

# HOMOTOPY IDEMPOTENTS ON MANIFOLDS AND BASS' CONJECTURES

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ABSTRACT. The Bass trace conjectures are placed in the setting of homotopy idempotent selfmaps of manifolds. For the strong conjecture, this is achieved via a formulation of Geoghegan. The weaker form of the conjecture is reformulated as a comparison of ordinary and  $L^2$ -Lefschetz numbers.

## PREFACE

This note has its origins in talks discussing Bass' trace conjecture. After one such lecture (by IC), R. Geoghegan kindly mentioned his geometric perspective on the matter. Then, when another of us (AJB) spoke about the conjecture at the Kinosaki conference, he thought that a topological audience might like to hear about that geometric aspect. Thus, it seemed desirable to attempt to put the conjecture (and its weaker version) in a setting that would be as motivating as possible to topologists. The result of that attempt appears below. The authors warmly thank both R. Geoghegan and B. J. Jiang for their interest in this work.

## 0. INTRODUCTION

In 1976, H. Bass [2] conjectured that for any discrete group  $G$ , the Hattori-Stallings trace of a finitely generated projective module over the integral group ring of  $G$  should be supported on the identity component only. Despite numerous advances (see e.g. [6], [10] and [3]), this conjecture remains open in general. In [12], R. Geoghegan gave the first topological interpretation, in terms of Nielsen numbers (stated as Theorem 3.4 below). In the setting of selfmaps on manifolds, this translates to the following.

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**Theorem 1.** *The following are equivalent.*

- (a) *The Bass conjecture is a theorem.*
- (b) *Every homotopy idempotent selfmap of a closed, smooth and oriented manifold of dimension greater than 2 is homotopic to one that has precisely one fixed point.*

Throughout this paper we use the word “closed” to refer to a connected, compact manifold without boundary. Background material on homotopy idempotents and related invariants will be discussed in Section 2 and 3. A weaker version of Bass’ conjecture amounts to saying that for any group  $G$ , the coefficients of the non-identity components of the Hattori-Stallings trace of a finitely generated projective module over the integral group ring of  $G$  should sum to zero (and not necessarily be individually zero).

**Theorem 2.** *The following are equivalent.*

- (a) *The weak Bass conjecture is a theorem.*
- (b) *Every pointed homotopy idempotent selfmap of a closed, smooth and oriented manifold inducing the identity on the fundamental group, has its Lefschetz number equal to the  $L^2$ -Lefschetz number of the induced map on its universal cover.*

Background material regarding  $L^2$ -Lefschetz numbers is explained in Section 6. The implication (a)  $\Rightarrow$  (b) had already been observed by Eckmann in [7] in a slightly different form. The proofs of these two theorems proceed as follows. Theorem 1 is derived from the analogous statement for finite CW-complexes (involving Nielsen numbers) from Geoghegan’s work, which is explained in Section 3. The transition from CW-complexes to manifolds is done in Section 4. The proof of Theorem 1 as well as some applications is discussed in Section 5. For Theorem 2 the strategy is somewhat similar: namely, we first prove the statements for finitely presented groups instead of arbitrary groups and for finite complexes instead of manifolds (see Section 7). To deduce Bass’ conjectures (weak and classical) for arbitrary groups we use a remark due to Bass (Lemma 5.3).

## 1. REVIEW OF BASS’ CONJECTURES

We briefly recall Bass’ conjectures. Let  $\mathbb{Z}G$  denote the integral group ring of a group  $G$ . The *augmentation trace* is the  $\mathbb{Z}$ -linear map

$$\epsilon : \mathbb{Z}G \rightarrow \mathbb{Z}, \quad g \mapsto 1$$

induced by the trivial group homomorphism on  $G$ . Writing  $[\mathbb{Z}G, \mathbb{Z}G]$  for the additive subgroup of  $\mathbb{Z}G$  generated by the elements  $gh - hg$  ( $g, h \in G$ ), we identify  $\mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$  (the Hochschild homology group  $HH_0(\mathbb{Z}G)$ ) with  $\bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$ , where  $[G]$  is the set of conjugacy classes

$[s]$  of elements  $s$  of  $G$ . The *Hattori-Stallings trace* of  $M = \sum_{g \in G} m_g g \in \mathbb{Z}G$  is then defined by

$$\begin{aligned} \text{HS}(M) &= M + [\mathbb{Z}G, \mathbb{Z}G] \\ &= \sum_{[s] \in [G]} \epsilon_s(M)[s] \in \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s], \end{aligned}$$

where for  $[s] \in [G]$ ,  $\epsilon_s(M) = \sum_{g \in [s]} m_g$  is a partial augmentation. In particular, the component  $\epsilon_e$  of the identity element  $e \in G$  in the Hattori-Stallings trace is known as the *Kaplansky trace*

$$\kappa : \mathbb{Z}G \rightarrow \mathbb{Z}, \quad \sum m_g g \mapsto m_e.$$

Now, an element of  $K_0(\mathbb{Z}G)$  is represented by a difference of finitely generated projective  $\mathbb{Z}G$ -modules, each of which is determined by an idempotent matrix having entries in  $\mathbb{Z}G$ . Combining the usual trace map to  $\mathbb{Z}G$  of such a matrix with any of the above traces on  $\mathbb{Z}G$  turns out to induce a well-defined trace map on  $K_0(\mathbb{Z}G)$  that is given the same name and notation as before. Moreover, HS and  $\epsilon$  are natural with respect to all group homomorphisms (and  $\kappa$  with respect to group monomorphisms). In the case of a free module  $\mathbb{Z}G^n$ ,  $\epsilon$  takes the value  $n$  and so is just the rank of the module.

In [2], Bass conjectured the following.

**Conjecture 1** (Classical Bass conjecture). *For any group  $G$ , the induced map*

$$\text{HS} : K_0(\mathbb{Z}G) \rightarrow \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$$

*has image in  $\mathbb{Z} \cdot [e]$ .*

**Conjecture 2** (Weak Bass conjecture). *For any group  $G$ , the induced maps*

$$\epsilon, \kappa : K_0(\mathbb{Z}G) \rightarrow \mathbb{Z}$$

*coincide.*

To clarify the discussion below, it is helpful to consider also reduced  $K$ -groups. The inclusion  $\langle e \rangle \hookrightarrow G$  induces a natural homomorphism

$$\mathbb{Z} = K_0(\mathbb{Z}) = K_0(\mathbb{Z}\langle e \rangle) \longrightarrow K_0(\mathbb{Z}G)$$

whose cokernel is the *reduced  $K$ -group*  $\tilde{K}_0(\mathbb{Z}G)$ , equipped with natural epimorphism  $\eta : K_0(\mathbb{Z}G) \twoheadrightarrow \tilde{K}_0(\mathbb{Z}G)$ .

