B\mathrm{GL}R^+) = H_i(\mathrm{GL}R). \tag{2-2}$$

For example, it was by a computation of the homology groups in the case of a finite field $\mathbb{F}_q$ of q elements that Quillen was able to determine the K-theory.

Theorem 2.7 [117] *For $j \geq 1$,*

$$K_{2j}(\mathbb{F}_q) = 0,$$
$$K_{2j-1}(\mathbb{F}_q) = \mathbb{Z}/(q^j - 1)\mathbb{Z}.$$

He also used the Hurewicz map (relative to the Serre class of finitely generated abelian groups) to obtain the following.

Theorem 2.8 [119] *For the ring of integers $\mathcal{O}_F$ in a number field F, and each $i \geq 0$, the group $K_i(\mathcal{O}_F)$ is finitely generated.*

2.2.2 The plus-construction and fibrations

In general, the most important standard device for attacking homotopy groups is the exact homotopy sequence of a fibration. However, in order to be able to apply it, one needs to know when a fibration sequence $F \to E \xrightarrow{p} B$ (meaning that $F \to E$ factors through a homotopy equivalence from F to the homotopy fibre of p) gives rise to another fibration sequence $F^+ \to E^+ \xrightarrow{p^+} B^+$. In this event we say that the original sequence is *plus-constructive*, or is *preserved* by the plus-construction. Such sequences have been characterized [15]. However, it turns out to be possible to give the characterization for nullifications in general, the *fibration preservation theorem.*

Theorem 2.9 [19] *A fibration sequence is preserved by a nullification P_W if and only if its pullback over the homotopy fibre $A_{P_W}B$ of $B \to P_W B$ is preserved by P_W.*

The letter A is used here because we know that in the case when P_W is the plus-construction this homotopy fibre is acyclic. By using Lemma 1.3, one obtains a practical consequence.

Corollary 2.10 [19] *Suppose that $F \to E \xrightarrow{p} B$ is a fibration sequence of connected spaces, with the following properties.*

(i) *$P_W F$ is a nilpotent space.*

(ii) $\mathrm{Im}(\pi_1(A_{P_W}B) \to \pi_1(B))$ *acts nilpotently on $H_*(F; \mathbb{Z})$.*

(iii) *$A_{P_W}B$ is acyclic.*

Then p is preserved by P_W.

Among the fibration sequences that may easily be deduced from these results are the following:

$$BER^+ \to B\mathrm{GL}R^+ \to K(K_1R, 1)$$
$$K(K_2R, 1) \to B\mathrm{St}R^+ \to BER^+$$

The second is obtained from the *universal central extension* of groups [103]

$$K_2R \rightarrowtail \mathrm{St}R \twoheadrightarrow ER.$$

We can also easily derive, for commutative R, with unit group $U(R)$, the fibration induced from the determinant $\mathrm{GL}R \to U(R)$:

$$B\mathrm{SL}R^+ \to B\mathrm{GL}R^+ \longrightarrow K(U(R), 1)$$

which moreover has a section given by $U(R) = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}R$. Since, as we are about to see, each of these spaces is a loop space, this section gives rise (just like a splitting of an extension of abelian groups) to a direct product decomposition.

Proposition 2.11 *For a commutative ring R,*

$$B\mathrm{GL}R^{+} \simeq B\mathrm{SL}R^{+} \times K(U(R),1).$$

2.2.3 Delooping $B\mathrm{GL}R^{+}$

For the first major consequence of this technique (after Gersten and Wagoner [154]; see also [14] ch. 11), we apply it to the fibration sequence

$$B\mathrm{GL}R \to B\mathrm{GL}CR \to BESR.$$

This is obtained from the definition of the *suspension ring* SR as the quotient CR/mR, where we embed the nonunital ring mR of all finite matrices as a twosided ideal in the *cone ring* CR of all *locally finite* matrices over R (that is, those with only finitely many nonzero entries in each row and column), just as topological suspension is formed from the cone by collapsing an embedded copy of the original space. To get the sequence, one uses the Morita-type equivalence of $\mathrm{GL}mR$ with $\mathrm{GL}R$, and the fact that the group $\mathrm{GL}CR$ is perfect, so that it is equal to ECR. In fact, it is actually acyclic, so that the space $B\mathrm{GL}CR^{+}$ is contractible. Then it follows from the fibration sequence

$$B\mathrm{GL}R^{+} \to B\mathrm{GL}CR^{+} \to BESR^{+}$$

that $B\mathrm{GL}R^{+} \simeq \Omega BESR^{+} = \Omega(\widetilde{B\mathrm{GL}SR^{+}})$, which we can also write as $\Omega((B\mathrm{GL}SR)^{+}_{2})$, the r-connected cover being given the subscript $r+1$.

By taking higher and higher suspensions, one can iterate this process, 'delooping' $B\mathrm{GL}R^{+}$ as often as one wishes.

Theorem 2.12 *For each ring R, the space $B\mathrm{GL}R^{+}$ is an infinite loop space, with*

$$B\mathrm{GL}R^{+} \simeq \Omega^{r}(B\mathrm{GL}S^{r}R)^{+}_{r+1} \quad \text{for } r \geq 0.$$

This result immediately enables us to compare the higher K-groups of a ring with those of its suspension. We obtain that for all $i \geq 1$,

$$K_iR = \pi_i(B\mathrm{GL}R^{+}) \cong \pi_i(\Omega(B\mathrm{GL}SR)^{+}_{2}) = \pi_{i+1}((B\mathrm{GL}SR)^{+}_{2}).$$

However, because $(B\mathrm{GL}SR)^{+}_{2}$ is the universal cover of $B\mathrm{GL}SR^{+}$, this last group is just $\pi_{i+1}(B\mathrm{GL}SR^{+}) = K_{i+1}(SR)$.

Corollary 2.13 *For $i \geq 1$, $K_iR \cong K_{i+1}SR$.*

3 Other constructions for higher K

There are several alternative routes to higher K-theory, each useful for its own purposes, for example, for generalizing from the K-theory of rings to that of varieties or of schemes. Our main reasons for summarizing them here are to provide some historical balance, and to introduce a number of terms that we need later. A good reference for this material is [163].

3.1 Spectra, et cetera

We begin by noting the significance of the fact that $B\mathrm{GL}R^+$ is an infinite loop space. There is a long-known correspondence between infinite loop spaces, spectra and cohomology theories, as follows. (See [33], esp. §8, for an accessible introduction to this material.) A *spectrum* $\mathbf{E} = \{E_n\}_{n\in\mathbb{Z}}$ comprises a sequence of (based, always) spaces E_n with the property that each E_n is[3] in a canonical way the space of based loops ΩE_{n+1} on E_{n+1}. The choice of weak homotopy type of E_{n+1} is therefore determined to within number of path components, since for $i \geq 1$ we have

$$\pi_i(E_{n+1}) = \pi_{i-1}(\Omega E_{n+1}) = \pi_{i-1}(E_n).$$

One speaks of E_n *looping* E_{n+1}, and conversely of E_{n+1} *delooping* E_n. (Not quite 'conversely', in fact. Although the homotopy type of the looping of a given space is unique, that is not true of deloopings. Perhaps the simplest example is that S^2 and $S^3 \times \mathbb{C}P^\infty$ have homotopy equivalent loop spaces. To check uniqueness of deloopings in a given situation, one typically appeals to general machinery, such as [101].) When the structure maps $E_n \to \Omega E_{n+1}$ are not known to be equivalences, one calls $\mathbf{E}$ a *prespectrum*. A *map of spectra* $\mathbf{E} \to \mathbf{E}'$ is a sequence of (based) maps in each degree that give commutativity of all diagrams

$$\begin{array}{ccc} E_n & \longrightarrow & \Omega E_{n+1} \\ \downarrow & & \downarrow \\ E'_n & \longrightarrow & \Omega E'_{n+1} \end{array}$$

By taking homotopy fibres at each level, one obtains a new spectrum, the *homotopy fibre* of $\mathbf{E} \to \mathbf{E}'$.

[3] Historically, the word 'is' here denotes equivalence, which has been interpreted as referring to homeomorphism, homotopy equivalence or weak equivalence (inducing isomorphisms on all homotopy groups), according to the game one is playing. Also, historically this case was known as an Ω-spectrum; however, the language is still evolving.

Observe that in a spectrum each space E_n is $\Omega^k E_{n+k}$ for arbitrarily large k, and so an infinite loop space. In the other direction, any infinite loop space E, together with its iterated deloopings and loopings, can be the space E_n of a spectrum, for any choice of n ($n = 0$ is usual). The notion corresponding in this way to a map of spectra is that of an *infinite loop space map.*

The adjunction isomorphism

$$[\Sigma X,\, Y] \cong [X,\, \Omega Y],$$

in the form $\pi_i(\Omega E) \cong \pi_{i+1}(E)$, makes it possible to define the homotopy groups of a spectrum $\mathbf{E}$ as

$$\pi_n(\mathbf{E}) = \pi_n(E_0) \cong \pi_{n+1}(E_1) \cong \cdots$$

This makes sense for all $n \in \mathbb{Z}$; when the negatively indexed groups are trivial, the spectrum is said to be *connective.* (Since such a spectrum $\mathbf{E}$ is equivalent to choice of infinite loop space E_0, the notation sometimes vacillates between the two.) There is a standard procedure for converting a given spectrum $\mathbf{E}$ to its *connective cover*, denoted $\mathbf{E}[0, \infty)$. As the name suggests, $\mathbf{E}[0, \infty)$ maps to the original spectrum $\mathbf{E}$ in a way that preserves all homotopy groups in degrees ≥ 0.

A spectrum $\mathbf{E}$ gives rise to an (extraordinary) cohomology theory by, for each space X, adjoining a disjoint basepoint to form X_+, and then taking the homotopy classes of maps

$$E^n(X) = [X_+,\, E_n].$$

Brown's representability theorem assures us that this procedure is reversible: each cohomology theory with standard properties arises canonically in this way. Of more interest to us is the associated homology theory, defined by

$$E_n(X) = \lim_{k\to\infty} \pi_{n+k}(E_k \wedge X_+).$$

Some easily accessible examples include the following.

Examples 3.1 **1.** Using Theorem 2.12, for each ring R we have its *algebraic K-theory spectrum* $\mathbf{K}(R)$ given by the infinite loop space $B\mathrm{GL}R^+$ as the 0 th space. Sometimes, in order to give the traditional K_0R as π_0 of the spectrum, one artificially adjusts this by taking K_0R copies of $B\mathrm{GL}R^+$. Some other times, one starts with $\mathbb{Z}$ copies of $B\mathrm{GL}R^+$. These alternatives typically arise in situations (described below) where the construction is not directly the plus-construction. (For a plus-construction approach that incorporates K_0R, see [17].)

2. Considering Eilenberg-Mac Lane spaces $K(A,n)$ for an abelian group A and $n \geq 1$ (that is, $\pi_i(K(A,n)) = \delta_{in}A$), we deduce from the adjunction isomorphism that $K(A,n)$ is just $\Omega K(A, n+1)$. This gives the *Eilenberg-Mac Lane spectrum* $\mathbf{H}(A)$, whose cohomology theory is ordinary cohomology with coefficients in the group A, originally defined by means of simplicial or singular chain complexes. More generally, each *generalized Eilenberg-Mac Lane space* (GEM) is a product of Eilenberg-Mac Lane spaces and so also an infinite loop space.

3. By Bott periodicity, there is a *topological complex K-theory spectrum* $\mathbf{K}^{\text{top}}$. In odd dimensions, it is the unitary group U, and $BU \times \mathbb{Z}$ in even dimensions. Likewise, we may exploit Bott periodicity for the orthogonal group O, to define the *topological real K-theory spectrum* $\mathbf{KO}^{\text{top}}$ with $KO^{\text{top}}_{8j-i} = \Omega^i(BO \times \mathbb{Z})$ for $0 \leq i \leq 7$ and $j \in \mathbb{Z}$. Unlike the previous examples, these spectra are not connective. Their cohomology theories are the K-theories defined by Grothendieck, Atiyah and Hirzebruch by means of isomorphism classes of vector bundles over a space.

4. The *suspension spectrum* $\Sigma^\infty X$ of a (based) space X is the connective spectrum corresponding to the infinite loop space

$$\Omega^\infty \Sigma^\infty(X) = \lim_{k\to\infty} \Omega^k \Sigma^k(X)$$

via the natural inclusions

$$Y \to \Omega\Sigma(Y) = \Omega(S^1 \wedge Y)$$

given by passing along a meridian between the two (identified) poles.[4] Here the nth space is $\Omega^\infty\Sigma^\infty(\Sigma^n X)$, with

$$\begin{aligned}\Omega^\infty\Sigma^\infty(\Sigma^n X) &= \lim_{k\to\infty} \Omega^k \Sigma^{k-1}(\Sigma^{n+1} X)\\ &= \Omega \lim_{k\to\infty} \Omega^{k-1}\Sigma^{k-1}(\Sigma^{n+1}X)\\ &= \Omega(\Omega^\infty\Sigma^\infty(\Sigma^{n+1}X)).\end{aligned}$$

[4]The functor from spaces to spectra sending X to $\lim_{k\to\infty} \Omega^k\Sigma^k(X)$ is normally written as Q; however, there is a good reason for not doing so here.

The homotopy groups of this spectrum are just the stable homotopy groups of the space X:

$$\pi_n(\Sigma^\infty X) = \lim_{k\to\infty} \pi_{n+k}(\Sigma^k X).$$

(The fact that this sequence of groups becomes stationary is a consequence of the Freudenthal suspension theorem.) In particular, the example with $X = S^0$ is known as the *sphere spectrum*, whose homotopy groups are the notorious stable homotopy groups of spheres, $\lim_{k\to\infty} \pi_{n+k}(S^k)$ being called the *stable n stem* π_n^S.

The last of the examples above plays a special role in the theory of spectra. For, in any spectrum $\mathbf{E}$ there are structure maps $S^k \wedge E_n \to E_{n+k}$ that are adjoint to the equivalences from E_n to $\Omega^k E_{n+k}$. Applying Ω^k gives rise to canonical maps $\Omega^k\Sigma^k(E_n) \to \Omega^k E_{n+k} = E_n$ and so $\Omega^\infty\Sigma^\infty(E_n) \to E_n$ (indicating that repetition of $\Omega^\infty\Sigma^\infty$ fails to produce anything very new), and thus $\Omega^\infty\Sigma^\infty(S^m) \wedge E_n \to E_{n+m}$. This suggests that, if we think of the smash product as a topological counterpart of the tensor product, $\mathbf{E}$ may be regarded as a module over the sphere spectrum, an insightful idea that has had several implementations over the years [53]. Consistent with this is the notion of a *unital* spectrum, namely one that admits a map from the sphere spectrum, typically distinguished by further properties and called its *unit*. Various of our examples above are unital spectra. For instance, when A is cyclic there is a compatible family of maps $\{S^n \to K(A,n)\}_{n\in\mathbb{N}}$ yielding on the one hand generators of $H^n(S^n;\, A) \cong A$ and on the other, by the adjunction isomorphism, the unit

$$\lim_{k\to\infty} \Omega^k\Sigma^k(S^n) \to \lim_{k\to\infty} \Omega^k K(A, n+k) = K(A,n).$$

The algebraic K-theory spectrum is readily seen to be unital from the discussion in Example 3.5 below.

The first example above is of course central to these notes. Its construction is certainly not as well understood as one would wish. Why does the process of adding a single 2- and 3-cell to $B\mathrm{GL}R$ result in an infinite loop space? An interesting question here is: 'Why bother? Why not just settle for the suspension spectrum of $B\mathrm{GL}R$, in other words, stable homotopy groups of $B\mathrm{GL}R$ as higher K-groups?' The answer is that, although this gives the correct

$$\begin{aligned}\pi_1(\Sigma^\infty B\mathrm{GL}R) &= \pi_2(\Sigma B\mathrm{GL}R)\\ &= H_2(\Sigma B\mathrm{GL}R) = H_1(B\mathrm{GL}R) = K_1 R,\end{aligned}$$

the next group, $\pi_2(\Sigma^\infty B\mathrm{GL}R)$, fails to have the appropriate properties for K_2R established by Milnor [103].

One would love to be able to characterize those discrete groups G for which BG^+ is an infinite loop space. However, as the final lecture below makes clear, we are still far from reaching that goal.

3.1.1 Localization of spectra

Similar definitions to those seen above for localization of spaces also work for spectra. After [30], one defines in the following way the *localization* $L_{\mathbf{E}}(\mathbf{X})$ of a spectrum $\mathbf{X}$ with respect to a spectrum $\mathbf{E}$, or equivalently with respect to its associated homology theory E_* defined by $E_*(Z) = \pi_*(\mathbf{E} \wedge Z)$. A map $f : \mathbf{X} \to \mathbf{Y}$ of spectra is called an E_**-equivalence* if $E_*(f) : E_*(X) \to E_*(Y)$ is an isomorphism, while a spectrum $\mathbf{Z}$ is called E_**-local* if any E_*-equivalence f induces an isomorphism of groups of homotopy classes (of maps of spectra) $f^* : [\mathbf{Y}, \mathbf{Z}] \to [\mathbf{X}, \mathbf{Z}]$. Given $\mathbf{X}$, a characteristic property of $L_{\mathbf{E}}(\mathbf{X})$ is that it is E_*-local and there exists a co-augmentation map $\mathbf{X} \to L_{\mathbf{E}}(\mathbf{X})$ that is an E_*-equivalence.

Example 3.2 For each prime p, there is the p-adic completion $\mathbf{X}_p^\wedge$ of a spectrum $\mathbf{X}$. Likewise, one can localize and rationalize. However, because the fundamental group of an infinite loop space must be abelian, applying the plus-construction fails to yield anything new.

3.2 Category theory machinery

3.2.1 The classifying space of a category

The classifying space $B\mathcal{C}$ of a category $\mathcal{C}$ is obtained by composing two important functors. The first passes from the category to its nerve, a simplicial set. The second takes the geometric realization of a simplicial set to yield a space. The effect of this composition is that we obtain a CW-complex with cellular structure consisting of a vertex for each object of $\mathcal{C}$, a 1-cell for each nonidentity arrow in $\mathcal{C}$, a 2-cell for each commuting triangle in $\mathcal{C}$, and so on.