## 2. HOMOTOPY IDEMPOTENT SELFMAPS

Let  $X$  be a connected CW-complex. A selfmap  $f : X \rightarrow X$  is called *homotopy idempotent* if  $f^2 = f \circ f$  is *freely* homotopic to  $f$ . Since  $X$  is path-connected we can always assume that  $f$  fixes a basepoint  $x_0 \in X$ , so that  $f$  induces a (not necessarily idempotent) map  $f_{\sharp} :$

$\pi_1(X) \rightarrow \pi_1(X)$ . Given a homotopy idempotent selfmap  $f : X \rightarrow X$  on a *finite dimensional CW-complex*  $X$ , according to Hastings and Heller [14] there is a CW-complex  $Y$  and maps  $u : X \rightarrow Y$  and  $v : Y \rightarrow X$  such that the following diagram is (freely) homotopy commutative:

$$(2.1) \quad \begin{array}{ccccc} & & f & & \\ & \curvearrowright & & \curvearrowleft & \\ X & \xrightarrow{f} & X & \xrightarrow{f} & X \\ & \searrow u & \nearrow v & \searrow u & \nearrow v \\ & & Y & \xrightarrow{\text{id}} & Y \end{array}$$

In fact, in this diagram we can arrange that the outside triangles strictly commute. By replacing the maps by homotopic ones, we can (and do) choose the maps to preserve basepoints. We then get the following commutative diagram of groups:

$$\begin{array}{ccccc} \pi_1(X) & \xrightarrow{f_{\#}} & \pi_1(X) & \xrightarrow{f_{\#}} & \pi_1(X) \\ & \searrow u_{\#} & \nearrow v_{\#} & \searrow u_{\#} & \nearrow v_{\#} \\ & & \pi_1(Y) & \xrightarrow{\quad} & \pi_1(Y) \end{array}$$

Here the bottom arrow consists of conjugation by the class of a loop at the basepoint of  $Y$ . Looking at the middle triangle, we see that  $v_{\#}$  is an injective homomorphism while  $u_{\#}$  is surjective; hence we can make the identification

$$\pi_1(Y) \cong v_{\#}(\pi_1(Y)) = \text{Im}(f_{\#}) \leq \pi_1(X).$$

If the homotopy idempotent  $f$  is a pointed homotopy idempotent (meaning that  $f^2$  is pointed homotopic to  $f$ ), then  $u \circ v : Y \rightarrow Y$  will induce the identity on  $\pi_1(Y)$ . If we require that  $f_{\#} = \text{id}$ , we then get that  $\pi_1(Y)$  is isomorphic to  $\pi_1(X)$  via  $v_{\#} = u_{\#}^{-1}$ .

We now explain how, starting from a homotopy idempotent  $f : X \rightarrow X$  of a finite connected complex  $X$  with fundamental group  $G = \pi_1(X)$ , we obtain an element  $w(f) \in K_0(\mathbb{Z}G)$ . In the situation above,  $Y$  is called *finitely dominated*; then the singular chain complex of the universal cover  $\tilde{Y}$  of  $Y$  is chain homotopy equivalent to a complex of type FP over  $\mathbb{Z}\pi_1(Y)$

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}$$

with each  $P_i$  a finitely generated projective  $\mathbb{Z}\pi_1(Y)$ -module. We then look at the *Wall element*

$$w(Y) = \sum_{i=0}^n (-1)^i [P_i] \in K_0(\mathbb{Z}\pi_1(Y))$$

(where we follow the notation of [25]). Its image  $\tilde{w}(Y) = \eta(w(Y)) \in \tilde{K}_0(\mathbb{Z}\pi_1(Y))$  is known as Wall's *finiteness obstruction*, and  $\tilde{w}(Y) = 0$

exactly when  $Y$  is homotopy equivalent to a finite complex. Finally, we define

$$w(f) = v_{\#}(w(Y)) \in K_0(\mathbb{Z}G),$$

whose reduction  $\tilde{w}(f) \in \tilde{K}_0(\mathbb{Z}G)$  was first considered by Geoghegan [12]. As he notes, the element  $\tilde{w}(f)$  “can be interpreted as the obstruction to splitting  $f$  through a finite complex”. Before proceeding, we check that  $w(f)$  is well-defined. First, we observe a form of naturality of Wall elements.

**Lemma 2.1.** *Let  $X$  be a finite  $n$ -dimensional complex, and suppose that there are maps (of spaces having the homotopy type of a connected CW-complex)*

$$X \xrightarrow{u} W \xrightarrow{a} V \xrightarrow{v} X$$

*such that  $a : W \rightarrow V$  and  $u \circ v : V \rightarrow W$  are homotopy inverse. Then the Wall elements  $w(W) \in K_0(\mathbb{Z}\pi_1(W))$  and  $w(V) \in K_0(\mathbb{Z}\pi_1(V))$  are related by*

$$w(V) = a_{\#}w(W).$$

**Proof.** We use the fact that, because conjugation in  $G$  induces the identity map on  $K_0(\mathbb{Z}G)$ , homotopic maps induce the same homomorphism of  $K$ -groups. Recall from [29] that  $w(W)$  is defined (uniquely) by means of any  $n$ -connected map  $\psi : L \rightarrow W$  where  $L$  is a finite  $n$ -dimensional complex:

$$w(W) = (-1)^n [\pi_{n+1}(M_{\psi}, L)]$$

where  $M_{\psi}$  denotes the mapping cylinder of  $\psi$ , and the relative homotopy group is considered as a  $\pi_1(W)$ -module (finitely generated and projective because of the assumption that  $W$  is dominated by  $X$ ). Therefore, to define  $w(V)$ , we may take

$$w(V) = (-1)^n [\pi_{n+1}(M_{p\psi}, L)].$$

The result then follows from the natural isomorphism of the exact homotopy sequences (of  $\pi_1$ -modules) of the pairs  $(M_{\psi}, L)$  and  $(M_{p\psi}, L)$  induced by  $a$ .  $\square$

We now can see that the obstruction to splitting a homotopy idempotent through a finite complex is well defined.

**Lemma 2.2.** *Let  $X$  be a finite complex with fundamental group  $G$ , and for  $i = 1, 2$  let  $X \xrightarrow{u_i} Y_i \xrightarrow{v_i} X$  be a (homotopy) splitting of a homotopy idempotent map  $f : X \rightarrow X$ . Then in  $K_0(\mathbb{Z}G)$*

$$v_{1\#}(w(Y_1)) = v_{2\#}(w(Y_2)).$$

**Proof.** From the homotopy commutative diagram

$$\begin{array}{ccccc}
 & & Y_1 & \xrightarrow{\text{id}} & Y_1 & \xrightarrow{\text{id}} & Y_1 \\
 & \nearrow & & \searrow & \nearrow & \searrow & \nearrow \\
 X & \xrightarrow{f} & X & \xrightarrow{f} & X & \xrightarrow{f} & X \\
 & \searrow & & \nearrow & \searrow & \nearrow & \searrow \\
 & & Y_2 & \xrightarrow{\text{id}} & Y_2 & \xrightarrow{\text{id}} & Y_2
 \end{array}$$

we deduce from a simple diagram chase that  $a := u_2 \circ v_1 : Y_1 \rightarrow Y_2$  and  $b := u_1 \circ v_2 : Y_2 \rightarrow Y_1$  are mutually inverse homotopy equivalences. Therefore

$$v_2 \circ a \circ u_1 : X \rightarrow Y_1 \rightarrow Y_2 \rightarrow X$$

is such that  $a$  is homotopy inverse to  $u_1 \circ v_2 : Y_2 \rightarrow Y_1$ ; and thus, by Lemma 2.1,  $a_{\#}(w(Y_1)) = w(Y_2)$ . It then follows that

$$v_{1\#}(w(Y_1)) = v_{1\#} \circ a_{\#}^{-1}(w(Y_2)) = v_{1\#} \circ u_{1\#} \circ v_{2\#}(w(Y_2)) = f_{\#} \circ v_{2\#}(w(Y_2)).$$

Similarly

$$v_{2\#}(w(Y_2)) = f_{\#} \circ v_{1\#}(w(Y_1)).$$

Then substituting in the previous formula, and using idempotency of  $f_{\#}$ , gives the result.  $\square$

The key fact for giving a topological meaning to the Bass conjecture is the following, which can be extracted from [29] Theorem F in the light of the above. It is also shown explicitly in [24].

**Theorem 2.3.** *Let  $G$  be a finitely presented group, let  $\tilde{\alpha} \in \tilde{K}_0(\mathbb{Z}G)$ , and let  $n \geq 2$ . Then there is a finite  $n$ -dimensional complex  $X^n$  with fundamental group  $G$  and a pointed homotopy idempotent selfmap  $f$  of  $X^n$  inducing the identity on  $\pi_1$ , such that  $\tilde{w}(f)$  is equal to  $\tilde{\alpha}$ .  $\square$*

**Remark 2.4.** For  $n \geq 3$ , the unreduced version of this result also holds. For, given  $\alpha \in K_0(\mathbb{Z}G)$ , then choose a map  $f$  as in the theorem with respect to  $\tilde{\alpha} = \eta(\alpha)$ . It follows that for some nonnegative  $r, s$  we have  $w(f) = \alpha + [\mathbb{Z}G]^r - [\mathbb{Z}G]^s$ . Replacing  $f$  by  $f \vee \text{id}_W$  where  $W = (\bigvee_r S^3) \vee (\bigvee_s S^2)$  then gives the desired selfmap. When  $n = 2$ , the method fails, as without the possibility of adjoining a simply-connected space of non-positive Euler characteristic we can only increase the rank of the Wall element. (Recall from [25] Lemma 5.1 that for any finitely dominated space  $Y$ , the rank of  $w(Y)$  equals  $\chi(Y)$ .)

### 3. INVARIANTS FOR SELFMAPS OF COMPLEXES

We recall from for example [12], [13] the definition of the *Nielsen number*  $N(f)$  of a selfmap  $f : X \rightarrow X$  of a finite connected complex (assumed, as discussed above, to fix a basepoint of  $X$ ). Let  $f_{\#}$  be the

endomorphism of  $G = \pi_1(X, x)$  induced by  $f$ . Define elements  $\alpha$  and  $\beta$  of  $G$  to be  $f_{\#}$ -conjugate if for some  $z \in G$

$$\alpha = z \cdot \beta \cdot (f_{\#}z)^{-1},$$

and let  $G_{f_{\#}}$  denote the set of  $f_{\#}$ -conjugacy classes, making  $\mathbb{Z}G_{f_{\#}}$  a quotient of  $\mathbb{Z}G$ . Now  $N(f)$  is defined to be the number of nonzero coefficients in the formula for the *Reidemeister trace* of  $f$  at  $x \in X$ :

$$R(f, x) = \sum_{C \in G_{f_{\#}}} n_C \cdot C \in \mathbb{Z}G_{f_{\#}}.$$

The coefficient  $n_C$  can be described geometrically as the fixed-point index of a fixed-point class of  $f$ , and homologically in terms of traces of the homomorphisms induced by  $f$  on the chain complex of the universal cover of  $X$ . In the literature,  $R(f, x)$  is also known as the *generalized Lefschetz number*.

When, as prompted by Theorem 2.3 above, we take  $f$  to induce the identity on  $\pi_1$ , then  $R(f, x) \in \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$  and the following holds (cf. [13], p. 505).

**Lemma 3.1.** (Geoghegan). *In the setting of diagram (2.1) of Section 2 where  $X$  is a finite connected complex, and  $f$  is a pointed homotopy idempotent selfmap inducing the identity map on the fundamental group,*

$$\text{HS}(w(f)) = R(f, x).$$

For the computation of Nielsen numbers, the following result also proved in [17] p. 20, attributed to Fadell, is useful.

**Lemma 3.2.** *Suppose that the diagram of finite connected complexes and based maps*

$$\begin{array}{ccccc} \bar{T} & \xrightarrow{\bar{g}} & \bar{T} & & \\ \downarrow r & & \uparrow s & \searrow r & \\ T & \xrightarrow{g} & T & \xrightarrow{\text{id}} & T \end{array}$$

*is commutative up to (free) homotopy. Then  $N(g) = N(\bar{g})$ .*

**Proof.** We use the definition and notation for  $N(g)$  and  $N(\bar{g})$  given above; we also put  $\bar{G} = \pi_1(\bar{T}, s(t_0))$  where  $t_0$  is the basepoint of  $T$ . Evidently, because  $s \circ g \simeq \bar{g} \circ s$ , there is a well-defined function

$$s_{\#} : G_{g_{\#}} \longrightarrow \bar{G}_{\bar{g}_{\#}}$$

which, because also  $r \circ \bar{g} \simeq g \circ r$ , while  $r \circ s \simeq \text{id}_T$ , has left inverse  $r_{\#} : \bar{G}_{\bar{g}_{\#}} \rightarrow G_{g_{\#}}$ . In particular,  $s_{\#}$  is injective. (With reference to the sentence before [12] (2.2), note that this is not true in general without some condition on  $s$ , such as its having a left inverse.)

The formula for the Reidemeister trace of  $g$  at  $t_0 \in T$  is:

$$R(g, t_0) = \sum_{C \in G_{g_{\sharp}}} n_C \cdot C \in \mathbb{Z}G_{g_{\sharp}}.$$

According to [12] (2.2), we also have

$$R(\bar{g}, s(t_0)) = \sum_{C \in G_{g_{\sharp}}} n_C \cdot s_{\sharp}(C) \in \mathbb{Z}\bar{G}_{\bar{g}_{\sharp}}.$$

Because  $s_{\sharp}$  is injective, the two sums have the same number of nonzero coefficients; that is, the Nielsen numbers agree.  $\square$

The next lemma permits us in our discussion of Nielsen numbers to restrict to the case of those homotopy idempotents that are pointed homotopy idempotents and induce the identity on the fundamental group.

**Lemma 3.3.** *Suppose that  $f : X \rightarrow X$  is a homotopy idempotent on a finite connected complex  $X$ , fixing  $x_0 \in X$ , with  $G = \pi_1(X, x_0)$  and  $H := f_{\sharp}(G)$ . Then there is a finite connected complex  $K$  with fundamental group isomorphic to  $H$  and a pointed homotopy idempotent  $g : K \rightarrow K$ , inducing the identity map on  $H$ , such that  $N(f) = N(g)$ . Furthermore, if  $G$  satisfies the Bass conjecture, then so does  $H$ .*

**Proof.** Let  $X \xrightarrow{u} Y \xrightarrow{v} X$  be a splitting for  $f$ . Then, by [25] Corollary 5.5,  $Y \times S^3$  is homotopy equivalent to a finite connected complex  $K$ , because  $Y$  is finitely dominated and the Euler characteristic of  $S^3$  is zero. Let  $h : Y \times S^3 \rightarrow K$  be a pointed homotopy equivalence, with pointed homotopy inverse  $k$ ; define  $g : K \rightarrow K$  to be the map  $h \circ (\text{id}_Y \times \{*\}) \circ k$ , where  $\text{id}_Y \times \{*\} : Y \times S^3 \rightarrow Y \times S^3$  denotes the idempotent on  $Y \times S^3$  given by the projection onto  $Y$ . Clearly  $g$  is a pointed homotopy idempotent, inducing the identity on the fundamental group of  $K$ , and  $\pi_1(K) \cong \pi_1(Y) \cong H$ .