The intermediate gadget, a simplicial set, can be thought of as a combinatorial model for a complex. As so often, it helps to have a prototype. Let $\mathbf{\Delta}$ be the category whose objects are the finite sets $[0] = \{0\}$, $[1] = \{0, 1\}$, ..., $[n] = \{0, 1, \ldots, n\}$, etc. and whose morphisms preserve the order $\leq$. In fact, every morphism is a composition of functions of two kinds. Among the injective morphisms (*coface maps*) are those of the form $d^i : [n-1] \to [n]$, while among the surjective morphisms (*codegeneracy maps*) are those of the form $s^i : [n+1] \to [n]$. In both cases $0 \le i \le n$ and the map takes each value other than i exactly once. These generating morphisms satisfy some well-known (but ill-remembered) relations. A *simplicial set* $Z_\bullet$ is then simply a contravariant functor from $\mathbf{\Delta}$ to the category of sets (within some universe). More commonly, it is thought of as a sequence of sets $\{Z_n\}_{n\ge 0}$ known as *n-simplices* with extra structure afforded by the images of the d^i and s^i (known respectively as *face maps* d_i and *degeneracy maps* s_i). There's no need to restrict to mere sets here: a simplicial nonce is a contravariant functor from $\mathbf{\Delta}$ to the category of nonces, where a nonce might be a group, a monoid, a ring, etc.

The *nerve* of a (small) category $\mathcal{C}$ may be thought of as the canonical contravariant functor from $\mathbf{\Delta}$ to the category of (covariant) functors from $\mathbf{\Delta}$ to $\mathcal{C}$. Less abstractly, it is the simplicial set with an n-simplex for each composition

$$X_0 \to X_1 \to \cdots \to X_i \to \cdots \to X_n$$

of n morphisms in $\mathcal{C}$, with i th face d_i obtained by omitting mention of X_i, and i th degeneracy s_i given by insertion of the identity morphism at X_i.

Given a simplicial set $Z_\bullet$, its *geometric realization* $|Z_\bullet|$ is the complex whose n-simplices correspond to those of $Z_\bullet$, by attaching an n-cell for each nondegenerate n-simplex, its boundary faces corresponding to the $n+1$ $(n-1)$-simplices that are its images under the d_i. A little more formally, let the topological space

$$\Delta_n = \{\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ 0 \le x_i \le 1\}$$

denote the *geometric n-simplex*, equipped with the obvious geometric face and degeneracy maps that gave rise to the terminology. There is then a covariant functor from $\mathbf{\Delta}$ to the category described by these Δ_n and maps. Then

$$|Z_\bullet| = \left(\coprod_{n=0}^{\infty} (\Delta_n \times Z_n) \right) \Big/ \sim$$

where, for each morphism f of $\mathbf{\Delta}$, $(f_*\mathbf{x}, z) \sim (\mathbf{x}, f^*z)$, and one takes the quotient topology.

3.2.2 Classifying spaces of monoids

A *monoid* M can be thought of as (the morphisms of) a category with only one object. Then the classifying space BM of this category has a single 0-cell, a 1-cell for each element of M, and n-cells corresponding to products of length n in M. Under favourable conditions, its fundamental group $\pi_1(BM)$ is the *group completion* (or enveloping group, or Grothendieck group in the abelian situation) $U(M)$ of the original monoid M, characterized by the universal property that every homomorphism from M to a group factors uniquely through $U(M)$ [11]. The proto-example is $\mathbb{Z}$ as the group completion of the additive monoid of nonnegative integers $\mathbb{N}_+$. Thus the circle S^1 is a model for $B\mathbb{N}_+$. In the case of an abelian monoid M, its group completion $U(M)$ may be constructed as the quotient of $M \times M$ by the relation that (m_1, m_2) is equivalent to (m'_1, m'_2) whenever there exists $m \in M$ with $m_1 + m'_2 + m = m'_1 + m_2 + m$. An important example here is the abelian monoid consisting of isomorphism classes of finitely generated projective (say, right) modules over a given ring R, with direct sum of modules yielding the monoid addition. Its group completion is $K_0(R)$.

More generally, when M is a *topological monoid*, one would like the group $\pi_1(BM)$ to be the group completion of the (discrete) monoid $\pi_0 M$. For any topological monoid M, the above description of BM yields as inclusion of its 1-skeleton a map

$$\Sigma M \approx BM^{(1)} \hookrightarrow BM$$

whose adjoint $M \to \Omega BM$ can be group completion in a sense discussed below. In particular, when $\pi_0(M)$ is already a group, the spaces M and ΩBM are weakly homotopy equivalent.

Classifying space of a group. The above procedure applies to a group G, thought of as a category, again with a unique object, in which each morphism is an isomorphism. It yields a classifying space BG whose fundamental group, being the group-completion of G, is just G itself.

It's instructive to note some of the details of the construction, which works under the assumption that M is a topological monoid, acting on both sides of an M-space V (see [3]ch.2 for a discussion of attempts at further generality). There is then a simplicial space $N^{\text{cy}}_\bullet(M;V)$, with $N^{\text{cy}}_n(M;V) =$

$V \times M^n$,

$$\begin{aligned} d_i : N^{\mathrm{cy}}_n(M;V) &\longrightarrow N^{\mathrm{cy}}_{n-1}(M;V) \\ (v, m_1, \ldots, m_n) &\longmapsto \begin{cases} (vm_1, m_2, \ldots, m_n) & i = 0, \\ (v, m_1, \ldots, m_i m_{i+1}, \ldots, m_n) & 0 < i < n, \\ (m_n v, m_1, \ldots, m_{n-1}) & i = n, \end{cases} \end{aligned}$$

and

$$\begin{aligned} s_i : N^{\mathrm{cy}}_{n-1}(M;V) &\longrightarrow N^{\mathrm{cy}}_n(M;V) \\ (v, m_1, \ldots, m_{n-1}) &\longmapsto (v, m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_{n-1}) \quad 0 \le i \le n-1. \end{aligned}$$

The case where $V = \mathrm{pt}$ gives the classifying space BM as geometric realization. The other important specialization is $V = M$, when one writes $N^{\mathrm{cy}}_\bullet(M) = N^{\mathrm{cy}}_\bullet(M;M)$.

Example 3.3 The direct sum of matrices over a ring R:

$$S \oplus T = \begin{bmatrix} S & O \\ O & T \end{bmatrix}$$

combines with the canonical equivalences

$$B\mathrm{GL}_i R \times B\mathrm{GL}_j R \simeq B(\mathrm{GL}_i R \times \mathrm{GL}_j R)$$

to induce a monoid operation

$$B(\oplus) : B\mathrm{GL}_i R \times B\mathrm{GL}_j R \longrightarrow B\mathrm{GL}_{i+j} R$$

on $M = \coprod_{k \ge 0} B\mathrm{GL}_k R$.

3.2.3 The group completion theorem

As we saw with the plus-construction, there is a need for a machine in the context of simplicial monoids whose output is a fibration. There are numerous versions of the underlying homotopy-theoretic result here, which go by the name of the *group completion theorem*. In this situation one thinks of homotopy-theoretic group completion. Roughly, given a topological monoid M that enjoys some kind of homotopy-theoretic commutativity, this is the initial construction of a loop space from M (see [100], [122] for more precise wording, and [3]ch.3 for a more leisurely discussion). Here is the formulation of [76] after [92] (see also [129]).

Theorem 3.4 *Suppose that M is a topological monoid, acting on both sides of a simplicial M-space V such that multiplication by each vertex of M induces an integral homology isomorphism. Then the induced map from $|V|$ to the homotopy fibre of the projection $|N_\bullet^{\mathrm{cy}}(M;V)| \to BM$ is also an integral homology equivalence.*

This is most useful when it can be coupled with the Whitehead theorem so as to promote the homology equivalence to a homotopy equivalence. For this it suffices to know that the map in question has both its domain and codomain as, say, H-spaces, and induces an isomorphism of their fundamental groups. In other applications one has enough information to conclude that the homology equivalence holds for any local coefficient system, so that the induced map is acyclic, in other words, a plus-construction.

Examples 3.5 1. Consider $M = \coprod_{k\geq 0} B\mathrm{GL}_k R$, with $V = B\mathrm{GL}R^+ \times \mathbb{Z}$. Here $\pi_0 M = \mathbb{N}_+$, so that $\pi_1(BM)$ is its group completion $\mathbb{Z}$. In fact, one has

$$\Omega B(\coprod_{k\geq 0} B\mathrm{GL}_k R) \simeq B\mathrm{GL}R^+ \times \mathbb{Z}.$$

2. Starting with the simplicial monoid $\coprod_{k\geq 0} B\Sigma_k$ obtained from the symmetric groups Σ_k, and considering its action on $B\Sigma_\infty \times \mathbb{Z}$, one ends up with the theorem of [11] and [132] that

$$B\Sigma_\infty^+ \times \mathbb{Z} \simeq \Omega^\infty \Sigma^\infty (S^0).$$

For an alternative proof that uses a fibration preservation theorem, see [154]. For generalizations to other families of subgroups of Σ_k, see [83].

It is useful to compare these two examples. For, the natural embeddings of permutation matrices in general linear groups define a canonical map $B\Sigma_\infty^+ \to B\mathrm{GL}R^+$ that determines the unit for the algebraic K-theory spectrum, referred to after Example 3.1 above.

3.2.4 Quillen's Theorem B

A functor is called a *homotopy equivalence* or *weak homotopy equivalence* according to whether its induced map on classifying spaces has a homotopy inverse or induces isomorphisms of all homotopy groups. In particular, adjoint functors are homotopy equivalences. (It follows that one can generalize from small categories to the skeletally small categories that one tends to meet in K-theory.) At the other extreme, a functor having a natural transformation to a constant functor induces a nullhomotopic map on classifying spaces.

Example 3.6 A category with an initial object (such as an additive category) must be contractible. In particular, the category $\mathrm{P{\scriptsize ROJ}}_R$ of finitely generated projective right R-modules, in which the zero module is initial, has $B(\mathrm{P{\scriptsize ROJ}}_R)$ contractible.

The key result for translating category-theoretic data into homotopy-theoretic information about the classifying spaces is Quillen's Theorem B, which is the counterpart in this context to theorems above on preservation of fibrations by the plus-construction and other nullifications. To state it, we need another definition.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, and d an object of $\mathcal{D}$. Then the objects of the *left-fibre* or *comma category* F/d are pairs (c, f) consisting of an object c of $\mathcal{C}$ and morphism f of $\mathcal{D}$ from $F(c)$ to d. A morphism $(c, f) \to (c', f')$ consists of a morphism $g : c \to c'$ in $\mathcal{C}$ and a commuting diagram in $\mathcal{D}$:

$$\begin{array}{ccc} F(c) & \xrightarrow{f} & d \\ \downarrow^{F(g)} & & \downarrow^{\mathrm{id}} \\ F(c') & \xrightarrow{f'} & d \end{array}$$

Clearly, there is a forgetful functor $\digamma : F/d \to \mathcal{C}$ sending (c, f) to c. Since its composite with F gives an object $F(c)$ that always maps to d via f, on classifying spaces $BF \circ B\digamma$ is nullhomotopic. This means that it factorizes through the homotopy fibre of $BF : B\mathcal{C} \to B\mathcal{D}$.

Theorem 3.7 [118] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor such that for every morphism $d \to d'$ in $\mathcal{D}$ the induced functor $F/d \to F/d'$ is a homotopy equivalence. Then*

$$B(F/d) \xrightarrow{B\digamma} B\mathcal{C} \xrightarrow{BF} B\mathcal{D}$$

is a fibration sequence.

For alternative versions and proofs of this result, see [56] and [76].

3.2.5 Symmetric monoidal categories

The definition of a *symmetric monoidal category* $\mathcal{S}$ is just category-theoretic mimicry of the axioms for an abelian monoid. So $\mathcal{S}$ is endowed with a functor $\Box : \mathcal{S}\times\mathcal{S} \to \mathcal{S}$ and unit functor from the terminal category to $\mathcal{S}$ (equivalently, a unit object e), satisfying axioms corresponding to associativity, unitality and commutativity. Each of these axioms is expressed by a natural isomorphism (for example, $e\Box A \cong A \cong A\Box e$), together with commuting coherence diagrams [93] (for example, reconciling the various possible evaluations of $A\Box e\Box B$).

Examples 3.8 **1.** Any additive category is symmetric monoidal. The key examples are the categories PROJ_R of finitely generated projective right R-modules and MOD_R of all finitely generated right R-modules over a ring R, where $\Box$ is just direct sum $\oplus$ and e is the zero module 0.

2. In other examples, $\Box$ may represent tensor product of modules or disjoint union of sets having a group action.

3. In an additive category $\mathcal{P}$ in which every exact sequence is split (for example, PROJ_R), one obtains a symmetric monoidal category $\mathrm{ISO}\mathcal{P}$ by ignoring all morphisms other than the isomorphisms of $\mathcal{P}$.

On passage to classifying spaces, we obtain a product operation $B\Box$ giving a commutative H-space structure on $B\mathcal{S}$, and thus an abelian monoid structure on $\pi_0(B\mathcal{S})$. For example, in case (3) above, $\pi_0(\mathrm{ISO}\mathcal{P})$ is the abelian monoid of isomorphism classes of objects of $\mathcal{P}$, under the operation given by direct sum. In general, this situation lends itself nicely to application of the group completion theorem, with the following consequence.

Theorem 3.9 [160] *Let $\mathcal{S}$ be a symmetric monoidal category. Then $B\mathcal{S}$ is an infinite loop space if and only if $\pi_0(B\mathcal{S})$ is a group.*

Thus, to obtain an infinite loop space from a symmetric monoidal category $\mathcal{S}$, we need a construction on $\mathcal{S}$ whose effect on π_0 is group completion of the abelian monoid $\pi_0(B\mathcal{S})$. This is achieved by Quillen's localization $\mathcal{S} \mapsto \mathcal{S}^{-1}\mathcal{S}$ [58], under mild further conditions on $\mathcal{S}$. In the other direction, starting with any infinite loop space X, [146] shows that X may be regarded as the classifying space of a symmetric monoidal category, obtained from the category of weakly contractible spaces over X.

Example 3.10 When $\mathcal{S}$ =Iso(Proj$_R$), then $\pi_0(B\mathcal{S})$ is the monoid of isomorphism classes of finitely generated projective R-modules under direct sum, so its group completion is K_0R. Hence $B\mathcal{S}^{-1}\mathcal{S}$ is an infinite loop space with π_0 as K_0R.

Theorem 3.11 [58] *For* $\mathcal{S}$ =Iso(Proj$_R$),

$$K_0R \times B\mathrm{GL}R^+ \simeq B\mathcal{S}^{-1}\mathcal{S}.$$

Thus this gives another way of constructing algebraic K-theory, which has the advantage that it gives K_0 too. Its disadvantage is that it is less suited to homotopy theory arguments than is the plus-construction.

3.3 The Q-construction

Here, very briefly, is another approach, also from [118], that has the advantage that it is based on the category Proj$_R$ and so captures K_0 information. Another bonus is that it may be applied to more general categories than Proj$_R$ and so works more generally than for K-theory of rings.

Let M be an *exact category*, that is, a full additive subcategory (assumed small) of an abelian category that is closed under formation of short exact sequences with both kernel and quotient in M. In such a short exact sequence the maps are called *admissible.* Quillen's Q-construction then creates its category of subquotients QM, with the same objects as M. A morphism in QM from A to B is an equivalence class of diagrams in M

$$A \twoheadleftarrow S_{AB} \rightarrowtail B$$

($\rightarrowtail$ and $\twoheadrightarrow$ respectively denoting admissible monomorphism and admissible epimorphism), where equivalence corresponds to an isomorphism of diagrams

$$\begin{array}{ccccc} A & \twoheadleftarrow & S_{AB} & \rightarrowtail & B \\ \downarrow^{\mathrm{id}} & & \downarrow & & \downarrow^{\mathrm{id}} \\ A & \twoheadleftarrow & S'_{AB} & \rightarrowtail & B \end{array}$$

The composition of morphisms from A to B and from B to C is given by the pullback

$$\begin{array}{ccccccc} & & S_{AC} & \rightarrowtail & S_{BC} & \rightarrowtail & C \\ & & \downarrow & \lrcorner & \downarrow & & \\ A & \twoheadleftarrow & S_{AB} & \rightarrowtail & B & & \end{array}$$

(with both vertical arrows admissible epimorphisms), or $S_{AC} = S_{AB} \times_B S_{BC}$.

For $i \geq 0$, the algebraic K-groups of the exact category M may be defined as $K_i\mathrm{M} = \pi_i(\Omega BQ\mathrm{M})$, where the basepoint of $BQ\mathrm{M}$ comes from the zero object of M.

In particular, for the case M $=$PROJ$_R$ sitting inside the abelian category MOD$_R$, this definition agrees with the K-theory of the ring R. The proof of this result, in [58], actually involves showing the following.

Theorem 3.12 *For* M *the exact category* PROJ$_R$ *and* $\mathcal{S}$ *the symmetric monoidal category* ISO(M)*,*

$$\Omega BQ\mathrm{M} \simeq B\mathcal{S}^{-1}\mathcal{S}.$$

We remark that Quillen's Theorem B is well-suited to situations involving the Q-construction.

3.4 Other constructions

Even more briefly (see, for example, [163] for a fuller introduction), we mention two other constructions devised for greater generality or specific properties.