Writing  $u' = u \times \text{id}_{S^3}$  and  $v' = v \times \text{id}_{S^3}$ , we now apply Lemma 3.2 with  $\bar{T} = X \times S^3$  and  $T = K$ , and maps  $\bar{g} = f \times \{*\}$ ,  $r = h \circ u'$ ,  $s = v' \circ k$  and  $g$  as defined already. This yields the following homotopy commutative diagram

$$\begin{array}{ccccc}
 X \times S^3 & \xrightarrow{f \times \{*\}} & X \times S^3 & & \\
 \downarrow u' & & \downarrow v' & \searrow u' & \nearrow r \\
 Y \times S^3 & & Y \times S^3 & \xrightarrow{\text{id}} & Y \times S^3 \\
 \downarrow h & & \downarrow k & & \downarrow h \\
 K & \xrightarrow{g} & K & \xrightarrow{\text{id}} & K
 \end{array}$$

We conclude that  $N(g) = N(f \times \{*\})$ .

On the other hand,  $N(f \times \{*\}) = N(f)$ , as can be seen by again applying Lemma 3.2, with the top part of the diagram as before, but

$T = X$ ,  $r = \text{pr}_X$  and  $s$  the inclusion  $x \mapsto (x, *)$

$$\begin{array}{ccccc}
 X \times S^3 & \xrightarrow{f \times \{*\}} & X \times S^3 & & \\
 \downarrow & & \uparrow & \searrow & \\
 X & \xrightarrow{f} & X & \xrightarrow{\text{id}} & X.
 \end{array}$$

Therefore  $N(f) = N(g)$ . That  $H \cong \pi_1(Y)$  satisfies the Bass conjecture if  $G$  does follows by observing that  $v_{\sharp} : \pi_1(Y) \rightarrow \pi_1(X)$  is a split injection, and therefore the induced map  $HH_0(\mathbb{Z}\pi_1(Y)) \rightarrow HH_0(\mathbb{Z}\pi_1(X))$  is a split injection too.  $\square$

We are now able to obtain a restatement of the theorem of Geoghegan referred to in the Introduction ([12] Theorem 4.1' (i)'  $\Leftrightarrow$  (iii)'), in a form suitable to the present treatment.

**Theorem 3.4.** (Geoghegan). *Let  $G$  be a finitely presented group. The following are equivalent.*

- (a)  $G$  satisfies Bass' Conjecture 1.
- (b) Every homotopy idempotent selfmap  $f$  on a finite connected complex with fundamental group  $G$  has Nielsen number either zero or one.

**Proof.** We start with the implication (a)  $\Rightarrow$  (b). Let  $f$  be as in (b). Then by Lemma 3.3 we can assume that  $f : X \rightarrow X$  is actually a pointed homotopy idempotent on a finite connected complex  $X$  that induces the identity map on  $\pi_1(X, x_0) \cong G$ . Because  $G$  satisfies the Bass conjecture, we have  $\text{HS}(w(f)) \in \mathbb{Z} \cdot [e]$ . Then, by Lemma 3.1,  $R(f, x_0)$  has at most one nonzero coefficient, and  $N(f) \leq 1$ .

In the other direction, we of course use Theorem 2.3 and Remark 2.4. Then, given  $\alpha \in K_0(\mathbb{Z}G)$ , there is a finite  $n$ -dimensional complex  $X$  ( $n \geq 3$ ) with fundamental group  $G$  and a pointed homotopy idempotent selfmap  $f$  of  $X$  inducing the identity on  $\pi_1(X) = G$ , such that  $w(f)$  is equal to  $\alpha$ . So, from Lemma 3.1 we deduce that  $\text{HS}(\alpha) = R(f, x)$ . This last term vanishes when  $N(f) = 0$ ; if  $N(f) = 1$ ,  $R(f, x)$  is a nonzero multiple of some class  $[s]$ , and we are done in case  $[s] = [e]$ . So, the remaining case is where  $R(f, x)$  is a nonzero multiple (necessarily  $\chi(Y)$ ) of some class  $[s] \neq [e]$ . In that event we may turn instead to  $f' = f \vee \text{id}_{S^2}$  with corresponding  $Y' = Y \vee S^2$  having  $w(f') = w(f) + [\mathbb{Z}G]$ . However, this implies the contradiction that  $N(f') = 2$ , and can therefore be eliminated.  $\square$

**Remark 3.5.** The actual wording of [12] Theorem 4.1' is in terms of the Bass conjecture for a particular element  $\alpha$  of  $K_0(\mathbb{Z}G)$ , rather than for  $G$  itself.

## 4. SELFMAPS OF MANIFOLDS

From Wecken's work [31], one knows that  $N(f)$  serves as a lower bound for the number of fixed points of any map homotopic to  $f$ . Thus, the implication (ii)  $\Rightarrow$  (i) in the next result is immediate.

**Lemma 4.1.** *Suppose that  $f : M \rightarrow M$  is a selfmap of a closed manifold  $M$  of dimension at least 3. Then the following are equivalent:*

- (i) *the Nielsen number of  $f$  is 0 or 1;*
- (ii)  *$f$  is homotopic to a map having an arbitrarily chosen unique fixed point.*

**Proof.** We need only prove that (i) implies (ii).

First, various results in the literature (see in particular Wecken [31], and [4], [27] for the PL case; [16] for the smooth case) show that every selfmap of  $M$  with Nielsen number  $N$  is homotopic to a map with exactly  $N$  fixed points. By [26] Lemma 2, these fixed points may be chosen arbitrarily.

Second, recall from [26] that every fixed-point-free selfmap of a connected compact PL manifold of dimension at least 3 is homotopic to a selfmap having an arbitrary unique fixed point. The argument there can be adapted as follows.

Choose  $a \in M$ , and consider the closure  $\bar{B}$  of an open ball  $B$  around  $a$  lying in a chart domain for  $M$ . Since  $f$  is fixed-point-free, we choose the ball  $\bar{B}$  to be small enough so that it is disjoint from its image under  $f$ . For convenience, we consider points of  $\bar{B}$  with coordinates so that  $a = \mathbf{0}$  and  $\bar{B}$  consists of those points  $x$  with  $\|x\| \leq 1$ . Now let  $\gamma$  be any path from  $a$  to  $f(a)$  that continues a unit-speed ray from  $a$  to the boundary of  $\bar{B}_{1/2}$  (the closed ball of points of  $B$  of norm at most  $1/2$ ) and never re-enters  $\bar{B}_{1/2}$ . Also, let  $\lambda : M \rightarrow [0, 1]$  be a function having  $\lambda^{-1}(1) = M - B$  and  $\lambda^{-1}(0) = \bar{B}_{1/2}$ . Then the desired map  $g : M \rightarrow M$  homotopic to  $f$  is given as follows.

$$g(x) = \begin{cases} \gamma(t_x) & 0 \leq 2\|x\| < 1 \\ f((1 - \lambda(x))a + \lambda(x)x) & 1 \leq 2\|x\| \leq 2 \\ f(x) & x \in M - \bar{B} \end{cases},$$

where  $t_x = 1 - \exp(-2\|x\|/(1 - 2\|x\|))$ . Here, recall the standard inequality

$$\ln(1 - u) > -u/(1 - u)$$

for  $0 < u < 1$ . It implies that, whenever  $t_x \neq 0$  and  $\gamma(t_x) \in \bar{B}_{1/2}$ , so that  $\|\gamma(t_x)\| = t_x$ ,

$$\|\gamma(t_x)\| > 2\|x\|.$$

Hence  $a$  is the unique fixed point of  $g$ .

Note that it is possible to make  $g$  smooth. For, since every map is homotopic to a smooth map and homotopy does not change Nielsen numbers, there is no loss of generality in assuming  $f$  to be smooth.

Then, by our taking both  $\gamma$  and  $\lambda$  to be smooth functions in the above argument, a smooth map  $g$  results.  $\square$

**Remark 4.2.** Note that this result cannot be extended to dimension 2 in general. Indeed, for every connected, closed surface of negative Euler characteristic and every natural number  $n$ , [18] Theorem 2 exhibits a selfmap  $f_n$  of the surface having  $N(f_n) = 1$ , but with every map homotopic to  $f_n$  having more than  $n$  fixed points. For some particular results on selfmaps on surfaces, see also [19].

We next observe that selfmaps of complexes may be studied by means of selfmaps of manifolds without changing the Nielsen number.

**Lemma 4.3.** *Let  $X$  be a finite connected complex. Then the following hold.*

- (a) *There is a closed, oriented and smooth manifold  $M$  of dimension at least 3 with maps  $r : M \rightarrow X$  and  $s : X \rightarrow M$  having  $r \circ s$  pointed homotopic to  $\text{id}_X$  and inducing isomorphisms of fundamental groups.*
- (b) *For any selfmap  $f : X \rightarrow X$ , the selfmap  $\bar{f} = s \circ f \circ r : M \rightarrow M$  has Nielsen number*

$$N(\bar{f}) = N(f).$$

- (c) *If  $f$  is either homotopy idempotent or pointed homotopy idempotent, then so is  $\bar{f}$ .*

**Proof.** (a) Working up to pointed homotopy type, we may assume that  $X$  is a finite simplicial complex of dimension  $n \geq 2$ . By Theorem 1.4 of [30] we can do surgery on the constant map  $S^{2n} \rightarrow X$  to obtain a smooth, oriented (indeed, stably parallelizable) closed  $2n$ -manifold  $M$  and an  $n$ -connected map (called an  $n$ -equivalence in [28])  $r : M \rightarrow X$ . Because  $n \geq 2$ , the map  $r$  is a  $\pi_1$ -isomorphism. Moreover, since the obstruction groups  $H^i(Y; \pi_i(r))$  all vanish (or by [28] (7.6.13)), the identity map  $X \rightarrow X$  factors up to pointed homotopy through  $M \rightarrow X$ , and the result follows.

(b) This result is immediate from Lemma 3.2 above, on putting  $\bar{T} = M$ ,  $T = X$ ,  $g = f$  and  $\bar{g} = \bar{f}$ .

(c) Obviously,

$$\begin{aligned} \bar{f} \circ \bar{f} &\simeq s \circ f \circ r \circ s \circ f \circ r \\ &\simeq s \circ f \circ f \circ r \simeq s \circ f \circ r \simeq \bar{f}, \end{aligned}$$

and  $\bar{f}$  is a pointed idempotent if  $f$  is.  $\square$

**Example 4.4.** For any connected non-contractible space  $X$ , the monoid of homotopy classes of selfmaps of  $X$  always contains at least two idempotents, the class of nullhomotopic maps and the class of maps homotopic to the identity. Each constant map in the former class contains

exactly one fixed point (which by connectivity is arbitrary), and obviously has Nielsen number 1.

On the other hand, for  $X$  a finite complex the identity map has Nielsen number equal to  $\min\{1, |\chi(X)|\}$ . When  $X$  is also a smooth manifold, it admits a smooth vector field whose only singularity is an arbitrarily chosen point  $x_0 \in X$ . Its associated flow provides a homotopy from the identity map to a smooth map with sole fixed point  $x_0$ .

## 5. PROOF OF THEOREM 1 AND APPLICATIONS

The discussion above now allows a reformulation of statement (b) of Theorem 3.4. Lemma 4.3 combines with Lemma 4.1 to yield a manifold version of statement (b), as follows.

**Proposition 5.1.** *Let  $G$  be a finitely presented group. The following are equivalent.*

- (a)  *$G$  satisfies Bass' Conjecture 1.*
- (b) *Given any closed, smooth and oriented manifold  $M$  of dimension at least 3 with  $G = \pi_1(M)$ , every homotopy idempotent selfmap  $f$  on  $M$  is homotopic to one that has a single fixed point.  $\square$*

**Remark 5.2.** The following facts combine to show that the dimension 3 in (b) above is best possible. For  $F$  a closed surface of negative Euler characteristic and  $n \geq 2$ , Kelly [20] constructs a homotopy idempotent selfmap  $f_n : F \rightarrow F$  such that every map homotopic to  $f_n$  has at least  $n$  fixed points. On the other hand, the fundamental groups of surfaces are well-known to satisfy Bass' Conjecture 1 (see e.g. [8]).

The following argument of Bass, reported by R. Geoghegan, shows that it suffices to consider finitely presented groups in considering Bass' conjectures.

**Lemma 5.3.** *(Bass) Conjectures 1 and 2 hold for all groups if they hold for all finitely presented groups.*

**Proof.** Fix a group  $G$ . We show that any idempotent  $\mathbb{Z}G$ -matrix  $A$  lifts to an idempotent matrix  $A_1$  over the group ring of a finitely presented group  $G_1$ . There is a finitely generated subgroup  $G_0$  of  $G$  such that the entries of  $A$  lie in  $\mathbb{Z}G_0$ . Write  $G_0$  as  $F/R$  where  $F$  is a finitely generated free group; and let  $B$  be a lift of  $A$  to  $\mathbb{Z}F$ . Then there is a finite subset  $W$  of  $R$  such that the matrix  $B^2 - B$  has all its entries in the ideal of  $\mathbb{Z}F$  generated by  $\{1 - r \mid r \in W\}$ . Now let  $R_1 \leq R$  be the normal closure of  $W$  in  $F$ . Then we have  $G_1 := F/R_1$  finitely presented, and the image  $A_1$  of  $B$ , with entries in  $\mathbb{Z}G_1$ , is an idempotent matrix. The map  $G_1 \twoheadrightarrow G_0 \hookrightarrow G$  takes  $A_1$  to  $A$ . Therefore

$[A_1] \mapsto [A]$  under the induced map  $K_0(\mathbb{Z}G_1) \rightarrow K_0(\mathbb{Z}G)$ . Then the result follows from naturality of HS.  $\square$

After Proposition 5.1, it is now straightforward to deduce Theorem 1.  $\square$

**Remark 5.4.** Our arguments lead to variations on Theorem 1. First, one can sharpen the implication (b)  $\Rightarrow$  (a) by referring in (b) to a smaller class of manifolds. Because, by Theorem 2.3 above, for a finitely presented group  $G$  any  $\tilde{\alpha} \in \tilde{K}_0(\mathbb{Z}G)$  can be realized by a homotopy idempotent selfmap  $f$  of a 2-dimensional complex with fundamental group  $G$  (so  $\tilde{w}(f) = \tilde{\alpha}$ ), the Bass conjecture is equivalent to the following: *Every homotopy idempotent selfmap of a closed, stably parallelizable smooth 4-manifold is homotopic to one with a single fixed point.*

In the other direction, one can strengthen (a)  $\Rightarrow$  (b) by enlarging the class of spaces to which (b) applies. There is no need to restrict attention to oriented, smooth manifolds; one can also apply to PL manifolds and other, possibly bounded, Wecken spaces [15].

As an application of Proposition 5.1 we obtain the following.