First, there is Waldhausen's generalization of the Q -construction, applicable to a category M endowed with *cofibrations* (generalized admissible monomorphisms) and *weak equivalences* (generalized isomorphisms). [159] then constructs a simplicial category $s_\bullet\mathrm{M}$. For $n \leq 1$, we have $s_0\mathrm{M}$ the trivial category and $s_1\mathrm{M} = \mathrm{M}$. For $n \geq 2$, the objects of $s_n\mathrm{M}$ are chains of cofibrations of length $n-1$ together with a commuting diagram involving all cofibrations in the chain and their cokernels; morphisms are natural transformations. When each component of a morphism is a cofibration we can call the morphism a cofibration too, and likewise for weak equivalences. Thus the beauty of this construction is that, unlike the Q-construction, it can be iterated. This means that when we pass to geometric realizations we obtain not just one, but an infinite sequence of deloopings. This can be called the K-theory spectrum of the original category M. In particular, when M is exact, $s_\bullet\mathrm{M}$ is naturally homotopy equivalent to $Q\mathrm{M}$.

Then, generalizing Quillen's category $S^{-1}S$ (for the case in which all exact sequences of M split), [56] defines a simplicial set called $G\mathrm{M} = \Omega s_\bullet\mathrm{M}$, which is essentially the construction of Waldhausen done doubly. The justification for the notation is a natural homotopy equivalence between the realization $|G\mathrm{M}|$ and the loop space of $|s_\bullet \mathrm{M}|$. The argument provides a simplification of the proof for the split exact sequence case. The study of bisimplicial sets and $GG\mathrm{M}$ gives rise to a description of products in K-theory. Further, exterior power λ-operations are also obtainable from the G-construction [59].

4 Topological Hochschild and cyclic homology - methods

As the name suggests, this is modelled on the algebraic Hochschild and cyclic homologies of a ring. So we begin by recalling these.

4.1 Hochschild and cyclic homology of rings, Dennis trace

4.1.1 Hochschild homology

Fix an underlying commutative ring k (typically $k = \mathbb{Z}$ or, as in Hochschild's original definition [72], a field). By the *Hochschild homology* of a unital, associative k-algebra A, projective over k, with coefficients in an A-bimodule V, we mean the torsion groups

$$HH_*(A, V) = \mathrm{Tor}_*^{A\text{-}A}(V, A)$$

between A-bimodules A and V. For calculation purposes, one usually proceeds as in [34](IX.6), and associates to any k-algebra A the *bar complex* $B_*(A)$. Here, for $n \geq -1$, $B_n(A) = A^{\otimes(n+2)}$ (always tensoring over k). The multiplication $\mu : A \otimes A \to A$ ensures that $A^{\otimes n}$ is also an A-bimodule; the extra two factors in the tensor product can be viewed as bookkeeping to take this property into account. To form a chain complex, endow $B_*(A)$ with the differential (meaning $d_B \circ d_B = 0$)

$$d_B : B_n(A) \to B_{n-1}(A), \qquad d_B = \sum_{i=0}^{n} (-1)^i (\mathrm{id}_i \otimes \mu \otimes \mathrm{id}_{n-i}).$$

The bar complex $B_*(A)$ is in fact acyclic for any ring A, that is, each homology group $\mathrm{Ker} d_B/\mathrm{Im} d_B$ is zero. Thus we have the (*standard*) projective resolution of the A-bimodule $A = B_{-1}(A)$.

This is used to give homology with coefficients in the A-bimodule V, by taking $H_*(A, V)$ as the homology of the *Hochschild complex* $\{Z_n(A; V) = V \otimes_k A^{\otimes n}\}$, whose differential is given by

$$\begin{aligned} &d(v \otimes a_1 \otimes \cdots \otimes a_n) = \qquad\qquad (4\text{-}3) \\ &va_1 \otimes \cdots \otimes a_n - v \otimes d_B(a_1 \otimes \cdots \otimes a_n) + (-1)^n a_n v \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

Just as before, with help from the bimodule axioms, we have $dd = 0$. We write $HH_*(A) = H_*(A, A)$.

When A is the group ring kG of a group G, we can relate $HH_*(kG)$ to the usual homology of the group G. For, the Hochschild complex contains the subcomplex spanned (over k) by *normalized* tensors $g_0 \otimes \cdots \otimes g_n$ having $g_0 \cdots g_n = 1 \in G$. Inclusion of complexes gives rise to a split injection $H_*(G;k) \to HH_*(kG)$, known as an *assembly map*, as is its composite with $H_*(G;\mathbb{Z}) \to H_*(G;k)$ induced from the canonical map $\mathbb{Z} \to k$. An example we use below arises when a k-algebra A is given and we take $G = \mathrm{GL}_r(A)$, leading to the assembly map $H_*(\mathrm{GL}_r(A)) \to HH_*(k\mathrm{GL}_r(A))$.

4.1.2 Traces

As usual, it's helpful to look at the lowest dimensions. Clearly

$$HH_0(A) = A/dA^{\otimes 2} = A/[A,A] = A/\sim$$

where $[A,A]$ is the k-submodule of A generated by terms $a_0a_1 - a_1a_0$, and $a_0a_1 \sim a_1a_0$. In general, a k-linear map on A that respects the equivalence relation $\sim$ is known as a *trace map*, and it follows that $HH_0(A)$ enjoys a universal property with respect to such maps. The most familiar of traces is of course that on matrices over a commutative ring A:

$$\mathrm{tr} : M_r(A) \to A, \qquad (a_{ij}) \mapsto \sum a_{ii}$$

or equally, exploiting k-linearity, $\mathrm{tr}(E_{ij}(a)) = \delta_{ij}a$ where the matrix $E_{ij}(a)$ has a as (i,j)-entry, 0 elsewhere, and δ_{ij} is the Kronecker delta function. When A is not necessarily commutative, one obtains the usual trace property only on passing further to $HH_0(A)$. Evidently this extends to a map of Hochschild complexes, via

$$\mathrm{tr} : M_r(A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$$
$$E_{i_0j_0}(a_0) \otimes E_{i_1j_1}(a_0) \otimes \cdots \otimes E_{i_nj_n}(a_n) \mapsto \delta_{j_0i_1}\delta_{j_1i_2} \cdots \delta_{j_ni_0}a_0 \otimes \cdots \otimes a_n\,,$$

and thereby to $\mathrm{tr}_* : HH_i(M_r(A)) \longrightarrow HH_i(A)$. Less obviously, we have the *Morita invariance* of Hochschild homology.

Theorem 4.1 $\mathrm{tr}_* : HH_*(M_r(A)) \to HH_*(A)$ *is a natural isomorphism.*

Relating to K-theory, because we can regard a finitely generated projective A-module as determined by a finite idempotent matrix over A, which is unique up to stabilization and conjugation, its trace in $A/[A,A]$ is easily seen to be well-defined. Thus there is a trace map from $K_0(A)$ to $HH_0(A)$, often called the *Hattori-Stallings rank*. (When A is a commutative domain, it indeed reduces to the rank of a finitely generated projective A-module; in effect, a celebrated conjecture of Bass [13] is: when A is the integral group ring $\mathbb{Z}G$ of a discrete group G, then this trace map likewise conveys little information.)

The higher-dimensional counterpart of this map is due to R. K. Dennis [42]. It exploits the *fusion map* $k\mathrm{GL}_r(A) \to M_r(A)$, the unique k-algebra map extending the inclusion $\mathrm{GL}_r(A) \hookrightarrow M_r(A)$. This just sends a formal sum of A-matrices, with coefficients in k, to the actual sum of A-matrices, each multiplied (using the fact that A is a k-module) by its coefficient. We also employ the Hurewicz map defined above (2-2). Putting them together, we have the maps

$$H_i(\mathrm{GL}_r(A)) \stackrel{\text{ass}}{\to} HH_i(k\mathrm{GL}_r(A)) \stackrel{\text{fus}}{\to} HH_i(M_r(A)) \stackrel{\mathrm{tr}_*}{\to} HH_i(A).$$

Even though the fusion map does not successfully stabilize as r increases, the composite does. Then the composite, or its stabilization $H_i(\mathrm{GL}(A)) \to HH_i(A)$, or the further composite

$$K_i(A) \stackrel{\text{Hur}}{\to} H_i(\mathrm{GL}(A)) \to HH_i(A),$$

is known as the *Dennis trace map*. For a more category-theoretic approach to this map, involving Waldhausen's $s_\bullet$ construction, see [79]. As well as the Bass conjecture problem referred to above with this map in dimension zero, there is also the discouraging fact that in the key case $A = \mathbb{Z}$ we have $HH_i(\mathbb{Z}) = 0$ for $i > 0$. Some of the methods we now describe have been designed with the purpose of overcoming these limitations.

4.1.3 Cyclic homology

The action of a_n on v in formula 4-3 above suggests (with hindsight – in fact, the process took three decades) the presence of a cyclic group action here. In the simplest case, with $V = A$, this enriches the bar complex by having the generator t of $C_n = \mathbb{Z}/n\mathbb{Z}$ send $a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$ to $(-1)^{n-1}a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{n-2}$ (the sign $(-1)^{n-1}$ being the sign of the cyclic permutation on n letters). The above differential then induces a differential on the cokernels of the action of the endomorphism $1-t$, giving the *Connes complex* [41]

$$\cdots \xrightarrow{b} A^{\otimes(n+1)}/(1-t) \xrightarrow{b} A^{\otimes n}/(1-t) \xrightarrow{b} \cdots$$

The homology of this complex is the *cyclic homology* $HC_*(A)$ of A, and the map $A^{\otimes n} \to A^{\otimes n}/(1-t)$ induces a homomorphism $I : HH_*(A) \to HC_*(A)$ which is id in dimension 0. This is when $\mathbb{Q} \subseteq k$; more generally, it is conventional to consider instead the (first quadrant) double complex obtained by combining d (for $M = A$) and d_B vertically (in alternate columns) with the horizontal differential given by the observation that

$$(1-t)N = N(1-t) = 0$$

where $N = 1 + t + t^2 + \cdots + t^n$ [90], [150]. Then the map I is induced from the inclusion of the Hochschild complex in the double complex as the 0 th column. The inclusion of complexes formally induces a long exact sequence of homology groups, Connes' periodicity exact sequence, which is the lower row in the following natural commuting diagram.

$$\begin{array}{ccccccccc}
\cdots \to & HC_{n-1}(A) & \xrightarrow{B} & HC_n^-(A) & \xrightarrow{I} & HP_n(A) & \xrightarrow{S} & HC_{n-2}(A) & \to \cdots \\
 & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & \\
\cdots \to & HC_{n-1}(A) & \xrightarrow{B} & HH_n(A) & \xrightarrow{I} & HC_n(A) & \xrightarrow{S} & HC_{n-2}(A) & \to \cdots
\end{array}$$

The upper exact sequence is obtained similarly by extending the first quadrant double complex over the whole upper half plane (in the case of HP), and then truncating to the right of the zeroth column [5] (in the case of HC^-). Homology groups in dimension n are derived (as Ker/Im) from the differentials on the (cartesian) product of all groups whose horizontal and vertical dimensions sum to n. The new terms in this exact sequence are called *negative cyclic homology* and *periodic cyclic homology* (this last theory having period 2).

A bit more specifically, from the double complex with $A_{pq} = A^{\otimes(q+1)}$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle -d_B} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d_B} & & \\
\cdots & \xleftarrow{N} & A_{-1,2} = A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & \cdots \\
 & & \downarrow{\scriptstyle -d_B} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d_B} & & \\
\cdots & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{N} & A_{01} = A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & \cdots \\
 & & \downarrow{\scriptstyle -d_B} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle -d_B} & & \\
\cdots & \xleftarrow{N} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A_{10} = A & \xleftarrow{N} & \cdots \\
\text{col. } p : & & -1 & & 0 & & 1 & &
\end{array}$$

[5] Note that labelling of arrows and columns varies slightly in the literature.

we form abelian groups

$$T_n^{[\alpha,\beta]}(A) = \prod_{p=\alpha}^{\beta} A_{p,n-p}$$

(so, for example, groups in $T_1^{[\alpha,\beta]}$ have been labelled in the diagram above), with differentials

$$\prod(1-t-d_B) : T_{2k}^{[\alpha,\beta]}(A) \to T_{2k-1}^{[\alpha,\beta]}(A)$$

and

$$\prod(N+d) : T_{2k+1}^{[\alpha,\beta]}(A) \to T_{2k}^{[\alpha,\beta]}(A),$$

giving the homology theories:

$$\begin{array}{lll} & \beta = 0: & \beta = \infty: \\ \alpha = 0: & HH_*(A) & HC_*(A) \\ \alpha = -\infty: & HC_*^-(A) & HP_*(A) \end{array}$$

After Goodwillie [57], Hood and Jones [73], one sharpens the Dennis trace map by lifting it through π, into $HC_*^-(A)$. To a large extent, this is by mimicking Dennis' composition. For example, the trace maps considered above are clearly invariant under permutations of indices, and so give rise to a homomorphism from $HC_*^-(M_r(A))$ to $HC_*^-(A)$. Again we have Morita invariance (valid too for each of the other homology theories above).

Theorem 4.2 $\mathrm{tr}_* : HC_*^-(M_r(A)) \to HC_*^-(A)$ *is a natural isomorphism.*

The homology of a group embeds in its corresponding negative cyclic homology to yield an assembly map $H_*(\mathrm{GL}_r(A)) \to HC_*^-(k\mathrm{GL}_r(A))$. By composing with the effect of the fusion map in negative cyclic homology, one obtains, as the cyclic homology counterpart of the Dennis trace map, the *Chern character* (or *Goodwillie trace*) $\mathrm{ch} : K_i(A) \to HC_i^-(A)$. Since we are using the plus-construction, this is really only for $i \geq 1$; when $i = 0$ the Chern character from $K_0(A)$ to $HC_0^-(A)$ maps via π to $HC_0(A) = HH_0(A)$ as the Hattori-Stallings rank. The above trace retains considerably more information than the Dennis trace, as witnessed by the theorem of [57] that it induces a rational isomorphism of the relative K groups with the relative negative cyclic homology groups, arising from a $\mathbb{Q}$-algebra epimorphism with nilpotent kernel.

Extensive treatments of the above material may be found in Chapter 9 of [161], or the whole of [89].

4.2 Hochschild and cyclic homology of spectra, cyclotomic trace

In the early 1980s, T. Goodwillie conjectured that it should be possible to rework the above algebraic treatment of Hochschild homology in a topological setting. It would involve, for example, replacing tensor product of k-algebras over the ground ring k by smash product of spectra over the sphere spectrum $QS^0 = \lim_{n\to\infty} \Omega^n S^n$. This was initially done with the invention of functors with smash product in 1985 by M. Bökstedt [26] (with contribution from F. Waldhausen), leading to topological Hochschild homology. The cyclic homology version was constructed about four years later in [27].

4.2.1 Simplicial approach

The first step in the topologization of the above, purely algebraic, treatments is to embody the data of the Hochschild and cyclic complexes simplicially. The homology groups of the complex then coincide with the homotopy groups of a simplicial abelian group $Z_\bullet$, or equally, the homotopy groups of its geometric realization $|Z_\bullet|$.

The abelian groups $Z_n(A;V) := V \otimes A^{\otimes n}$ form the n-simplices, with face operators

$$d_i(v \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} va_1 \otimes \cdots \otimes a_n & i = 0, \\ v \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & 1 \le i \le n-1, \\ a_n v \otimes a_1 \otimes \cdots \otimes a_{n-1} & i = n, \end{cases}$$

and degeneracy operators

$$s_i(v \otimes a_1 \otimes \cdots \otimes a_n) = v \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \qquad 0 \le i \le n.$$

The homotopy groups of the geometric realization

$$\mathrm{HH}(A;V) = |Z_\bullet(A;V)|$$

of $Z_\bullet(A;V)$ (a GEM) are obtained as before as the homology of the chain complex $(Z_*(A;V), d)$ where

$$d : Z_n(A;V) \to Z_{n-1}(A;V), \qquad d = \sum_{i=0}^{n} (-1)^i d_i.$$

In the important case where $V = A$, there is further structure, given for each n by the action of the finite cyclic group C_n on $Z_{n-1}(A;A)$:

$$t(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}) = a_{n-1} \otimes a_0 \otimes \cdots \otimes a_{n-2}.$$

The interactions of t with the d_i and s_i fit together to form what Connes calls a *cyclic set*. Just as a simplicial object in a category may be thought of as a contravariant functor from the category $\mathbf{\Delta}$ of finite sets, so a cyclic object may be considered as a contravariant functor from the category $\mathbf{\Lambda}$ of finite sets with compatible cyclic group action, which has morphism sets

$$\mathrm{Mor}_{\mathbf{\Lambda}}([m],[n]) = \mathrm{Mor}_{\mathbf{\Delta}}([m],[n]) \times C_{m+1}.$$

A consequence of this is that the realization of any cyclic set, and in particular the realization $\mathrm{HH}(A) = \mathrm{HH}(A;A)$, admits a continuous action by the circle group $\mathbb{T}$.

Example 4.3 The above is of course analogous to the simplicial set $N^{\mathrm{cy}}_\bullet(M;V)$ that we constructed for a monoid M acting on a set V. In the case where $V = M$, then $N^{\mathrm{cy}}_\bullet(M) = N^{\mathrm{cy}}_\bullet(M;M)$ forms another example of a cyclic set, with realization denoted $B^{\mathrm{cy}}M$. When $\pi_0 M$ is a group, $B^{\mathrm{cy}}M$ is (weakly homotopy) equivalent to the *free loop space* $\Lambda BM = \mathrm{map}(S^1, BM)$ in such a way that for each finite subgroup C of $\mathbb{T}$ there is an equivalence of fixed point sets $(B^{\mathrm{cy}}M)^C \simeq (\Lambda BM)^C$. An equivalence of this sort is known as a C_∞ *equivalence.*

4.2.2 Functors with smash product

A *functor with smash product* (*FSP*) is an endofunctor L of the category of pointed spaces (or, if one prefers, a simplicial functor of the category of pointed simplicial sets), together with natural transformations $1_X : X \to L(X)$ and $\mu_{X,Y} : L(X) \wedge L(Y) \to L(X \wedge Y)$, satisfying associativity and unitality conditions that involve an *assembly map* $X \wedge L(Y) \to L(X \wedge Y)$. We also need the functor L to be continuous, in that the induced function

$$\mathrm{Mor}(X,Y) \longrightarrow \mathrm{Mor}(L(X), L(Y))$$

is a pointed map of pointed spaces, and preserve the connectivity of spaces, and the assembly map to behave well with respect to suspensions.