**Corollary 5.5.** *Any homotopy idempotent selfmap on a closed, smooth and oriented 3-dimensional manifold  $M$  is homotopic to one with a single fixed point.*

**Proof.** It is enough to show that the fundamental group  $G$  of a closed smooth oriented 3-dimensional manifold  $M$  satisfies Bass' conjecture; the Corollary then follows from Proposition 5.1. By Kneser's result (see [23]),  $M$  is a connected sum of prime manifolds  $M_i$ , where each  $M_i$  belongs to one of the following classes:

- 1)  $M_i$  with finite fundamental group;
- 2)  $M_i$  with fundamental group  $\mathbb{Z}$ ;
- 3)  $M_i$  a  $K(\pi, 1)$  manifold (so  $\pi$  is a Poincaré duality group).

Note that the fundamental group of  $M$  is the free product of the fundamental groups of the various  $M_i$ . By Gersten's result [11], given two groups  $\Gamma$  and  $H$ , the reduced projective class group of the free product  $\Gamma * H$  reads

$$\tilde{K}_0(\mathbb{Z}(\Gamma * H)) \cong \tilde{K}_0(\mathbb{Z}\Gamma) \oplus \tilde{K}_0(\mathbb{Z}H).$$

Thus, every element in  $\tilde{K}_0(\mathbb{Z}(\Gamma * H))$  is an integral linear combination of projectives induced up from  $\Gamma$  and  $H$  respectively. It follows that if Bass' conjecture holds for both  $\Gamma$  and  $H$ , then it holds for  $\Gamma * H$  as well.

In the list above, clearly finite groups and  $\mathbb{Z}$  satisfy Bass' conjecture. That 3-dimensional Poincaré duality groups satisfy Bass' conjecture follows from Eckmann's work (see p. 247 of [8]) on groups of rational cohomological dimension 2.  $\square$

Since the Bass conjecture is known for instance for the fundamental groups of manifolds in the class below [8], we have another consequence.

**Corollary 5.6.** *Any homotopy idempotent selfmap of a non-positively curved, oriented closed manifold of dimension at least 3 is homotopic to a map with a single fixed point.*  $\square$

It would be interesting to see geometric proofs of these facts.

## 6. LEFSCHETZ NUMBERS

Let  $X$  be a CW-complex and  $f : X \rightarrow X$  a continuous selfmap. Then  $f$  induces for each  $n \in \mathbb{N}$  a map

$$f_n : H_n(X; \mathbb{Q}) \rightarrow H_n(X; \mathbb{Q})$$

of  $\mathbb{Q}$ -vector spaces. If the sum of the dimensions of the vector spaces  $H_n(X; \mathbb{Q})$  is finite, the *Lefschetz number* of  $f$  is defined as:

$$L(f) = \sum_{n \geq 0} (-1)^n \text{Tr}(f_n) \in \mathbb{Z}.$$

In cases where the CW-complex  $X$  is finite or finitely dominated, the Lefschetz number of a selfmap is obviously always defined, and for  $f = \text{id}_X$ ,  $L(f) = \chi(X)$  the Euler characteristic of  $X$ . One extends this definition to the case of  $G$ -CW-complexes as follows.

Let  $\mathcal{N}G$  denote the von Neumann algebra of the discrete group  $G$  (i.e. the double commutant of  $\mathbb{C}G$  considered as a subalgebra of the algebra of bounded operators on the Hilbert space  $\ell^2 G$  – see for example [21]). With  $e$  as the neutral element of  $G$ , write  $e \in G \subset \ell^2 G$  for the delta-function

$$e : G \rightarrow \mathbb{C}, \quad g \mapsto \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

The standard trace

$$\text{tr}_G : \mathcal{N}G \rightarrow \mathbb{C}, \quad x \mapsto \langle xe, e \rangle_{\ell^2 G} \in \mathbb{C}$$

extends to a trace  $\text{tr}_G(\phi) \in \mathbb{C}$  for  $\phi : M \rightarrow M$  a map of finitely presented  $\mathcal{N}G$ -modules as follows. Recall that a finitely presented  $\mathcal{N}G$ -module  $M$  is of the form  $S \oplus T$  with  $S$  projective and  $T$  of von Neumann dimension 0; the trace of  $\phi$  is then defined as the usual von Neumann trace of the composite  $S \rightarrow M \rightarrow M \rightarrow S$  (for the trace of self-maps of finitely generated projective  $\mathcal{N}G$ -modules see [21]). It follows that  $\text{tr}_G(\text{id}_M) = \dim_G(M)$ , the von Neumann dimension of the finitely presented  $\mathcal{N}G$ -module  $M$  (a non-negative real number cf. [21]). The Kaplansky trace (as defined in Section 1) induces a trace on  $G$ -maps  $\psi : P \rightarrow P$  of finitely generated projective  $\mathbb{Z}G$ -modules, and

$$\kappa(\psi) = \text{tr}_G(\text{id}_{\mathcal{N}G} \otimes \psi)$$

where  $\text{id}_{\mathcal{N}G} \otimes \psi : \mathcal{N}G \otimes_{\mathbb{Z}G} P \rightarrow \mathcal{N}G \otimes_{\mathbb{Z}G} P$ .

Now let  $Z$  be a free  $G$ -CW-complex that is dominated by a cocompact  $G$ -CW-complex (for example, the universal cover of a finitely dominated CW-complex with fundamental group  $G$ ). A  $G$ -map  $\tilde{f} : Z \rightarrow Z$  induces a map of singular chain complexes  $C_*(Z) \rightarrow C_*(Z)$  and of  $L^2$ -chain complexes

$$C_*^{(2)}(Z) := \mathcal{N}G \otimes_{\mathbb{Z}G} C_*(Z) \rightarrow C_*^{(2)}(Z),$$

and therefore of  $L^2$ -homology groups

$$H_n^{(2)}(Z) := H_n(Z; \mathcal{N}G) \rightarrow H_n^{(2)}(Z).$$

The groups  $H_n^{(2)}(Z)$  are finitely presented  $\mathcal{N}G$ -modules, because the complex  $C_*^{(2)}(Z)$  is chain homotopy equivalent to a complex of type FP over  $\mathcal{N}G$  and because the category of finitely presented  $\mathcal{N}G$ -modules is known to be abelian [21]. Thus the induced map

$$\tilde{f}_n : H_n^{(2)}(Z) \rightarrow H_n^{(2)}(Z)$$

is a selfmap of a finitely presented  $\mathcal{N}G$ -module and has, therefore, a well defined trace  $\text{tr}_G(\tilde{f}_n)$  as explained in the beginning of this section; we also write  $\beta_n^{(2)}(Z; G)$  for the von Neumann dimension of  $H_n^{(2)}(Z)$ .

Let  $Y$  be a finitely dominated CW-complex with fundamental group  $G$  and universal cover  $Z$ . Then the  $n$ th  $L^2$ -Betti number  $\beta_n^{(2)}(Y)$  of  $Y$  is defined to be  $\beta_n^{(2)}(Z; G)$ . (If  $Y$  happens to be a finite complex, this reduces to the usual  $L^2$ -Betti number of  $Y$  as defined for instance in [1] and [9].) By definition, the alternating sum  $\sum (-1)^i \beta_i^{(2)}(Y) = \chi^{(2)}(Y)$  is the  $L^2$ -Euler characteristic of  $Y$ . Recall that for  $Y$  a finite complex,  $\chi(Y) = \chi^{(2)}(Y)$  by Atiyah's formula [1]; see also [5] and Lemma 7.1 below for more general results. We now define  $L^2$ -Lefschetz numbers as follows.

**Definition 6.1.** Let  $Z$  be a free  $G$ -CW-complex that is dominated by a cocompact  $G$ -CW-complex and let  $\tilde{f} : Z \rightarrow Z$  be a  $G$ -map. Denote by  $\tilde{f}_n : H_n^{(2)}(Z) \rightarrow H_n^{(2)}(Z)$  the induced map in  $L^2$ -homology. Then the  $L^2$ -Lefschetz number of  $\tilde{f}$  is given by

$$L^{(2)}(\tilde{f}) := \sum_{n \geq 0} (-1)^n \text{tr}_G(\tilde{f}_n) \in \mathbb{R}.$$

In case  $Z$  is cocompact our  $L^2$ -Lefschetz number agrees with the one defined in [22] (*cf.* Remark 1.7 *loc.cit.*). If  $Y$  is a finitely dominated connected CW-complex with fundamental group  $G$ , and with universal cover the free  $G$ -space  $\tilde{Y}$ , then the  $L^2$ -Lefschetz number of the identity map of  $\tilde{Y}$  is  $\chi^{(2)}(\tilde{Y}; G) = \chi^{(2)}(Y)$ , the  $L^2$ -Euler characteristic of  $Y$ .

## 7. PROOF OF THEOREM 2

Let  $Y$  is a finitely dominated connected CW-complex. Thus  $\chi(Y)$  and  $\chi^{(2)}(Y)$  are defined as above, and are related as follows.

**Lemma 7.1.** *Let  $G$  be a finitely presented group. Then the following holds.*

- (a) *Let  $Y$  be a finitely dominated connected CW-complex with fundamental group  $G$ . If the finiteness obstruction  $\tilde{w}(Y) \in \tilde{K}_0(\mathbb{Z}G)$  is a torsion element, then  $\chi^{(2)}(Y) = \chi(Y)$ .*
- (b) *The following are equivalent.*
  - (i) *The weak Bass conjecture holds for  $G$ .*
  - (ii) *For any finitely dominated connected CW-complex  $Y$  with  $\pi_1(Y) = G$ , we have*

$$\chi^{(2)}(Y) = \chi(Y).$$

**Proof.** As in Section 2, for  $Y$  finitely dominated, the chain complex  $C_*(\tilde{Y})$  is chain homotopy equivalent to a chain complex  $P_*$  of type FP over  $\mathbb{Z}G$ ,  $G = \pi_1(Y)$ , and we have the Wall element

$$w(Y) = \sum_{i=0}^n (-1)^i [P_i] \in K_0(\mathbb{Z}G).$$

As  $\mathcal{N}G \otimes_{\mathbb{Z}G} P_* \simeq C^{(2)}(\tilde{Y})$  and  $\mathrm{tr}_G(\mathcal{N}G \otimes_{\mathbb{Z}G} P_i) = \kappa(P_i)$ , we see that

$$\chi^{(2)}(Y) = \sum (-1)^i \mathrm{tr}_G(\mathcal{N}G \otimes_{\mathbb{Z}G} P_i) = \sum (-1)^i \kappa(P_i) = \kappa(w(Y)).$$

On the other hand,  $\mathbb{Z} \otimes_{\mathbb{Z}G} P_* \simeq C_*(Y)$  and  $\epsilon(P_i) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} P_i)$  so that

$$\chi(Y) = \sum (-1)^i \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}G} P_i) = \sum (-1)^i \epsilon(P_i) = \epsilon(w(Y)).$$

(a) Again we observe that for  $n > 1$ ,

$$\chi(Y \vee (\vee^k S^n)) = \chi(Y) + (-1)^n k$$

and

$$\chi^{(2)}(Y \vee (\vee^k S^n)) = \chi^{(2)}(Y) + (-1)^n k.$$

On the other hand,  $w(Y \vee (\vee^k S^n)) = w(Y) + (-1)^n k$ , so that without loss of generality we may assume that actually  $w(Y)$  is a torsion element. But then, since the range of the Hattori-Stallings trace is torsion-free, Bass' conjectures are valid for torsion elements of  $K_0(\mathbb{Z}G)$  and we have

$$\kappa(w(Y)) = \epsilon(w(Y)).$$

Thus,

$$\chi(Y) = \chi^{(2)}(Y),$$

proving the claim.

(b) (i)  $\Rightarrow$  (ii): Assuming the weak Bass conjecture, we have

$$\chi^{(2)}(Y) = \kappa(w(Y)) = \epsilon(w(Y)) = \chi(Y).$$

Assuming the Bass conjecture, this implication has also been proved by Eckmann in [7].

(ii)  $\Rightarrow$  (i): Recall from Theorem 2.3 that for a finitely generated projective  $\mathbb{Z}G$ -module  $P$  there is always a finitely dominated CW-complex  $Y$  whose Wall element  $w(Y)$  equals  $[P] \in K_0(\mathbb{Z}G)$ . We then have that

$$\kappa(P) = \kappa(w(Y)) = \chi^{(2)}(Y) = \chi(Y) = \epsilon(w(Y)) = \epsilon(P).$$

□

Next, we need an intermediate result.

**Lemma 7.2.** *Let  $X$  be a finite connected complex, and  $f : X \rightarrow X$  be a homotopy idempotent. Let  $Y$  be a finitely dominated CW-complex determined by  $f$  as in Section 2.*

- (a) *Then  $L(f) = \chi(Y)$ .*
- (b) *If moreover,  $f$  is a pointed homotopy idempotent inducing the identity on  $G = \pi_1(X)$  and  $\tilde{f}$  denotes the induced  $G$ -map on the universal cover of  $X$ , then  $L^{(2)}(\tilde{f}) = \chi^{(2)}(Y)$ .*

**Proof.** (a) Applying  $H_i(-; \mathbb{Q})$  to the diagram (2.1) of Section 2 yields the following commutative diagram of groups:

$$\begin{array}{ccccc}
 & & & & f_i \\
 & & & & \curvearrowright \\
 H_i(X) & \xrightarrow{f_i} & H_i(X) & \xrightarrow{f_i} & H_i(X) \\
 & \searrow u_i & & \searrow u_i & \\
 & & H_i(Y) & \xrightarrow{\text{id}_i} & H_i(Y) \\
 & & \nearrow v_i & & \nearrow v_i
 \end{array}$$

We can now compute that

$$\text{Tr}(f_i) = \text{Tr}(v_i u_i) = \text{Tr}(u_i v_i) = \text{Tr}(\text{id}_i) = \dim(H_i(Y; \mathbb{Q})),$$

so that (a) follows by taking alternating sums.