When applied to spheres, the assembly maps $S^1 \wedge L(S^n) \to L(S^{n+1})$ combine with the extra structure on L to define a ring prespectrum L^S, whose homotopy groups form a graded ring.

For example, the ring A gives rise to the FSP $\tilde{A}$ that sends a based space X to the configuration space of X with labels in A:

$$\tilde{A}(X) = \left\{ \sum_{\text{finite}} a_i \cdot x_i \mid a_i \in A,\, x_i \in X \right\} \Big/ \{a \cdot * = 0 \cdot x = *\}.$$

In general, this is a GEM. A classical theorem of Dold and Thom is that the homotopy groups of this space are the reduced homology groups of X with coefficients in A. In particular, $\tilde{A}(S^n)$ is an Eilenberg-Mac Lane space $K(A, n)$, so that $\tilde{A}^S$ is just the Eilenberg-Mac Lane spectrum $\mathbf{H}(A)$.

Another way of producing an FSP starts with a topological monoid G, and yields the FSP $\tilde{G}$ given by $\tilde{G}(X) = X \wedge G_+$, where as usual the subscript $_+$ indicates the adjunction of a disjoint basepoint. Evidently, $\tilde{G}^S$ is the suspension spectrum $\Sigma^\infty(G_+)$.

For another example, give the set $[n] = \{0, 1, \ldots, n\}$ the basepoint 0. Then the FSP M_n is defined by $M_n(X) = \mathrm{map}_*([n], [n] \wedge X)$.

The composite of FSPs is also an FSP. This enables us, for example, to produce an FSP $M_n(L)$ for each FSP L.

4.2.3 Algebraic K-theory of an FSP

The idea here is to mimic the construction of the K-theory spectrum of a ring R as $\Omega B(\coprod_{k \geq 0} B\mathrm{GL}_k R)$. First, $\mathrm{GL}_r(L)$ is taken to be the set of invertible components in the monoid $\operatorname{hocolim} \Omega^n M_r(L(S^n))$. Observe that $\pi_0(\mathrm{GL}_r(L))$ is the group $\mathrm{GL}_r(\pi_0(\operatorname{hocolim} \Omega^n L(S^n)))$. Then one puts

$$\mathbf{K}(L) = \Omega B(\coprod B\mathrm{GL}_r(L)).$$

The validity of this analogy is confirmed by checking that the case $L = \tilde{A}$ gives a spectrum equivalent to $\mathbf{K}(A)$, as next.

Examples 4.4 1. The FSP $L = \tilde{A}$ above defines an infinite loop space $K(\tilde{A}) = \Omega B(\coprod_{k \geq 0} B\mathrm{GL}_k \tilde{A})$ with the property that

$$K(\tilde{A}) \simeq \Omega B(\coprod_{k \geq 0} BGL_k A) \simeq BGLA^+ \times \mathbb{Z}.$$

2. For a monoid G with the property that $\pi_0 G$ is already a group, the space $K(\tilde{G})$ is the classifying space $A(BG)$ for Waldhausen's A-theory of BG. More generally [27], Waldhausen's $A(X)$ is obtained from the FSP associated with the monoid of *Moore loops* on X (loops of arbitrary positive length, rather than unit length).

4.2.4 Topological Hochschild homology of an FSP

The material here originates in work of [26], [31].

Write I for the category of finite sets and injections. For each X we now define $\mathbf{THH}^X(L)$ as the geometric realization of the simplicial space $\mathrm{THH}^X_\bullet(L)$, with

$$\mathrm{THH}^X_n(L) = \mathrm{holim}_{\mathrm{I}^{n+1}} \mathrm{map}_*(S^{m_0} \wedge \cdots \wedge S^{m_n},\, L(S^{m_0}) \wedge \cdots \wedge L(S^{m_n}) \wedge X).$$

In fact, because of the axioms for an FSP L, the homotopy groups of $\mathrm{THH}^X_n(L)$ may be calculated without passing to the limit in the above formula, merely by choosing instead $m_0, \ldots, m_n$ to be sufficiently large (for a proof, see [94]). Again, cyclic permutation of the factors gives $\mathrm{THH}^X_\bullet(L)$ the structure of a cyclic space, and its geometric realization $\mathbf{THH}^X(L)$ a $\mathbb{T}$ action. In particular, $\mathbf{THH}(L) := \mathbf{THH}^{S^0}(L)$.

Starting with a ring A and its associated FSP $\tilde{A}$ gives the spectrum $\mathbf{THH}(A)$. We observed above that $\tilde{A}^S$ is the Eilenberg-Mac Lane spectrum $\mathbf{H}(A)$; thereby $\mathrm{THH}(\tilde{A})$ becomes isomorphic to the algebraic Mac Lane homology of A [114].

4.2.5 Cyclotomic spectra

We now move into the realm of S^1-spectra [84] by noticing the example of the above where $X = S^V$, the one-point compactification of a complex $\mathbb{T}$-representation space V. Suppose that $V \subset W$, a larger representation space, with orthogonal complement $W - V$. Then, because $S^{W-V} \wedge S^V = S^W$, we have

$$\begin{aligned} &S^{W-V} \wedge \mathrm{map}_*(S^{m_0} \wedge \cdots \wedge S^{m_n},\, L(S^{m_0}) \wedge \cdots \wedge L(S^{m_n}) \wedge S^V) \\ &\longrightarrow \mathrm{map}_*(S^{m_0} \wedge \cdots \wedge S^{m_n},\, L(S^{m_0}) \wedge \cdots \wedge L(S^{m_n}) \wedge (S^{W-V} \wedge S^V)) \end{aligned}$$

inducing

$$S^{W-V} \wedge \mathrm{THH}^{S^V}(L) \longrightarrow \mathrm{THH}^{S^W}(L),$$

whose adjoint

$$\mathrm{THH}^{S^V}(L) \longrightarrow \Omega^{W-V}\mathrm{THH}^{S^W}(L)$$

is a C_∞ equivalence, and which has good transitivity properties (for $W \subset W'$). So, there is a C_∞ equivalence between $\mathrm{THH}^{S^V}(L)$ and

$$T(L)(V) := \operatorname*{colim}_{W\ (\supset V)} \Omega^{W-V}\mathrm{THH}^{S^W}(L),$$

where one takes the colimit within some universal $\mathbb{T}$-space that contains each finite-dimensional representation of $\mathbb{T}$. Now for each finite subgroup C of $\mathbb{T}$, restriction to the C-fixed points (and in particular with $V = V^C$) makes

$$\Phi^C T(L)(V) := \operatorname*{colim}_{W\ (\supset V)} \Omega^{W^C-V}\mathrm{THH}^{S^W}(L)^C$$

a $\mathbb{T}/C$-space, and so, by extraction of $|C|$ th roots, again gives a $\mathbb{T}$-space, denoted $\rho_C^\# \Phi^C T(L)(V)$. The key property [66] that the spectra $\mathbf{T}(L)$ and $\rho_C^\# \Phi^C \mathbf{T}(L)$ are $\mathbb{T}$-equivalent for each C, in a way that respects enlargement of C within $\mathbb{T}$, is what makes $\mathbf{T}(L)$ a *cyclotomic spectrum.*

4.2.6 Topological cyclic homology

We begin with the remark that any C_{p^n}-equivariant map gives rise to a $C_{p^{n-1}}$-equivariant map in two ways. The first passage, called *Frobenius* [6], consists in using $C_{p^{n-1}} \subset C_{p^n}$ to regard the map as being equivariant with respect to the subgroup. The second, called *restriction*, is obtained by restricting the map to the C_p-fixed points of the domain, on which there is induced a $C_{p^{n-1}} \cong C_{p^n}/C_p$ action. These two processes commute with each other and behave nicely with respect to enlargements of finite cyclic groups (which in fact need not be p-groups). A prototype category for this setup is the category $\mathbb{I}$ whose objects are natural numbers, and with two commuting morphisms $F, R : rm \to m$, for all $m, r \in \mathbb{N}$, that model this good behaviour. Likewise, such structure can be seen to hold in any cyclotomic spectrum $\mathbf{T}$ with respect to its associated fixed point spectra $\mathbf{T}^C$.

For any cyclotomic spectrum T, one then defines

$$\mathbf{TC}(T) = \operatorname*{holim}_{\overleftarrow{\mathbb{I}}} \mathbf{T}^{C_n}$$

In particular, when $\mathbf{T}$ has the form $\mathbf{T}(L)$ for an FSP L, we write this as $\mathbf{TC}(L)$; and when further $L = \tilde{A}$ for a ring A, as $\mathbf{TC}(A)$.

[6]This term is plundered from the language of Witt vectors, for reasons that become apparent when one performs calculations – see below.

4.2.7 The cyclotomic trace

For an FSP L [27] constructs the cyclotomic trace

$$\mathrm{trc} : \mathbf{K}(L) \to \mathbf{TC}(L),$$

which is a map of spectra. The actual construction of the cyclotomic trace involves counterparts for FSPs of the assembly, fusion and trace (Morita invariance) maps we constructed before. These give a topological Dennis trace map, which lifts to $\mathbf{TC}(L)$ to yield the cyclotomic trace.

A generalization of this procedure has been undertaken in [47], in terms of *ring functors*, intuitively described as 'FSPs with several objects'; a ring functor on a single morphism category is an FSP (with extra structure). This enables exact categories to be used as the starting-point. Then for a ring A one recovers topological Hochschild and cyclic homology of A as that of the category $\textsc{Proj}_A$. In this framework the lifting of the Bökstedt trace to the cyclotomic trace appears rather naturally. More recently, the theory has been reworked in [45] for symmetric monoidal categories.

The following commutative diagram depicts the relationship between some of the constructions referred to above.

$$\begin{array}{ccccc}
\mathbf{TC}(A) & & \longrightarrow & & \mathbf{THH}(A) \\
& \nwarrow^{\mathrm{trc}} & & \nearrow^{\text{Bökstedt}} & \\
\downarrow & & \mathbf{K}(A) & & \downarrow \\
& \swarrow_{\mathrm{ch}} & & \searrow^{\text{Dennis}} & \\
\mathbf{HC}^{-}(A) & & \longrightarrow & & \mathbf{HH}(A)
\end{array}$$

A more detailed version of the account above appears in [94].

5 Topological cyclic homology - results

5.1 General results

In order to be able to use calculations in topological Hochschild and cyclic homology to obtain results about the K-theory of rings, one first needs to establish some general theorems linking the theories.

5.1.1 Goodwillie's conjecture

The best that one can hope for is for the trace to yield an isomorphism, in some sense. That was what Goodwillie had conjectured, initiating the above program. Here, 'in some sense' means after p-completion where p is a suitable prime; for example, in the case of a ring of algebraic integers in a local field, p would be the (positive) characteristic of the residue field.

When V is an A-bimodule, one can form the semiproduct ring $A \ltimes V$ (e.g. the 'unitalization' $\mathbb{Z} \ltimes R$ of a nonunital ring R) and associated algebraic K-theory *tangent space* $\mathbf{K}(A \ltimes V \to A)$, the homotopy fibre of the map of spectra $\mathbf{K}(A \ltimes V) \to \mathbf{K}(A)$. Following [157], the *stable K-theory* is defined to be

$$\mathbf{K}^{s}(A; V) = \underset{\overrightarrow{n}}{\operatorname{holim}}\, \Omega^{n+1} K(A \ltimes \tilde{V}(S^n_\bullet) \to A)$$

where $\tilde{V}(S^n_\bullet)$ is an $(n-1)$-connected simplicial A-bimodule. Generalizing to FSPs, there is the Dundas-McCarthy solution of Goodwillie's conjecture.

Theorem 5.1 [46] *Let L be an FSP and M an L-module. Then the topological Dennis trace* $\mathrm{tr} : \mathbf{K}(L; M) \to \mathbf{T}(L; M)$ *factors through a natural weak equivalence* $\mathbf{K}^{s}(L; M) \to \mathbf{T}(L; M)$.

5.1.2 The relative trace isomorphism

This result was used by McCarthy to derive a p-adic analogue of Goodwillie's relationship [57] between relative rational algebraic K-theory and relative rational cyclic homology.

Theorem 5.2 [91] *For any nilpotent ideal I of A and any prime p, the diagram*

$$\begin{array}{ccc} \mathbf{K}(A) & \xrightarrow{\mathrm{trc}} & \mathbf{TC}(A) \\ \downarrow & & \downarrow \\ \mathbf{K}(A/I) & \xrightarrow{\mathrm{trc}} & \mathbf{TC}(A/I) \end{array}$$

becomes homotopy Cartesian after p-adic completion.

In particular, the homotopy fibres $\mathbf{K}(A \to A/I)^\wedge_p$ and $\mathbf{TC}(A \to A/I)^\wedge_p$ are equivalent. The key case is when $I^2 = 0$. Moreover, arguments from Goodwillie's calculus of functors reduce to the study of semidirect products. In this way, the previous theorem applies. Again, there is an FSP generalization of this result [44].

Now suppose that k is a perfect field of positive characteristic p and A is a finitely generated algebra over the Witt vectors of k. In this situation the cyclotomic trace is an 'absolute' equivalence.

Theorem 5.3 [66] *For A and p as above, the cyclotomic trace induces a weak equivalence*

$$\mathrm{trc} : \mathbf{K}(A)^\wedge_p \to \mathbf{TC}(A)[0, \infty)^\wedge_p.$$

To complete this circle, there is a result linking a stabilized version of topological cyclic homology with topological Hochschild homology.

Theorem 5.4 [65] *Let L be an FSP and M an L-module. Then there is a natural weak equivalence*

$$\mathbf{TC}^{\mathrm{s}}(L; M)^\wedge_p \to \mathbf{T}(L; M)^\wedge_p.$$

5.1.3 Novikov conjecture

Another way of looking at the assembly map is by means of the product

$$\begin{aligned} G \times \mathrm{GL}_r(k) &\longrightarrow \mathrm{GL}_r(kG) \\ (g, (a_{ij})) &\longmapsto (a_{ij} g) \end{aligned}$$

After stabilizing, passing to classifying spaces, applying the plus-construction, and adjusting basepoints, one obtains

$$BG_+ \wedge B\mathrm{GL}(k)^+ \longrightarrow B\mathrm{GL}(kG)^+.$$

By taking instead the deloopings that give the K-theory spectra, we thereby obtain a K-theoretic assembly map

$$K(k)_i(BG) = \varinjlim_j \pi_{i+j}(BG_+ \wedge \mathbf{K}(k)_j) \longrightarrow K_i(kG).$$

Conjecture 5.5 *The K-theoretic assembly map is rationally injective for all commutative rings k and discrete groups G.*

This conjecture is considered to be the analogue in algebraic K-theory of Novikov's conjecture about the oriented homotopy invariance of the higher signature for connected, closed, oriented manifolds; a strong version of which is that the natural map $K_*(BG) \to K_*^{\mathrm{top}}(C_{\mathrm{r}}^*(G))$ is rationally injective.

Theorem 5.6 [27] *Conjecture 5.5 holds for $k = \mathbb{Z}$ whenever $H_i(G;\mathbb{Z})$ is finitely generated for each i.*

In this area one meets technical problems such as the effect on the K-theory of group rings induced by passage to a subgroup of finite index. This can be resolved by applying topological Hochschild homology in the setting of the topological theory of transfer maps [130]. The same work, by slightly modifying the original construction of THH, exhibits an explicit trace map that yields Morita equivalence for the theory.

5.1.4 Lichtenbaum-Quillen conjecture

In order to describe the Dwyer-Friedlander-Thomason formulation of this conjecture [48], discussed later, we start with the topological K-theory spectrum $\mathbf{K}^{\mathrm{top}}$ (the unitary group U in odd dimensions, $BU \times \mathbb{Z}$ in even dimensions). Next, employ the Bousfield localization $\hat{L}_p$ [30] of a spectrum $\mathbf{X}$ with respect to p-adic topological K-theory. Thus $\hat{L}_p(\mathbf{X}) = (\mathbf{X} \wedge J_p)_p^\wedge$ where J_p is just $\hat{L}_p$ of the sphere spectrum, in other words the fibre of

$$\psi^k - 1 : (\mathbf{K}^{\mathrm{top}})_p^\wedge \longrightarrow (\mathbf{K}^{\mathrm{top}})_p^\wedge$$

where 1 means id, and k is a topological generator of $\mathbb{Z}_p^\times$, that is, a generator of $U(\mathbb{Z}/p^2)$.

Conjecture 5.7 *Let F be a number field or p-local number field, with ring of integers $\mathcal{O}_F$. Then for $i \geq 2$ localization induces an isomorphism*

$$\pi_i(\mathbf{K}(\mathcal{O}_F);\, \mathbb{Z}_p) \longrightarrow \pi_i(\hat{L}_p\mathbf{K}(\mathcal{O}_F);\, \mathbb{Z}_p).$$

For a helpful discussion of this issue, see [156].

5.2 Calculations

5.2.1 Rings of integers

One of the first calculations of Bökstedt provides significant information about the K-theory of the integers.

Theorem 5.8 [28], [123] *For any prime p, the topological Dennis trace map induces a surjection* $K_{2p-1}(\mathbb{Z}) \longrightarrow T_{2p-1}(\mathbb{Z})$.

This is a stepping-stone to the next result, which relates to McCarthy's theorem that $\mathbf{K}(A)_p^\wedge$ is isomorphic to $\mathbf{TC}(A)_p^\wedge$ if A is the ring of integers in a p-local number field. For the p-adic integers $\mathbb{Z}_p$, where p is an odd prime, Bökstedt and Madsen identify, after p-adic completion, these spectra in terms of the spectrum $\mathbf{bu}$ of connective topological K-theory and the homotopy fibre $F\Psi^k$ of the Adams operation $\psi^k - 1 : \mathbf{K}^{\mathrm{top}} \longrightarrow \mathbf{K}^{\mathrm{top}}$ ($k \in \mathbb{Z}_p^\times$ a topological generator).