(b) Here we need to know that  $f$  induces the identity map on  $\pi_1(X)$ , in order to obtain equivariance of the induced maps on the universal covers  $\tilde{X}$  and  $\tilde{Y}$ . We apply  $H_i^{(2)}$  to diagram (2.1):

$$\begin{array}{ccccc}
 & & & & \tilde{f}_i \\
 & & & & \curvearrowright \\
 H_i^{(2)}(\tilde{X}) & \xrightarrow{\tilde{f}_i} & H_i^{(2)}(\tilde{X}) & \xrightarrow{\tilde{f}_i} & H_i^{(2)}(\tilde{X}) \\
 & \searrow \tilde{u}_i & & \searrow \tilde{u}_i & \\
 & & H_i^{(2)}(\tilde{Y}) & \xrightarrow{\text{id}_i} & H_i^{(2)}(\tilde{Y}) \\
 & & \nearrow \tilde{v}_i & & \nearrow \tilde{v}_i
 \end{array}$$

and compute

$$\text{tr}_G(\tilde{f}_i) = \text{tr}_G(\tilde{v}_i \tilde{u}_i) = \text{tr}_G(\tilde{u}_i \tilde{v}_i) = \text{tr}_G(\text{id}_i) = \dim_G(H_i^{(2)}(\tilde{Y})),$$

and take alternating sums. The desired equality uses the fact that, given two finitely presented  $\mathcal{N}G$ -modules  $A$  and  $B$ , with two maps  $\phi : A \rightarrow B$  and  $\psi : B \rightarrow A$ , then  $\mathrm{tr}_G(\phi\psi) = \mathrm{tr}_G(\psi\phi)$ .  $\square$

**Proposition 7.3.** *Let  $G$  be a finitely presented group. The following are equivalent.*

- (a)  $G$  satisfies the weak Bass conjecture.
- (b) Every pointed homotopy idempotent selfmap of a closed, smooth and oriented manifold  $M$  with  $\pi_1(M) = G$  and inducing the identity on  $G$  has its Lefschetz number equal to the  $L^2$ -Lefschetz number of the induced  $G$ -map on the universal cover of  $M$ .

**Proof.** That (a) implies (b) follows from the implication (a)  $\Rightarrow$  (b) in Lemma 7.1, combined with Lemma 7.2. To prove that (b) implies (a), namely that the Lefschetz number information on manifolds is enough to imply the weak Bass conjecture, it suffices to see that for a connected simplicial complex  $X$  of dimension  $n \geq 2$  there are a closed smooth oriented manifold  $M$  and maps  $X \rightarrow M$ ,  $M \rightarrow X$  inducing isomorphisms of the fundamental groups, and such that  $X \rightarrow M \rightarrow X$  is pointed homotopic to  $\mathrm{id}_X$ . However, this was already discussed in Lemma 4.3. We then conclude by combining the implication (b)  $\Rightarrow$  (a) in Lemma 7.1 with Lemma 7.2.  $\square$

Finally, we turn to the proof of Theorem 2. That (a) implies (b) follows from the previous proposition, which also shows that (b) implies (a) for all finitely presented groups, and therefore for all groups via Lemma 5.3.  $\square$

**Remark 7.4.** For each group  $G$ , it is evident that the algebraic statement of Conjecture 1 for  $G$  implies the statement of Conjecture 2. For our geometric formulations of the conjectures, the implication is less clear. One can approach this problem via work of Lück and Rosenberg on computing  $L^2$ -Lefschetz numbers and local degrees [22].

## REFERENCES

- [1] M. F. Atiyah. *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32-3** (1976), 43–72.
- [2] H. Bass. *Euler characteristics and characters of discrete groups*, Invent. Math. **35** (1976), 155–196.
- [3] A. J. Berrick, I. Chatterji and G. Mislin. *From acyclic groups to the Bass conjecture for amenable groups*, Math. Annalen **329** (2004), 597–621.
- [4] R. F. Brown. *The Lefschetz Fixed Point Theorem*, Scott Foresman (Glenview, 1971).
- [5] I. Chatterji and G. Mislin. *Atiyah’s  $L^2$ -index theorem*, Enseign. Math. (2) **49** (2003), 85–93.
- [6] B. Eckmann. *Cyclic homology of groups and the Bass conjecture*, Comment. Math. Helv. **61** (1986), 193–202.

- [7] B. Eckmann. *Projective and Hilbert modules over group algebras, and finitely dominated spaces*, Comment. Math. Helv. **71** (1996), 453–462.
- [8] B. Eckmann. *Idempotents in a complex group algebra, projective modules, and the von Neumann algebra*, Arch. Math. **76** (2001), 241–249.
- [9] B. Eckmann. *Introduction to  $l_2$ -methods in topology: reduced  $l_2$ -homology, harmonic chains,  $l_2$ -Betti numbers*, Notes prepared by Guido Mislin. Israel J. Math. **117** (2000), 183–219.
- [10] I. Emmanouil. *On a class of groups satisfying Bass' conjecture*, Invent. Math. **132** (1998), 307–330.
- [11] S. M. Gersten. *On class groups of free products*, Annals of Math. (2) **87** (1968) 392–398.
- [12] R. Geoghegan. *Fixed points in finitely dominated compacta: the geometric meaning of a conjecture of H. Bass*, in “Shape Theory and Geometric Topology (Dubrovnik, 1981)”, Lecture Notes in Math. **870**, Springer (Berlin-New York, 1981), 6–22.
- [13] R. Geoghegan. *Nielsen fixed point theory*, in “Handbook of Geometric Topology”, North-Holland (Amsterdam, 2002), 499–521.
- [14] H. M. Hastings and A. Heller. *Homotopy idempotents on finite-dimensional complexes split*, Proc. Amer. Math. Soc. **85** (1982), 619–622.
- [15] B. J. Jiang. *On the least number of fixed points*, Amer. J. Math. **102** (1980), 749–763.
- [16] B. J. Jiang. *Fixed point classes from a differential viewpoint*, in “Fixed Point Theory (Sherbrooke, Que., 1980)”, Lecture Notes in Math. 886, Springer (1981), 163–170.
- [17] B. J. Jiang. *Lectures on Nielsen Fixed Point Theory*, Contemp. Math. 14, Amer. Math. Soc. (Providence R.I., 1983).
- [18] B. J. Jiang. *Commutativity and Wecken properties for fixed points on surfaces and 3-manifolds*, Topology Appl. **53** (1993), 221–228.
- [19] M. R. Kelly. *Minimizing the number of fixed points for self-maps of compact surfaces*, Pacific J. Math. **126** (1987), 81–123.
- [20] M. R. Kelly. *The Nielsen fixed point structure for homotopy idempotents on surfaces*, Contemp. Math., to appear.
- [21] W. Lück.  *$L^2$ -Invariants: Theory and Applications to Geometry and K-Theory*, Ergebn. Math. Grenzgeb. 44, Springer (Berlin, 2002).
- [22] W. Lück and J. Rosenberg. *The equivariant Lefschetz fixed point theorem for proper cocompact  $G$ -manifolds*, Proceedings of the conference “Topology of High-dimensional Manifolds: ICTP June 2001”, World Scientific (Trieste, 2003).
- [23] J. Milnor. *A unique decomposition theorem for 3-manifolds*, Amer. J. Math. **84** (1962), 1–7.
- [24] G. Mislin. *The geometric realization of Wall obstructions by nilpotent and simple spaces*, Math. Proc. Camb. Phil. Soc. **87** (1980), 199–206.
- [25] G. Mislin. *Wall's finiteness obstruction*, in “Handbook of Algebraic Topology”, Elsevier (Amsterdam, 1995), 1259–1291.
- [26] H. Schirmer. *Mappings of polyhedra with prescribed fixed points and fixed point indices*, Pacific J. Math. **63** (1976), 521–530.
- [27] G.-H. Shi. *On the least number of fixed points and Nielsen numbers*, Chinese Math, **8** (1966), 234–243.
- [28] E. H. Spanier. *Algebraic Topology*, McGraw-Hill (New York, 1966).
- [29] C. T. C. Wall. *Finiteness conditions for CW-complexes*, Annals of Math. (2) **81** (1965), 55–69.

- [30] C. T. C. Wall. *Surgery of non-simply-connected manifolds*, Annals of Math. (2) **84** (1966), 217–276.
- [31] F. Wecken. *Fixpunktklassen III*, Math. Ann. **118** (1942), 544–577.

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