Theorem 5.9 [28] *Let p be an odd prime. Then*

$$\mathbf{TC}(\mathbb{Z}_p)_p^\wedge \simeq (F\Psi^k \times \mathbb{Z}_p)_p^\wedge \times B(F\Psi^k \times \mathbb{Z}_p)_p^\wedge \times (\Sigma\mathbf{bu})_p^\wedge.$$

In fact, they prove a more general result of this form, handling the case of unramified extensions of $\mathbb{Q}_p$, p odd. (For this to become a theorem, it first was necessary to affirm a conjectural step en route, in [149].) After considering the action of the mod 4 homotopy of $\mathbf{T}(\mathbb{Z})$ on its mod 2 homotopy [124], Rognes performed the calculation for $\mathbf{TC}(\mathbb{Z}_2)_2^\wedge$ [125], obtaining not a splitting as above, but fibrations linking the corresponding spaces.

The case of an arbitrary finite extension (again with p odd) is handled in [69], which is able to affirm the Lichtenbaum-Quillen conjecture in this context, specifically, for complete discrete valuation fields of characteristic zero having a perfect residue field of odd characteristic.

As one would expect (if one views topological Hochschild homology as a tool towards algebraic K-theory), THH calculations have often proven more fruitful than those of their K counterparts. Thus, after Bökstedt, one knows the following (see [55]).

Theorem 5.10 $\pi_i(\mathrm{THH}(\mathbb{Z})) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 2j \geq 2, \\ \mathbb{Z}/j\mathbb{Z} & i = 2j-1 \geq 1. \end{cases}$

This result has been extended to rings of integers in number fields in [86] and to certain algebras over $\mathbb{Z}$ or $\mathbb{F}_p$ in [113] and [80].

5.2.2 Perfect fields

Results here use the ring $W(A)$ of (p-typical) *Witt vectors* $\mathbf{a} = (a_0, a_1, \ldots)$ $a_i \in A$, unit $(1, 0, \ldots)$, subject to

$$\mathbf{a} + \mathbf{a}' = (s_0(a_0; a_0'),\ s_1(a_0, a_1; a_0', a_1'), \ldots),$$
$$\mathbf{a} \cdot \mathbf{a}' = (p_0(a_0; a_0'),\ p_1(a_0, a_1; a_0', a_1'), \ldots),$$

for certain integral polynomials s_i, p_i that depend on variables a_j, a_j' with $j \le i$.

So $W(A)$ can be viewed as the inverse limit of quotient rings $W_n(A) = \{(a_0, a_1, \ldots, a_{n-1})\}$ formed with respect to the restriction maps

$$R : W_n(A) \to W_{n-1}(A) \qquad (a_0, a_1, \ldots, a_{n-1}) \longmapsto (a_0, a_1, \ldots, a_{n-2}).$$

When A is an algebra over the prime field $\mathbb{F}_p$, $W(A)$ admits the *Frobenius* endomorphism $F : (a_0, a_1, \ldots) \longmapsto (a_0^p, a_1^p, \ldots)$. In the case of $\mathbb{F}_p$ itself, $W(\mathbb{F}_p) = \mathbb{Z}_p$. More generally, suppose that k is a *perfect* field of positive characteristic p (that is, $a \mapsto a^p$ is an automorphism, as when k is finite or algebraically closed). Then $W(k)$ is classically known to be a complete discrete valuation ring with residue field k and uniformizer p [134]. The generalization of F to the situation of **THH** arises from the following.

Theorem 5.11 [66] *For any commutative ring A,*

$$\pi_0 \mathbf{THH}(A)^{C_{p^n}} \cong W_{n+1}(A).$$

By means of an induction using McCarthy's theorem, [66] also establishes an equivalence between the p-adic completions of $\mathbf{K}(A)$ and $\mathbf{TC}(A)$ (after taking the connective cover in the latter case), when A is a $W(k)$-algebra that is finitely generated as a $W(k)$-module.

This is in turn used to describe the *tangent space* to algebraic K-theory at the perfect field k. For the ring of dual numbers $k[\varepsilon]$ as the truncated polynomial ring $k[x]/(x^2)$, this is the homotopy fibre of the map from $\mathbf{K}(k[\varepsilon])$ to $\mathbf{K}(k)$ induced by $\varepsilon \mapsto 0$. We describe the subsequent generalization to truncated polynomial algebras over k [67] and thence to truncated polynomial A-algebras, where A is a smooth algebra over k [68]. The description invokes the ring $\underline{\mathrm{W}}(A)$ of *big Witt vectors* of a commutative ring A (see, for example, [22]). This comprises sequences $(a_i) = (a_1, a_2, \ldots)$ $a_i \in A$, subject to

$$(a_i) + (a'_i) = (S_1(a_1; a'_1),\ S_2(a_1, a_2; a'_1, a'_2), \ldots),$$
$$(a_i) \cdot (a'_i) = (P_1(a_1; a'_2),\ P_2(a_1, a_2; a'_1, a'_2), \ldots),$$

where the polynomials S_i, P_i involve only those indices that are divisors of i. As an additive group, $\underline{\mathrm{W}}(A)$ is just the multiplicative group $(1+xA[[x]])^\times$ of power series with constant term 1. Again, we write $\underline{\mathrm{W}}_m(A)$ for the quotient ring of vectors of length m. Then, as an abelian group,

$$\underline{\mathrm{W}}_m(A) \cong (1 + xA[[x]])^\times / (1 + x^{m+1}A[[x]])^\times.$$

Finally, we use the (additive) *Verschiebung*[7] maps

$$V_d : \underline{\mathrm{W}}(A) \longrightarrow \underline{\mathrm{W}}(A) \qquad (a_j) \longmapsto (V_d a_j)$$

where

$$V_d a_j = \begin{cases} a_{j/d} & d \mid j, \\ 0 & d \nmid j. \end{cases}$$

Thus V_d is induced from $x \mapsto x^d$. With this notation, here is the relative K-theory of the truncated polynomial ring $k[x]/(x^d)$.

Theorem 5.12 [67] *For $j \geq 1$ there are isomorphisms*

$$K_{2j-1}(k[x]/(x^d),\ (x)) \cong \underline{W}_{dj}(k) / V_d\, \underline{W}_j(k),$$
$$K_{2j}(k[x]/(x^d),\ (x)) = 0.$$

5.3 Application: The spaces that define K-theory

By choosing special test rings R, we are now able to give a complete characterization of the spaces W that define algebraic K-theory.

Theorem 5.13 [20] *A CW-complex W defines algebraic K-theory if and only if both*

(i) *W is acyclic, and*

(ii) *there is a nontrivial homomorphism $\pi_1 W \to \mathrm{GL}\,\mathbb{Z}$.*

[7] According to my dictionary, this word means: shift, displacement, postponement, dislocation, or illicit sale. Take your pick.

Remark 5.14 In (ii) it is not necessary to have a nontrivial finite-dimensional representation. For example, the fundamental group of the acyclic fibre V of $B\,\mathrm{GL}\,\mathbb{Z} \to B\,\mathrm{GL}\,\mathbb{Z}^+$, namely the Steinberg group $\mathrm{St}\,\mathbb{Z}$, maps onto the group $E\mathbb{Z} \subset \mathrm{GL}\,\mathbb{Z}$ generated by elementary matrices, but has no nontrivial homomorphic image in any $\mathrm{GL}_n\,\mathbb{Z}$. It follows that V is an example of a space that defines algebraic K-theory but does not define the plus-construction. This is because if π is any finite group then

$$[V,\, B\pi] = \mathrm{Hom}(\mathrm{St}\,\mathbb{Z}, \pi)$$

is trivial, so that $B\pi$ is V-null and $P_V B\pi = B\pi$. One construction of V is the Volodin space for $\mathbb{Z}$ introduced in (1.4.1).

In section 2.1 above, proof of this theorem was reduced to supposing that every space of the form $B\mathrm{GL}R^+$ is W-null and deducing that W is acyclic.

An initial observation is that the test case $R = \mathbb{C}$ yields $H_1(W) = 0$. For otherwise, it admits a nontrivial homomorphism ψ to the group $\mathrm{GL}_1\mathbb{C}$. Now the composite map

$$W \xrightarrow{f_1} B\pi_1(W) \xrightarrow{f_2} BH_1(W) \xrightarrow{B\psi} B\mathrm{GL}_1\mathbb{C} \xrightarrow{f_3} B\mathrm{GL}\mathbb{C} \xrightarrow{f_4} B\mathrm{GL}\mathbb{C}^+$$

is homotopically trivial, but on H_1 each f_i is an isomorphism ($i \neq 3$) or split injection ($i = 3$).

Summarizing [20], we test W-nullity of $B\,\mathrm{GL}\,R^+$ on rings R related to rings of dual numbers $R[\varepsilon]$. Recall that the ring of dual numbers $R[\varepsilon]$ is the truncated polynomial ring $R[t]/(t^2)$, and the tangent space to algebraic K-theory at R is the homotopy fibre of the map from $B\,\mathrm{GL}(R[\varepsilon])^+$ to $B\,\mathrm{GL}\,R^+$ induced by $\varepsilon \mapsto 0$. When $R = \mathbb{F}_p$ (p any prime) we denote this homotopy fibre by F_p. The space F_1 is defined to be the corresponding fibre in the case $R = \mathbb{Z}$. We also write F_0 for the homotopy fibre of the map from $B\mathrm{GL}(\mathbb{Z} + \mathbb{Q}\varepsilon)^+$ to $B\mathrm{GL}\mathbb{Z}^+$ induced by $\varepsilon \mapsto 0$, with $\mathbb{Z} + \mathbb{Q}\varepsilon$ considered as a subring of $\mathbb{Q}[\varepsilon]$. As the fibre of a map between W-null spaces, each such F_n must also be W-null. Moreover, it turns out that the natural map $\gamma : F_1 \to F_0$ is rationalization.

From the computation in [139] of the torsion-free rank of the homotopy groups of F_1, we deduce that the homotopy groups of F_0 are $\mathbb{Q}$ in odd dimensions, zero otherwise. As the fibre of the infinite loop space map p_0^+, F_0 is itself an infinite loop space, and so a GEM, namely the product of all spaces $K(\mathbb{Q}, 2j-1)$. Since F_0 is W-null, so is each $K(\mathbb{Q}, 2j-1)$; then, via universal coefficients, one deduces that all the groups $\tilde{H}_i(W;\mathbb{Z})$ are torsion.

To study this torsion, we use the fact that for each prime p the tangent space F_p is W-null. Now F_p is shown in [66, Sections 1, 9] to be a GEM, with all its even-dimensional homotopy groups zero. By Theorem 5.12 above, for each $j \geq 1$, $\pi_{2j-1}(F_p)$ is a (nontrivial) finite abelian p-group of exponent p^n, where $n = 1$ when $p = 2$ and $n = 1 + [\log_p(2j-1)]$ otherwise.

We therefore have, whenever $1 \leq i \leq 2j-1$, the triviality of

$$[\Sigma^{2j-1-i}W,\, K(\pi_{2j-1}(F_p), 2j-1)] = \tilde{H}^i(W;\, \pi_{2j-1}(F_p)).$$

Using universal coefficients again, one deduces that for all primes p, multiplication by p on $\tilde{H}_*(W;\mathbb{Z})$ is an isomorphism. Since as above these reduced integral homology groups are torsion, W is acyclic after all.

6 Rings of integers

For a more extensive treatment of much of the following, see [4].

As the earliest history of the subject some 150 years ago concerns the ring of integers $\mathcal{O}_F$ in a number field F (in the guise of the ideal class group of the finite field extension F of $\mathbb{Q}$), the K-theory of these objects is a crucial benchmark for progress. After two somewhat frustrating decades, there is now good news to report.

6.1 Initial results

An early tool for calculation was the fibration [118]

$$\prod_{\mathfrak{m}\in \mathrm{Max}\mathcal{O}_F} (B\mathrm{GL}(\mathcal{O}_F/\mathfrak{m})^+ \times K_0(\mathcal{O}_F/\mathfrak{m})) \longrightarrow B\mathrm{GL}\mathcal{O}_F{}^+ \longrightarrow B\mathrm{GL}F^+$$

(actually obtained via the Q-construction, and valid more generally for Dedekind domains), where $\mathfrak{m}$ ranges over the maximal ideals of $\mathcal{O}_F$. Note that, apart from the F (finitely many in the case of quadratic extensions) for which $\mathcal{O}_F$ is a principal ideal domain, $\mathrm{Max}\mathcal{O}_F$ is infinite. In fact, the first map vanishes on all homotopy groups [140], and, since $\mathcal{O}_F/\mathfrak{m}$ is finite, the homotopy groups of its domain are given by Theorem 2.7. Applying these data to the homotopy exact sequence of the above fibration, we have the following.

Theorem 6.1 *For $j \geq 1$, the above fibration induces an exact sequence*

$$0 \to K_{2j}(\mathcal{O}_F) \longrightarrow K_{2j}(F) \longrightarrow \bigoplus_{\mathfrak{m}} K_{2j-1}(\mathcal{O}_F/\mathfrak{m}) \to 0$$

and isomorphism

$$K_{2j+1}(\mathcal{O}_F) \xrightarrow{\cong} K_{2j+1}(F).$$

In consequence, the last-named groups must be finitely generated, although the $K_{2j}(F)$ are almost always not so. On the other hand, by applying the rational Hurewicz homomorphism to the result of a homological calculation, [29] showed the groups $K_{2j}(F)$ to be torsion, with the other groups having torsion-free rank as below. Here and subsequently we assume that the number field F has r_1 real and r_2 complex places (that is, $F \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$), so that $r_1 = 0$ makes F *totally imaginary*, and $r_2 = 0$ corresponds to F *totally real.* Also, δ is the Kronecker delta.

Theorem 6.2 *For a number field F and $j \geq 0$, $K_{2j+1}(F)$ has torsion-free rank:*

$$\begin{cases} r_2 & j \text{ odd}, \\ r_1 + r_2 - \delta_{0j} & j \text{ even}. \end{cases}$$

From these facts it readily follows that each $K_{2j}(F)$ contains only finitely many infinitely divisible elements. As an example of (relatively recent) heavily number-theoretic calculations of [9], in $K_{22}(\mathbb{Q})$ they form a cyclic subgroup of order 691. The significance of 691 here is that it is an *irregular prime*. This means that it divides the numerator of some $\frac{B_m}{m}$, where the *Bernoulli numbers* B_m are the reduced fractions defined by

$$\frac{te^t}{e^t - 1} = \sum_{m=1}^{\infty} \frac{B_m}{m!} t^m.$$

In particular, $B_0 = 1$, $B_1 = \frac{1}{2}$; for $j \geq 1$, $B_{2j+1} = 0$; while $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_{12} = -\frac{691}{2730}$. The numerator of $\frac{B_m}{m}$ enters the picture via a theorem of [138], extended by [48].

Theorem 6.3 *For an even positive integer m, the numerator of $\frac{B_m}{m}$ divides the order of $K_{2m-2}(\mathbb{Z})$.*

This was used by [9] to derive information concerning divisibility in $K_{2m-2}(\mathbb{Q})$.

On the other hand, the denominator of $\frac{B_m}{2m}$ (as a reduced fraction) is long familiar in homotopy theory as the order of the (cyclic) image of the *J-homomorphism* $\pi_n(O) \to \lim\limits_{k \to \infty} \pi_{n+k}(S^k)$ in homotopy groups of spheres.[8] This homomorphism is the stabilization ($k \to \infty$) of the map that sends $\alpha : S^n \to O(k)$ (with, for each $x \in S^n$, the orthogonal transformation α_x on $\mathbb{R}^k$ preserving S^{k-1}) to

$$J(\alpha) : S^{n+k} \to S^k$$
$$(x\cos\theta,\ y\sin\theta) \longmapsto (\cos 2\theta,\ \alpha_x^{-1}(y)\sin 2\theta)$$

[8]Unfortunately, the notation for Bernoulli numbers used by topologists, after [71], has different indexing, given by the formula

$$\frac{te^t}{e^t - 1} = 1 + \frac{1}{2}t + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} t^{2k}.$$

See [104].

where $x \in S^n$, $y \in S^{k-1}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. For example, the case $n = 1$, and α the inclusion $S^1 = \mathrm{SO}(2) \hookrightarrow O(k)$, gives the Hopf map $S^3 \to S^2$ when $k = 2$, and the nontrivial element $\eta \in \pi_{1+k}(S^k) \cong \mathbb{Z}/2$ of the stable 1 stem π_1^S when $k > 2$.

Quillen [121] applied work of Adams on the e-invariant [1], viewed as a map from $B\Sigma_\infty^+$ to the fibre of the Chern character $BO \to \prod_{j\geq 1} K(\mathbb{Q}, 4j)$. As noted in the final lecture, because $\mathrm{GL}\mathbb{Z}$ is a discrete group, consideration of Pontrjagin classes implies that this map to the fibre factors through $B\mathrm{GL}\mathbb{Z}^+$. From a commuting diagram

$$\begin{array}{ccccc} B\Sigma_\infty^+ & \longrightarrow & B\mathrm{GL}\mathbb{Z}^+ & \longrightarrow & B\mathrm{GL}\mathbb{F}_p^+ \\ & & \downarrow & & \downarrow \\ & & BO & \longrightarrow & BU[p^{-1}] \end{array}$$

he used his previous calculation of $K_n(\mathbb{F}_p)$ to deduce the following.

Theorem 6.4 *For $n \equiv 3 \pmod 4$, the image of J maps injectively to the homotopy group $K_n(\mathbb{Z})$ of $B\mathrm{GL}\mathbb{Z}^+$, its image being a direct summand in dimensions $n \equiv 7 \pmod 8$ and mapping onto an odd-order summand otherwise.*

Further, [32] showed that for $n \equiv 3 \pmod 8$ $K_n(\mathbb{Z})$ contains an element of order 16, rather than the expected 8. Consequently, Theorem 6.3 has the following counterpart.

Corollary 6.5 *For an even positive integer m, $K_{2m-1}(\mathbb{Z})$ contains a cyclic direct summand of order equal to the denominator of*

$$\begin{cases} \frac{B_m}{2m} & m \equiv 0 \pmod 4, \\ \frac{B_m}{4m} & m \equiv 2 \pmod 4. \end{cases}$$

(These denominators are easily calculated from an 1845 formula of von Staudt:

$$\left(\mathrm{denom}\frac{B_m}{m}\right)_p = \left(\frac{pm}{p-1}\right)_p$$

where $(x)_p$ denotes the p-primary part of an integer x and is 1 otherwise. In fact, the denominator of B_m is always squarefree [104].) In the first case, $m = 2$, this is the whole story, since as [81] revealed, $K_3(\mathbb{Z}) = \mathbb{Z}/48$. In contrast, there is a result of Waldhausen [158], via his algebraic K-theory of spaces.

Theorem 6.6 *For $n \equiv 0, 1 \pmod 8$, the canonical map from* $\mathrm{Im}J \cong \mathbb{Z}/2$ *to* $K_n(\mathbb{Z})$ *is trivial.*

In general, concerning odd-dimensional groups, there is the following. With ξ_k as a primitive k th root of unity, define w_m to be the largest integer k for which the exponent of the Galois group of the extension $F(\xi_k)$ of F divides m. For example, when $F = \mathbb{Q}$, let us write $(m)_2$ for the highest 2-power dividing m. Then we have

$$w_m = 4(m)_2 = (|K_{2m-1}(\mathbb{F}_q)|)_2 \ ,$$

with the second equality holding precisely when the odd prime q is of the form $q \equiv \pm 3 \pmod 8$ (an amusing elementary exercise, given Theorem 2.7).

Theorem 6.7 [62] *For $m \geq 2$, $K_{2m-1}(\mathcal{O}_F)$ has a cyclic direct summand of order $2^{\nu} w_m$, where $\nu \in \{-1, 0, 1\}$.*

Comparing the tendencies above, it would seem that the Bernoulli numbers themselves are expressible in terms of the ratio of orders of successive K-groups of $\mathbb{Z}$. The more general formulation of this assertion is the original version of the Lichtenbaum-Quillen conjecture, which has spurred much of the research in this area.

Conjecture 6.8 [85] *For $n \geq 1$, the ratio of the orders of K-groups of $\mathbb{Z}$ is given by:*

$$|K_{4n-2}(\mathbb{Z})| \, / \, |K_{4n-1}(\mathbb{Z})| = 2^{\varepsilon} \, |B_{2n}| \, / 2n$$

where ε is an integer (small, one hopes).[9]

More generally, for any totally real number field F,

$$|K_{4n-2}(\mathcal{O}_F)| \, / \, |K_{4n-1}(\mathcal{O}_F)| = 2^{\rho} \zeta_F(-2n+1)$$

for some integer ρ.

Here $\zeta_F(s)$ is the Dedekind zeta function of F, the meromorphic continuation to the whole complex plane of the function

$$\zeta_F(s) = \sum\nolimits_{\text{ideal } \mathfrak{a}} (N\mathfrak{a})^{-s},$$

[9] *Caveat lector*: This and related conjectures have undergone a number of metamorphoses in the literature, some more mathematically and historically faithful than others.

whose series converges whenever $\mathrm{Re}(s) > 1$, $N\mathfrak{a} = \mathrm{card}(\mathcal{O}_F/\mathfrak{a})$ being the norm of an ideal $\mathfrak{a}$. By [135], the right-hand expression in the second equation of the conjecture is known to be rational. The reconciliation of the two equations is that, for even $m \geq 1$, the Riemann zeta function $\zeta_{\mathbb{Q}}(1-m) = -\frac{B_m}{m}$. A provocation for the above conjecture was Lichtenbaum's observation that Theorem 2.7 gives, for all $m \geq 1$,

$$|K_{2m}(\mathbb{F}_q)| \,/\, |K_{2m+1}(\mathbb{F}_q)| = \zeta_{\mathbb{F}_q}(-m).$$

Another early input, from topologists' interest in homotopy groups of spheres, was Adams' result that for $n \equiv 1, 2 \pmod 8$ the unique nontrivial element of $\pi_n(BO) = \pi_n(B\mathrm{GL}\mathbb{R})$ lies in the image of the stable n stem, $\pi_n(B\Sigma_\infty^+)$, induced by the inclusion homomorphism: every permutation matrix is orthogonal. Since the inclusion map factors through $B\mathrm{GL}\mathbb{Z}^+$, we have the following companion to Theorem 6.5.

Theorem 6.9 *For $n \equiv 1, 2 \pmod 8$, $K_n(\mathbb{Z})$ contains an element of order* 2.

For $n = 1$, this element corresponds both to the sign of a permutation and the determinant of an integral matrix. When $n = 2$, it represents the universal central extension of the infinite alternating group A_∞ [103]. For, since $A_\infty = \mathcal{P}\Sigma_\infty$, there is a commuting diagram

$$\begin{array}{ccc} BA_\infty^+ & \longrightarrow & BE\mathbb{Z}^+ \\ \downarrow & & \downarrow \\ B\Sigma_\infty^+ & \longrightarrow & B\mathrm{GL}\mathbb{Z}^+ \end{array}$$

and so a map of Schur multipliers $H_2(A_\infty) \to H_2(BE\mathbb{Z}^+) = K_2(\mathbb{Z})$, which is an isomorphism between groups of order 2 and leads to the map of universal central extensions

$$\begin{array}{ccc} H_2(A_\infty) = \mathbb{Z}/2 & \xrightarrow{\cong} & \mathbb{Z}/2 = K_2(\mathbb{Z}) \\ \downarrow & & \downarrow \\ A_\infty \times_{E\mathbb{Z}} \mathrm{St}\mathbb{Z} & \longrightarrow & \mathrm{St}\mathbb{Z} \\ \downarrow & \lrcorner & \downarrow \\ A_\infty & \longrightarrow & E\mathbb{Z} \end{array}$$

In both these cases, we have the only nontrivial element of $K_n(\mathbb{Z})$. By application of Bott periodicity, one then obtains corresponding elements for all n as in the theorem.

6.2 Odd torsion

The uncertainties concerning the prime 2 in statements above suggest that its behaviour in K-groups requires different analysis from that of odd primes. Indeed, that is the way the subject has evolved, especially in recent years.

One odd torsion topological consequence of Quillen's work on K-theory of finite fields [117] is that, for each odd prime p, the homotopy type of the p-localized space $B\mathrm{GL}(\mathbb{F}_q)^+_{(p)}$ is independent of choice of q from among the infinitely many primes that generate the unit group $U(\mathbb{Z}/p^2)$. He also established a fibration sequence

$$B\mathrm{GL}(\mathbb{F}_q)^+ \xrightarrow{\bar{b}} BU \xrightarrow{\psi^q - 1} BU \tag{6-4}$$

where the map $\bar{b}$ derives from representation theory (the 'Brauer lift'). In view of earlier work on the Adams conjecture, after p-localization the homotopy fibre is given the name $\mathrm{Im}J_{(p)}$ (sometimes written J_p), the name (among other things) reflecting its independence from appropriate choice of q. Collecting all these fibres together, we form the space

$$\mathrm{Im}J_{[1/2]} = \prod_{p \text{ odd}} \mathrm{Im}J_{(p)}$$

and combine all the maps

$$B\mathrm{GL}(\mathbb{Z})^+ \longrightarrow B\mathrm{GL}(\mathbb{F}_q)^+ \xrightarrow{p\text{-locn}} \mathrm{Im}J_{(p)}$$

to obtain a map $B\mathrm{GL}(\mathbb{Z})^+ \longrightarrow \mathrm{Im}J_{[1/2]}$ which it follows from Theorem 6.4 is a homotopy retraction, or in other words admits a right homotopy inverse. Thus odd primary information about the image of J gathered by Adams et al appears in $K_*(\mathbb{Z})$. The situation is in part described as follows.

Theorem 6.10 [107] *For each odd prime p there is a homotopy commutative diagram*

$$\begin{array}{ccc} B\Sigma_\infty^+ & \longrightarrow & B\mathrm{GL}(\mathbb{Z})^+ \\ \uparrow\downarrow & \nearrow & \downarrow \\ \mathrm{Im}J_{(p)} & \xrightarrow{\bar{b}} & BU \end{array}$$

Odd primary calculations have mostly occurred by means of topological Hochschild and topological cyclic homology, and have already been referred to. In low dimensions, other techniques have also been useful, as mentioned below.

6.3 2-Primary torsion

Here it is necessary to modify the odd prime definition of $\mathrm{Im}J_{(p)}$. For, whereas, for any odd prime q, $\pi_5(B\mathrm{GL}(\mathbb{F}_q)^+)$ has even order (2.7), yet $\pi_5(B\Sigma_\infty^+) = \pi_5^S = 0$ (3.5). In order that it be a homotopy retract of $B\Sigma_\infty^+$ (after localization) at 2, one defines $\mathrm{Im}J_{(2)}$ as the localized homotopy fibre of $\psi^3 - 1 : BO \to B\mathrm{Spin}$. This fibre may alternatively be described as $B\mathrm{NO}(\mathbb{F}_3)^+$ where $\mathrm{NO}(\mathbb{F}_3)$ comprises (stabilized) orthogonal matrices over $\mathbb{F}_3$ whose determinant coincides with the spinor norm [54](III.3.1). Now let $\bar{b}$ denote the composition

$$B\mathrm{NO}(\mathbb{F}_3)^+ \to B\mathrm{GL}(\mathbb{F}_3)^+ \to B\mathrm{GL}(\mathbb{C})) \simeq BU.$$

Theorem 6.11 [107] *Theorem 6.10 above also holds for $p = 2$.*

Starting with the Brauer lift fibration (6-4) above (for any prime q), and passing to the map b of universal covers, Bökstedt in effect defined a space $JK(\mathbb{Z}, q)$ as the pull-back

$$\begin{array}{ccc} JK(\mathbb{Z},q) & \longrightarrow & BO \\ \downarrow & \lrcorner & \downarrow^{c} \\ B\mathrm{SL}(\mathbb{F}_q)^+ & \xrightarrow{b} & BSU \end{array}$$

where c is given by the splitting $BO \to BSO$, of (2.11) above, followed by the complexification map that sends an orthogonal matrix to itself regarded as a unitary matrix. (Strictly speaking, $JK(\mathbb{Z}, q)$ is a covering of the space Bökstedt considered.)

Theorem 6.12 [25] *Let $q \equiv \pm 3 \pmod 8$. Then after 2-completion there is a map $\psi : (B\mathrm{GL}\mathbb{Z}^+)_2^\wedge \to JK(\mathbb{Z}, q)_2^\wedge$ which has $\Omega\psi$ a retraction, and therefore induces a split surjection on all homotopy groups.*

Because of the pull-back diagram above, in which all homotopy groups of the remaining three spaces are well understood, the homotopy groups of $JK(\mathbb{Z}, q)$, and so its 2-completion, are amenable to calculation.

The biggest of several breakthroughs in this area is provided by Voevodsky's proof of the Milnor conjecture for a number field F. [127] applies it to a finite coefficient version of the Bloch-Lichtenbaum spectral sequence [23] for any field of characteristic zero, conjectured in [120] and analogous to an Atiyah-Hirzebruch type spectral sequence of [48], from étale cohomology over

F to 2-primary K-theory of the field F, at least with coefficients in $\mathbb{Z}/2^\nu$. Its abutment can be related to $K_*(\mathcal{O}_F)\otimes\mathbb{Z}_2$. However, the E_2 page of this spectral sequence is highly nontrivial. To massage it into something computable one needs to know that the original Bloch higher Chow groups can be replaced by motivic homology groups [143], which in turn can be interpreted as étale cohomology groups [144] and thereby Milnor K-groups [152].

This enables a precise calculation of the Harris-Segal summand of Theorem 6.7.

Theorem 6.13 [127] *For $m \geq 2$, $K_{2m-1}(\mathcal{O}_F)$ has a cyclic direct summand of order:*

$$\begin{cases} 2w_m & r_1 \neq 0;\ m \equiv 2 \pmod 4, \\ 1 & r_1 \neq 0;\ m \equiv 3 \pmod 4, \\ w_m & \textit{otherwise.} \end{cases}$$

By performing the intricate calculations entailed, Rognes and Weibel were able to compute, in particular, $K_*(\mathbb{Q};\ \mathbb{Z}/2)$ and thereby, using (6.1), $K_*(\mathbb{Z};\ \mathbb{Z}/2)$. In principle, such a group should reveal limited information about $K_*(\mathbb{Z})$ because it detects only the existence of its cyclic 2-subgroups and not their orders. Fortunately however, this disclosed no further 2-subgroups beyond those already encountered in Theorems 6.5 and 6.9 above. This left the following state of affairs. (Recall that $(i+1)_2$ denotes the 2-primary part of $i+1$.)

Theorem 6.14 [127] *For $i \geq 2$, modulo a finite summand of odd order, $K_i(\mathbb{Z})$ is nonzero in precisely the following cases:*

$$K_i(\mathbb{Z})/\mathrm{odd} \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}/2 & i \equiv 1 \pmod 8, \\ \mathbb{Z}/2 & i \equiv 2 \pmod 8, \\ \mathbb{Z}/16 & i \equiv 3 \pmod 8, \\ \mathbb{Z} & i \equiv 5 \pmod 8, \\ \mathbb{Z}/(2(i+1)_2) & i \equiv 7 \pmod 8. \end{cases}$$

An earlier announcement of this result depended on certain assertions about the Bloch-Lichtenbaum spectral sequence that remain open [162]. Likewise [77] obtained generalizations to number fields F that depended on similar spectral sequence properties when $r_1 > 0$. For further advances, again heavily dependent on spectral sequence developments, we mention [50], [108], [109].

It follows from the theorem that the homotopy groups of $(B\mathrm{GL}\mathbb{Z}^+)^\wedge_2$ and $JK(\mathbb{Z},q)^\wedge_2$ agree ($q \equiv \pm 3 \pmod 8$), so that Bökstedt's map ψ is actually a homotopy equivalence. This has strong consequences for the homotopy type of the space $B\mathrm{GL}\mathbb{Z}^+$ at the prime 2. For, when applied to the defining cartesian square for $JK(\mathbb{Z},q)$ it gives the following.

Theorem 6.15 *For $q \equiv \pm 3 \pmod 8$, there is a pull-back of fibrations*

$$\begin{array}{ccccc} SU^\wedge_2 & \longrightarrow & (B\mathrm{GL}\mathbb{Z}^+)^\wedge_2 & \longrightarrow & BO^\wedge_2 \\ \downarrow\simeq & & \downarrow & \lrcorner & \downarrow c \\ SU^\wedge_2 & \longrightarrow & (B\mathrm{SL}(\mathbb{F}_q)^+)^\wedge_2 & \xrightarrow{b} & BSU^\wedge_2 \end{array}$$

Here are two applications of this result. First, by using known facts about the Hurewicz homomorphism for SU, Ausoni computes, at the prime 2, the Postnikov k-invariants that determine the homotopy type of $B\mathrm{GL}\mathbb{Z}^+$. Recall that for a simple CW complex X, one may for any i kill all homotopy groups of X in dimensions above i by adjoining cells of dimensions above $i+1$, thereby obtaining the i th *Postnikov section* $X[i]$ of X that approximates X up to dimension i. The fibration

$$K(\pi_i(X),i) \longrightarrow X[i] \longrightarrow X[i-1]$$

is then classified by the *k-invariant* $k^{i+1}(X) \in H^{i+1}(X[i-1];\ \pi_i(X))$. When X is an infinite loop space, this obstruction has finite order for $i \geq 2$.

Theorem 6.16 [8] *For $i \geq 2$, the 2-primary part of the order of the k-invariant $k^{i+1}(B\mathrm{GL}\mathbb{Z}^+)$ is:*

$((\frac{i-1}{2})!)_2$	$i \equiv 1 \pmod 4$,
2	$i \equiv 2 \pmod 8$ *and* $i > 2$, *or* $i = 3$ *or* $i = 7$,
16	$i \equiv 3 \pmod 8$ *and* $i > 3$, *or* $i = 15$,
$2(i+1)_2$	$i \equiv 7 \pmod 8$ *and* $i > 15$,
1	*otherwise.*

The second consequence is a complete computation of the 2-adic product structure on the ring $K_*(\mathbb{Z})$. In general, for a commutative ring A, the product structure on $K_*(A)$ arises from the tensor product of matrices [14]ch.13, [88]. This produces natural group homomorphisms

$$\mathrm{GL}_p A \times \mathrm{GL}_q A \to \mathrm{GL}(A \otimes A),$$

leading to

$$BGL_pA^+ \times BGL_qA^+ \to BGL(A \otimes A)^+.$$

One wishes to combine this with the multiplication map on A, and stabilization of the indices of the domain. The latter requires an adjustment that works on the smash product, giving a natural class of maps

$$\theta : BGLA^+ \wedge BGLA^+ \to BGLA^+.$$

By passing to homotopy groups one obtains the product

$$K_iA \otimes K_jA \to K_{i+j}A.$$

In the case of interest, $A = \mathbb{Z}$, the 2-*adic product* is the composite of this map with 2-completion $K_{i+j}(\mathbb{Z}) \to K_{i+j}(\mathbb{Z}) \otimes \mathbb{Z}_2$.

Theorem 6.17 [5] *The* 2-*adic product*

$$K_i(\mathbb{Z}) \otimes K_j(\mathbb{Z}) \to K_{i+j}(\mathbb{Z}) \otimes \mathbb{Z}_2$$

is trivial for all positive i and j unless the residues of i and j (mod 8) *form the set* $\{1\}$ *or* $\{1,2\}$. *In these exceptional cases the only nontrivial product is a product of two elements of order* 2.

6.4 Low-dimensional results

It is easy to see that $K_0(\mathbb{Z}) = \mathbb{Z}$ via the rank of a finitely generated projective $\mathbb{Z}$-module, while the determinant homomorphism yields the equalities $K_1(\mathbb{Z}) = GL_1\mathbb{Z} = \mathbb{Z}/2$. More difficult, although still elementary, is to see that $K_2(\mathbb{Z}) = \mathbb{Z}/2$ [103]. With a great deal more work, one obtains $K_3(\mathbb{Z}) = \mathbb{Z}/48$ [81], as previously noted; similar methods at that time showed that the only torsion in $K_5(\mathbb{Z})$ is 3-torsion [82], [136].

The next step is a much more recent development, although its origins trace back to Soulé's thesis, in which he showed that $K_4(\mathbb{Z}) = 0$ or $\mathbb{Z}/3$ modulo a finite abelian 2-group [137]. Returning to this problem in [141], he computes a spectral sequence converging to $H_1(SL_4(\mathbb{Z}); St_4)$ with coefficients in the Steinberg module St_4, corresponding to the top homology of the Tits building of SL_4 over $\mathbb{Q}$. He thereby shows that the homology group is a finite abelian 2-group. This information is an important ingredient in Rognes' calculation of the $E^1_{3,1}$ term of a spectral sequence that he constructs, based on the spectrum level *rank filtration*

$$* \simeq F_0\mathbf{K}(\mathbb{Z}) \subset \cdots \subset F_k\mathbf{K}(\mathbb{Z}) \subset \cdots \subset \mathbf{K}(\mathbb{Z})$$

of the K-theory spectrum $\mathbf{K}(\mathbb{Z})$. Here $F_k\mathbf{K}(\mathbb{Z})$ is the subspectrum of $\mathbf{K}(\mathbb{Z})$ whose n th space is built from the simplices of the n th space of $\mathbf{K}(\mathbb{Z})$ involving only free modules of rank at most k. With this input, and computation of other terms $E^1_{s,t}$, $s+t \leq 4$, he shows that the 3-torsion of $K_4(\mathbb{Z})$ is trivial. Thus $K_4(\mathbb{Z})$ must be a finite 2-group, and we are now in Rognes-Weibel territory. The ultimate conclusion is that $K_4(\mathbb{Z}) = 0$ [126].

Finally, some work released in July 2002 [52] is that computation of the Voronoi cell complex attached to real n-dimensional quadratic forms gives access to $H_*(\mathrm{GL}_n(\mathbb{Z}); \mathbb{Z})$ up to small primes. In consequence, $K_5(\mathbb{Z}) = \mathbb{Z}$, and $K_6(\mathbb{Z})$ is a finite 3-group.

7 The plus-construction and the stable mapping class group

7.1 Background

Write $F^p_{g,q}$ for (a copy of) an oriented smooth surface of genus g with p marked points (usually referred to as 'punctures') and q boundary components. Thus the boundary $\partial F^p_{g,q}$ of $F^p_{g,q}$ consists of the disjoint union of q circles, where $q \geq 0$. The (pure) *mapping class group* $\Gamma^p_{g,q}$ is the discrete group of components of the topological group $\mathrm{Diff}_+(F^p_{g,q})$ of orientation-preserving diffeomorphisms of $F^p_{g,q}$ that fix the marked points and the boundary pointwise. This can be seen to be equivalent to other geometric definitions; for special values of p, q there are also algebraic characterizations (as we shall see later). One reason for the importance of $\Gamma^p_{g,q}$ is that for $3g + 2p + 5q \geq 5$ the identity component of $\mathrm{Diff}_+(F^p_{g,q})$ is contractible [51], and so the classifying space $B\mathrm{Diff}_+(F^p_{g,q})$ is just the Eilenberg-Mac Lane space $B\Gamma^p_{g,q} = K(\Gamma^p_{g,q}, 1)$ [110].

What the space $B\Gamma^p_{g,q}$ classifies are oriented bundles with fibre $F^0_{g,0}$ that have $p + q$ disjoint cross-sections, the normal bundles of the last q of which are trivialized. Thus, elements of the cohomology of $\Gamma^p_{g,q}$ may be regarded as characteristic classes of surface bundles. A good introduction to characteristic classes for mapping class groups is [105].

A conjecture that has stimulated much of the research on this topic is as follows.

Conjecture 7.1 [111] *In dimensions less than* $g -$ constant, *the rational cohomology of* $\Gamma^0_{g,0}$ *is a polynomial algebra on even-dimensional generators.*

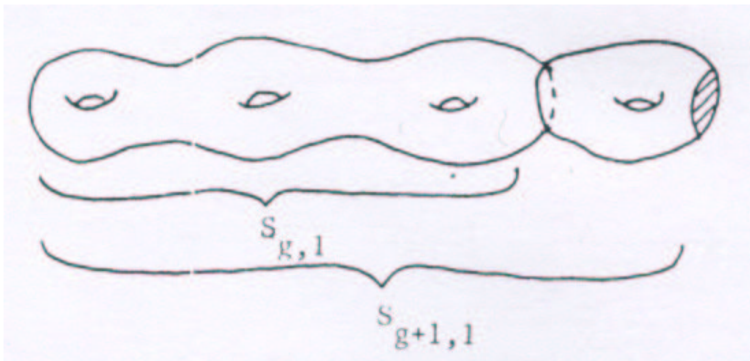
(Strictly speaking, [111] considered the rational Chow ring of the compactification of the moduli space of curves of genus g. However, by virtue of a properly discontinuous action of the mapping class group on Teichmüller space $T_g \approx \mathbb{R}^{6g-6}$ ($g \geq 2$), this object is naturally isomorphic to the cohomology stated above [63]. As we see below, current attacks on the conjecture focus rather on dimensions below a constant fraction of g.)

7.2 Stabilization of mapping class groups

For $q \geq 1$, there are some standard operations that enable one to vary the suffices [61], [102].

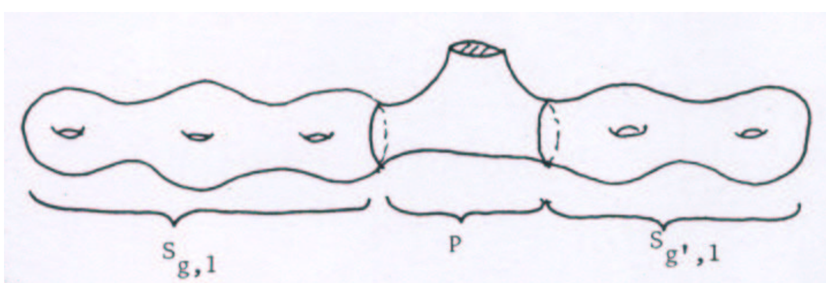
- To pass from g to $g+1$, glue a two-holed torus $F^0_{1,2}$ to $F^p_{g,q}$, yielding $F^p_{g+1,q}$.

Figure 1



- To pass from q to $q+1$, glue a pair-of-pants $F^0_{0,3}$ to a boundary circle of $F^p_{g,q}$, yielding $F^p_{g,q+1}$.
- In fact, the first operation is the composition of the second operation followed by the process that glues a pair-of-pants $P = F^0_{0,3}$ to a pair of boundary circles of $F^p_{g,q}$, to yield $F^p_{g+1,q-1}$.
- By taking two of the boundary circles of a pair-of-pants P, and gluing one to each of $F^p_{g,q}$ and $F^{p'}_{g',q'}$, we obtain the 'boundary connected sum' $F^{p+p'}_{g+g',q+q'-1}$.

Figure 2



The figures above are taken from [37], where $S_{g,1} = F^0_{g,1}$.
Extending diffeomorphisms by the identity produces respectively:

- a homomorphism from $\Gamma^p_{g,q}$ to $\Gamma^p_{g+1,q}$;
- a homomorphism from $\Gamma^p_{g,q}$ to $\Gamma^p_{g,q+1}$;
- a homomorphism from $\Gamma^p_{g,q}$ to $\Gamma^p_{g+1,q-1}$;
- a product from $\Gamma^p_{g,q} \times \Gamma^{p'}_{g',q'}$ to $\Gamma^{p+p'}_{g+g',q+q'-1}$.[10]

Motivated by homological stability phenomena for arithmetic groups such as $\mathrm{GL}_n\mathbb{Z}$, [61] showed that the three homomorphisms above induce isomorphisms of the integral homology of $\Gamma^p_{g,q}$ in homological dimensions below $g/3$ (later improved to $g/2$ in [75]).

Now fix $p = 0$ and simply write $\Gamma_{g,q} = \Gamma^0_{g,q}$. Combining with [115], one has that $\Gamma_{g,q}$ is a perfect group whenever $g \geq 3$. One can define $\Gamma_{\infty,q}$ as the direct limit of groups $\Gamma_{g,q+1}$ formed by means of the first homomorphism above (in the limit, one boundary circle moves off to infinity and disappears). In other words, one stabilizes with respect to the genus. Thus, for $q \geq 0$, $B(\Gamma_{\infty,q})^+$ is simply-connected, so the homology equivalences induced by the second family of homomorphisms above are homotopy equivalences.

Theorem 7.2 *For all $q \geq 0$, the homomorphism $\Gamma_{g,q+1} \to \Gamma_{g,q+2}$ induces on the stabilizations a homotopy equivalence*

$$B(\Gamma_{\infty,q})^+ \xrightarrow{\simeq} B(\Gamma_{\infty,q+1})^+.$$

Hence, in studying $B(\Gamma_{\infty,q})^+$ we may as well restrict attention to $B\Gamma^+_{\infty,0}$. In fact, $\Gamma_\infty = \Gamma_{\infty,0}$ is known as the *stable mapping class group*. Returning to the case $p > 0$, [24] observes that gluing p punctured disks onto the first p boundary circles of $F_{g,p+1}$ commutes with stabilization and so gives rise to a central extension

$$\mathbb{Z}^p \rightarrowtail \Gamma_{\infty,p} \twoheadrightarrow \Gamma^p_\infty,$$

which must be classified by a map $B\Gamma^p_\infty \to K(\mathbb{Z}^p, 2) = (\mathbb{C}P^\infty)^p$. Therefore there is a fibration

$$B\Gamma_{\infty,p} \longrightarrow B\Gamma^p_\infty \longrightarrow (\mathbb{C}P^\infty)^p$$

[10] This product is usually known as the *pair-of-pants multiplication*. More evocatively, it could also be called the *Beatles composition*: 'All we are saying is give each piece its pants.'

that is plus-constructive since the base is simply-connected. Now, the forgetful map $\Gamma^p_\infty \to \Gamma_\infty$ is left inverse to the composite $\Gamma_\infty \to \Gamma_{\infty,p} \twoheadrightarrow \Gamma^p_\infty$, so that, because of the theorem above, in the fibration sequence

$$B(\Gamma_{\infty,p})^+ \longrightarrow B(\Gamma^p_\infty)^+ \longrightarrow (\mathbb{C}P^\infty)^p$$

the inclusion of the homotopy fibre has a left homotopy inverse. Hence this fibration reduces to a product.

Theorem 7.3 *For all* $p, q \geq 0$,

$$B(\Gamma^p_{\infty,q})^+ \simeq B\Gamma^+_\infty \times (\mathbb{C}P^\infty)^p.$$

When the mapping class groups are decorated by means of symmetric group actions on the sets of punctures or boundary components, a more general version of this argument holds [24].

7.3 Topological field theory

Historically, these groups have been considered important for a number of reasons, such as the properly discontinuous action referred to above, of $\Gamma^p_{g,0}$ on Teichmüller space with quotient the moduli space of isometry classes of complete hyperbolic metrics of fixed finite area on $F^p_{g,0}$. Representations of the groups $\Gamma_{g,q}$ have also been studied, as discussed below. More recently, there has been another source of motivation, resulting from attempts to axiomatize topological field theories in mathematical physics.

For G. Segal [133], a $1+1$ dimensional field theory roughly consists of a Hilbert space H naturally associated to the standard circle, together with an operator $\Psi_\Sigma : H^{\otimes m} \to H^{\otimes q}$ for each Riemann surface Σ with m incoming and q outgoing boundary circles. Underlying this is a topological category $\mathcal{M}$ for which an object is a finite set of m disjoint circles ($m \geq 0$). The morphisms from one such compact 1-manifold to another are given by cobordisms. Thus the space of morphisms has one connected component for each topological type of Riemann surface with the appropriate number of boundary circles. In Tillmann's work [147], the morphism spaces are replaced by morphism categories in which the objects are surfaces with $m+q$ boundary circles and the morphisms are (isotopy classes of) diffeomorphisms that fix the boundary. The resulting structure is a 2-category (lax, because gluing of cobordisms satisfies only a weak form of associativity) that inherits a symmetric monoidal structure from operations on surfaces such as those described above. When the morphism categories are in turn replaced by their classifying spaces, then general arguments give a connected, symmetric monoidal topological category. It follows from Theorem 3.9 above that its classifying space is an infinite loop space.

7.4 The plus-construction on the classifying space

It remains to relate this classifying space to $B\Gamma_\infty^+$. To do this, one invokes a group completion theorem, or rather, an extension of the version given above that is applicable to 2-categories. The necessary local homological input is provided by the homological stability results cited above. Because $B\Gamma_\infty^+$ is simply-connected, this gives homotopy equivalence of $B\Gamma_\infty^+ \times \mathbb{Z}$ with the loop space of the infinite loop space previously obtained. So the upshot is the following.

Theorem 7.4 [147] *$B\Gamma_\infty^+ \times \mathbb{Z}$ is an infinite loop space.*

7.4.1 Operads and group completion

An alternative approach [148] to this space uses the theory of operads [99]. An *operad* C is a collection of spaces $C = \coprod_{n\geq 0} C_n$ with an action of the symmetric group Σ_n on C_n for each $n \geq 0$ and product maps

$$\gamma : C_k \times C_{j_1} \times \cdots \times C_{j_k} \longrightarrow C_{j_1+\cdots+j_k}$$

for $k \geq 1$ and each $j_h \geq 0$. These maps satisfy certain compatibility conditions corresponding to associativity, unitality (with respect to $1 \in C_1$), and equivariance. A morphism of operads is a collection of equivariant maps that commute with the respective product maps. Then a based space X is a C-space if for each $j \geq 0$ there is a structure map

$$\theta_j : C_j \times X^j \longrightarrow X$$

obeying certain compatibility conditions with respect to the product maps γ of C. In particular, C_0 has a basepoint such that both C_0 and C are C-spaces. A map of C-spaces is required to commute with the structure maps. If $C \to D$ is a morphism of operads, then every D-space pulls back to a C-space. Another construction of C-spaces is that the loop space of a C-space is also a C-space.

When each C_j is a contractible space on which Σ_j acts freely, C is termed an E_∞ *operad*. The prototype is the *little cubes operad* Σ for which C_j

comprises j disjoint cubes in Euclidean space. Any C-space for an E_∞ operad C is called an E_∞ *space*. By means of a group completion theorem of [122], [100] shows that E_∞ spaces group-complete to infinite loop spaces.

Here is a good method for constructing operads, given in [148]. Suppose that $\{G_n\}_{n\geq 0}$ is a family of groups such that G_n maps onto Σ_n with kernel H_n, and such that there is a coherent collection of wreath product maps

$$G_m \wr G_n \longrightarrow G_{mn}.$$

Now for any group G with subgroup H, define $\mathcal{C}_H^G$ to be the category whose objects are the left cosets of H and morphism sets are each H, with $h \in H$ corresponding to the arrow $aH \to bH$ given by left multiplication by bha^{-1}. Then the union of classifying spaces $\bigsqcup_{n\geq 0} B\mathcal{C}_{H_n}^{G_n}$ forms an operad with free action of the symmetric groups. Evident examples are the *symmetric operad* Γ [11] obtained from the family $\{\Sigma_n, 1\}$, and *braid operad* $\mathcal{B}$ built from $\{B_n, P_n\}$ comprising respectively the n-strand braid group and its subgroup of pure braids. The surjections $B_n \twoheadrightarrow \Sigma_n$ induce an operad map $\mathcal{B} \to \Gamma$.

7.4.2 The CFT operad

By considering the groupoids $H_n = \bigsqcup_{g\geq 0} \Gamma_{g,n+1}$ and the extensions

$$H_n \longrightarrow G_n \longrightarrow \Sigma_n$$

obtained by allowing diffeomorphisms to permute boundary circles, Tillmann [148] constructs an operad $\mathcal{M}$ associated to Segal's category, with factorization

$$\mathcal{B} \longrightarrow \mathcal{M} \longrightarrow \Gamma.$$

The term $\mathcal{M}_0$ of $\mathcal{M}$ is the monoid $\bigsqcup_{g\geq 0} B\Gamma_{g,1}$, where multiplication is given by pair-of-pants adjunction. On the one hand, the group completion theorem of [122] shows that $\mathcal{M}_0$ group-completes to $\mathbb{Z}\times B\Gamma_\infty^+$ [38]. On the other, it is shown that, up to homotopy, group completions of $\mathcal{M}$-spaces are the same thing as Γ-spaces, which are known to be the same as infinite loop spaces [10]. Since $\mathcal{M}_0$ is an $\mathcal{M}$-space, there follows an infinite delooping of $\mathbb{Z}\times B\Gamma_\infty^+$ that is compatible with the multiplication on $\mathcal{M}_0$. Moreover, it has subsequently been shown to agree with the previous infinite loop space structure [155].

[11] Notation historic, but, in the present context, unfortunate.

7.4.3 Thom space maps

Our purpose now is to set up a conjecture that topologically describes the space $\mathbb{Z} \times B\Gamma_\infty^+$. From a good description, Mumford's conjecture on the rational cohomology of $B\Gamma_\infty^+$ should follow readily.

Conjecture 7.5 [96] *There is a homotopy equivalence*

$$\alpha_\infty : \mathbb{Z} \times B\Gamma_\infty^+ \longrightarrow \Omega^\infty \mathbb{C}P_{-1}^\infty$$

where the space $\Omega^\infty \mathbb{C}P_{-1}^\infty$ is defined below.

In the eyes of Madsen, this conjecture bears the same kind of relationship to the Mumford conjecture that reformulations in terms of spaces have to the original Lichtenbaum-Quillen conjecture.

Let L_r be the canonical complex line (Hopf) bundle over $2r$-dimensional complex projective space $\mathbb{C}P^r$, and write ε_r for the trivial complex line bundle. Then for any integer m the class $m[L_r] + (r+1-m)[\varepsilon_r] \in K^0(\mathbb{C}P^r)$ has a unique (up to bundle equivalence) $(r+1)$-dimensional (over $\mathbb{C}$) bundle representative stbl_{mL}^r – its *stable bundle*. Next recall that the Thom space $\mathrm{Th}(E)$ of a vector bundle E over a paracompact space is obtained by collapsing all points with fibre norm greater than 1 (the complement of the unit disk bundle); over a compact space this is the same as the one-point compactification $E_{\sqcup\infty}$ of E. Since both the line bundles L and ε restrict to their counterparts via the inclusion $i : \mathbb{C}P^r \hookrightarrow \mathbb{C}P^{r+1}$, we have an inclusion of bundles

$$\begin{array}{ccc} \varepsilon_r \oplus \mathrm{stbl}_{mL}^r & \hookrightarrow & (\mathrm{stbl}_{mL}^{r+1}) \\ \downarrow & & \downarrow \\ \mathbb{C}P^r & \hookrightarrow & \mathbb{C}P^{r+1} \end{array}$$

and thereby a map of Thom spaces

$$\Sigma^2 \mathrm{Th}(\mathrm{stbl}_{mL}^r) = \mathrm{Th}(\varepsilon_r \oplus \mathrm{stbl}_{mL}^r) \longrightarrow \mathrm{Th}(\mathrm{stbl}_{mL}^{r+1})$$

with adjoint

$$\mathrm{Th}(\mathrm{stbl}_{mL}^r) \longrightarrow \Omega^2 \mathrm{Th}(\mathrm{stbl}_{mL}^{r+1}).$$

So for suitable integers a, the colimit $\underset{r\to\infty}{\mathrm{colim}}\,\Omega^{2r+a}\mathrm{Th}(\mathrm{stbl}_{mL}^r)$ is thus defined, and an infinite loop space. When $0 \leq m \leq r+1$, it can be expressed in terms of stunted projective spaces $\mathbb{C}P_m^{m+r} = \mathbb{C}P^{m+r}/\mathbb{C}P^{m-1}$, for by [6]

$$\mathrm{Th}(\mathrm{stbl}_{mL}^r) = \mathrm{Th}(mL_r \oplus (r+1-m)\varepsilon_r) = \Sigma^{2r+2-2m}\mathbb{C}P_m^{m+r}.$$

In the case of interest here, $m = -1$, stbl^r_{-L} (otherwise written as $L_r^\perp$ – see [95]) may be written as

$$\varepsilon_r \oplus \mathrm{Hom}(L_r,\ \tau(\mathbb{C}P^r)),$$

a form less amenable to calculation. Nevertheless, based no doubt on considerations of the easier cases above, the infinite loop space $\underset{r\to\infty}{\mathrm{colim}}\Omega^{2r+2}\mathrm{Th}(\mathrm{stbl}^r_{-L})$ receives the notation $\Omega^\infty\mathbb{C}P^\infty_{-1}$.

In the case of $F = F^0_{g,0}$ where $g \geq 2$, we now indicate how every map from a compact manifold B to $B\Gamma^0_{g,0}$ determines a map from B to $\Omega^\infty\mathbb{C}P^\infty_{-1}$. This gives an approximation to α above, just as, when A_1 is a f.g. projective A-module via a homomorphism $A \to A_1$, one constructs a transfer $K_*A_1 \to K_*A$ by means of a sequence of maps $B\mathrm{GL}_n A_1^+ \to B\mathrm{GL}A^+$ [14]. From the fact that $B\Gamma^0_{g,0} \simeq B\mathrm{Diff}_+(F)$ noted earlier, we may pull back the universal F-bundle

$$F \longrightarrow E\mathrm{Diff}_+(F) \times_{\mathrm{Diff}_+(F)} F \longrightarrow B\mathrm{Diff}_+(F)$$

via the map from B, inducing a smooth fibre bundle

$$F \longrightarrow E \xrightarrow{\pi} B.$$

Since E is also a compact manifold, it has a smooth embedding i in $\mathbb{C}^{r+1}$ for $2r > \dim E$. This gives a factorization of the projection π as

$$\pi : E \overset{\pi_0 = (\pi, i)}{\hookrightarrow} B \times \mathbb{C}^{r+1} \twoheadrightarrow B.$$

We then have over E the bundles

$$\pi^*\tau B \oplus \tau_\pi E \oplus \nu(\pi_0) = \tau E \oplus \nu(\pi_0) = \pi_0^*\tau(B \times \mathbb{C}^{r+1}) = \pi^*\tau B \oplus (r+1)\varepsilon_E$$

corresponding to pull-backs of tangent bundles τ, with normal bundle ν, and $\tau_\pi E$ denoting the tangent bundle along the fibres. Because $\dim E < 2r$ we may cancel the common summand $\pi^*\tau B$, to obtain a bundle trivialization

$$\tau_\pi E \oplus \nu(\pi_0) = (r+1)\varepsilon_E,$$

whence $\nu(\pi_0)$ may be thought of as representing the 'stable normal bundle along the fibres'. In this guise, while the complex line bundle $\tau_\pi E = f^*L_r$ for some classifying map $f : E \to \mathbb{C}P^r$, so $\nu(\pi_0) = f^*\mathrm{stbl}^r_{-L}$.

We now have two different descriptions of the vector bundle $\nu(\pi_0)$, inducing two maps of Thom spaces. First, by viewing it as the normal bundle for the embedding $\pi_0 : E \hookrightarrow B \times \mathbb{C}^{r+1}$, we may collapse the complement of a tubular neighbourhood of E in $B \times \mathbb{C}^{r+1}$. This gives a map

$$\Sigma^{2r+2} B_+ = (B \times \mathbb{C}^{r+1})_{\sqcup \infty} \longrightarrow \mathrm{Th}\nu(\pi_0).$$

Second, the bundle map over f induces a map of Thom spaces

$$\mathrm{Th}\nu(\pi_0) \longrightarrow \mathrm{Th}(\mathrm{stbl}^r_{-L}).$$

The composite $\Sigma^{2r+2} B_+ \longrightarrow \mathrm{Th}\nu(\pi_0) \longrightarrow \mathrm{Th}(\mathrm{stbl}^r_{-L})$ then has as adjoint a map $B \to \Omega^{2r+2}\mathrm{Th}(\mathrm{stbl}^r_{-L})$, which in turn passes to the stabilization $\Omega^\infty \mathbb{C}P^\infty_{-1}$ as required.

In June 2002, I. Madsen and M. Weiss announced a proof of this conjecture. However, the preprint, [97], is not yet available. One feature of their work is the gluing operation

$$\Gamma_{g,2} \times \Gamma_{g',2} \longrightarrow \Gamma_{g+g',2}$$

giving rise to a topological monoid $\coprod_g B\Gamma_{g,2}$ whose group completion is $\mathbb{Z} \times B\Gamma^+_\infty$. Other innovations draw heavily from differential topology.

7.5 Link to K-theory

The main reason for the significance of the study of the space $B\Gamma^+_\infty$ in this lecture is its map to $B\mathrm{GL}\mathbb{Z}^+$. This comes about from the action of the group of (diffeotopy classes of) diffeomorphisms of a manifold on its integral cohomology ring. In the case of $F^0_{g,\varepsilon}$ (where $\varepsilon \in \{0,1\}$), this is the same thing as the action of $\Gamma_{g,\varepsilon}$ on $H_1(F^0_{g,\varepsilon};\mathbb{Z}) = \mathbb{Z}^{2g}$ so as to preserve the intersection pairing. Thus the action gives representations

$$\Gamma_{g,\varepsilon} \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \hookrightarrow \mathrm{SL}_{2g}(\mathbb{Z}),$$

where, for a commutative ring R,

$$\mathrm{Sp}_{2g}(R) = \left\{ \alpha \in \mathrm{GL}_{2g}(R) \mid \sigma^{-1}\alpha\sigma = {}^{\mathrm{t}}\alpha^{-1} \right\}$$

with

$$\sigma = \begin{bmatrix} O & -I_g \\ I_g & O \end{bmatrix}$$

and ${}^{t}\alpha$ denoting the transpose of α; for surjectivity of the first map, see [98]p.178. Now the maximal compact subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ is the unitary group $U(g)$ embedded via

$$A+iB \longmapsto \begin{bmatrix} A & -B \\ B & A \end{bmatrix}.$$

Stabilization and passage to classifying spaces lead to the following diagram.

$$\begin{array}{ccccccccc} & & \Omega^\infty \mathbb{C}P^\infty_{-1} & \xrightarrow{\eta'} & BU & & & & \\ & \nearrow^{\alpha_\infty} & & & \downarrow^{\simeq} & & & & \\ B\Gamma^+_\infty & \longrightarrow & B\mathrm{Sp}\mathbb{Z}^+ & \longrightarrow & B\mathrm{Sp}\mathbb{R} & \longrightarrow & B\mathrm{Sp}\mathbb{C} & = & B\mathrm{Sp} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Im}J_{[1/2]} & \leftrightarrows & B\mathrm{GL}\mathbb{Z}^+ & \longrightarrow & B\mathrm{GL}\mathbb{R} & \longrightarrow & B\mathrm{GL}\mathbb{C} & \xleftarrow{\simeq} & BU \end{array} \tag{7-5}$$

Although there is no canonical way to perform the stabilization

$$\mathrm{Sp}_{2g}(R) \hookrightarrow \mathrm{Sp}_{2g+2}(R),$$

the various choices are conjugate and so induce homotopy equivalences of the resulting $B\mathrm{Sp}(R)^+$. The vertical arrows of the diagram are all induced from inclusions, except for the leftmost, which is by composition. The map η' of [96] is given by a compatible family of elements of $[\mathrm{Th}(\mathrm{stbl}^r_{-L}), BU] = \tilde{K}(\mathrm{Th}(\mathrm{stbl}^r_{-L}))$ corresponding under the Thom isomorphism to the class of L_r in $K(\mathbb{C}P^r)$. [96] conjectures that the top polygon commutes.

It is intriguing to compare the above diagram with the data from the previous lecture (see Theorem 6.10):

$$\begin{array}{ccccc} B\Sigma^+_\infty & \longrightarrow & BO\mathbb{Z}^+ & \longrightarrow & BO \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Im}J_{[1/2]} & \rightleftarrows & B\mathrm{GL}\mathbb{Z}^+ & \longrightarrow & B\mathrm{GL}\mathbb{R} \end{array}$$

Prompted by Mumford's conjecture, there has been much interest in the cohomology classes of $B\Gamma^+_\infty$ obtained from the universal Chern classes c_j of the top BU. Observe that on passage to $\mathrm{GL}_{2g}(\mathbb{C})$ an inner automorphism of $\mathrm{GL}_{2g}(\mathbb{C})$ takes the above embedding of $U(g)$ in $\mathrm{Sp}_{2g}(\mathbb{R})$ to

$$P = A+iB \longmapsto P \oplus \bar{P} \in \mathrm{GL}_{2g}(\mathbb{C}).$$

From this it follows that the c_{2i} can be expressed in terms of the Pontrjagin classes p_i coming from the lower BU and the classes c_{2k-1}. Now, because real Pontrjagin classes are definable in terms of curvature and so vanish on flat bundles, the p_i have finite order over the classifying space of the discrete group $\mathrm{Sp}\mathbb{Z}$. Hence the rational cohomology of $B\Gamma_\infty^+$ is detected by the c_{2k-1}. It is shown in [102] and [110] that the polynomial algebra on these classes injects into $H^*(B\Gamma_\infty^+;\mathbb{Q})$; moreover, the images of these classes is related to the generators posited by Mumford by means of formulae involving the Bernoulli numbers. This provides half of Mumford's conjectured generators. To obtain the others, [97] constructs an alternative map from $\Omega^\infty\mathbb{C}P_{-1}^\infty$ to BU.

On the other hand (via the above argument about Pontrjagin classes), it is through torsion that the lower half of diagram (7-5) above affects the homotopy of $B\Gamma_\infty^+$. For example, [37] shows that, just as $B\mathrm{GL}\mathbb{Z}^+$ splits off $\mathrm{Im}J_{[1/2]}$ (see (6.4), (6.2) above), so stably does $B\Gamma_\infty^+$. Hence, much of the torsion of $K_*(\mathbb{Z})$ reappears in $\pi_*(B\Gamma_\infty^+)$.

These results make $B\mathrm{Sp}\mathbb{Z}^+$ a hot topic for further study, and raise interesting questions. For example, to what extent is there a counterpart of Theorem 6.15 based on the pull-back of

$$\begin{array}{ccc} & & BU \\ & & \downarrow \\ B\mathrm{SL}(\mathbb{F}_q)^+ & \longrightarrow & B\mathrm{Sp}\,; \end{array}$$

does the topology give a natural construction of the map

$$\Omega^\infty\mathbb{C}P_{-1}^\infty \longrightarrow B\mathrm{Sp}\mathbb{Z}^+$$

indicated by [97]?

There are two other famous groups that one can bring into the picture here. Let Br_g denote the g-strand braid group. This has presentation

$$\mathrm{Br}_g = \langle \sigma_i \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1};\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ i+1<j\rangle_{1\le i,j<g}$$

It admits Artin's representation Artin of Br_g as automorphisms of the free group Fr_g on g generators. Algebraically, this sends each σ_i to the automorphism of the free group on $x_1, x_2, \ldots, x_g$ that maps

$$x_j \longmapsto \begin{cases} x_{i+1} & j=i \\ x_i & j=i+1 \\ x_j & j\ne i, i+1. \end{cases}$$

An adaptation of this embedding is the monomorphism

$$\psi_g : \mathrm{Br}_g \to \Gamma_{g,1}$$

constructed by Vershinin [151] as follows. This exploits the facts that $\Gamma_{g,q}$ corresponds to an outer automorphism group of $\pi_1(F_{g,q})$ and that we may write

$$\pi_1(F_{g,1}) = \mathrm{Fr}_{2g} = \langle a_1, b_1, \ldots, a_g, b_g \rangle\,;$$

in this case $\Gamma_{g,1} \cong \mathrm{Stab}_{\mathrm{Aut}(\mathrm{Fr}_{2g})}(c_1 \cdots c_g)$ where c_j is the commutator $a_j b_j a_j^{-1} b_j^{-1}$. Then ψ_g sends each σ_i to the automorphism of Fr_{2g} given by

$$a_j \longmapsto \begin{cases} a_{i+1} & j = i \\ c_{i+1}^{-1} a_i c_{i+1} & j = i+1 \\ a_j & j \neq i, i+1 \end{cases}$$

$$b_j \longmapsto \begin{cases} b_{i+1} & j = i \\ c_{i+1}^{-1} b_i c_{i+1} & j = i+1 \\ b_j & j \neq i, i+1, \end{cases}$$

which, being easily seen to stabilize $c_1 \cdots c_g$, may be considered as an element of $\Gamma_{g,1}$. Now define Δ to be the map sending a matrix A to the matrix $\begin{bmatrix} A & O \\ O & {}^{\mathrm{t}}A^{-1} \end{bmatrix}$; of course, when A is orthogonal this is just the diagonal. We thereby have the following commutative diagram.

$$\begin{array}{ccccccc}
 & & \Gamma_{g,1} & \xrightarrow{\mathrm{id}} & \Gamma_{g,1} & & \\
 & \nearrow \psi_g & & & & \searrow & \\
\mathrm{Br}_g & \twoheadrightarrow & \Sigma_g & \rightarrowtail & O_g(\mathbb{Z}) & \xrightarrow{\Delta} & \mathrm{Sp}_{2g}(\mathbb{Z}) \\
 & \searrow \mathrm{Artin} & & & \downarrow & & \downarrow \\
 & & \mathrm{Aut}(\mathrm{Fr}_g) & \twoheadrightarrow & \mathrm{GL}_g(\mathbb{Z}) & \xrightarrow{\Delta} & \mathrm{GL}_{2g}(\mathbb{Z})
\end{array}$$

Here all maps are injective except for the surjections $\mathrm{Aut}(\mathrm{Fr}_g) \twoheadrightarrow \mathrm{GL}_g(\mathbb{Z})$, $\mathrm{Br}_g \twoheadrightarrow \Sigma_g$ and $\Gamma_{g,1} \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$.

Now, combine with the inclusions in the symplectic and general linear real and complex groups, stabilize, and take $B(\,\cdot\,)^+$, to get

$$
\begin{array}{ccccccccccc}
 & & B\Gamma_\infty^+ & \stackrel{\mathrm{id}}{\to} & B\Gamma_\infty^+ & & & & BU & & B\mathrm{Sp} \\
 & \nearrow & & & & \searrow & & & \downarrow^{\simeq} & & \downarrow^{=} \\
B\mathrm{Br}_\infty^+ & \to & B\Sigma_\infty^+ & \to & BO(\mathbb{Z})^+ & \to & B\mathrm{Sp}(\mathbb{Z})^+ & \to & B\mathrm{Sp}\mathbb{R} & \to & B\mathrm{Sp}\mathbb{C} \\
 & \searrow & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & B\mathrm{Aut}(\mathrm{Fr}_\infty)^+ & \to & B\mathrm{GL}(\mathbb{Z})^+ & \to & B\mathrm{GL}(\mathbb{Z})^+ & \to & B\mathrm{GL}\mathbb{R} & \to & B\mathrm{GL}\mathbb{C} \\
 & & & & & & & & & & \uparrow^{\simeq} \\
 & & & & & & & & & & BU
\end{array}
$$

Here for instance, after [131] one has $B\mathrm{Br}_\infty^+ \simeq \Omega^2 S^3$, with the map $B\mathrm{Br}_\infty^+ \to B\Gamma_\infty^+$ a double loop space map [151]. Moreover, Greenberg and Sergiescu [60] echoed the procedure for delooping $B\mathrm{GL}R^+$ in Theorem 2.12 above, to set up a group extension

$$\mathrm{Br}_\infty \rightarrowtail A \twoheadrightarrow F'.$$

Here the group A is acyclic such that, on passing to the associated fibration of classifying spaces and taking the plus-construction, one obtains the path fibration

$$\Omega^2 S^3 \longrightarrow P\Omega S^3 \longrightarrow \Omega S^3.$$

More broadly, the challenge is to describe the spaces and maps of the above diagram in homotopy theory.

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