SYMMETRIC AXIAL MAPS AND EMBEDDINGS OF PROJECTIVE SPACES

I. Introduction

The first named author wishes to express his appreciation to the Chief of the I.P.P. for their hospitality during the summer of 1976.

By A. Bereczki, S. F. Feder, and S. Gitler

1. Introduction

We consider smooth embeddings of real projective n-space \( \mathbb{P}^n \) into \( \mathbb{R}^{2n+1} \). Let \( S^{2n+1} \) denote the unit sphere in \( \mathbb{R}^{2n+1} \) and let \( \mathbb{P}^n \) denote the projective space associated with \( S^{2n+1} \). The natural projection map\(^{1}\) \( \mathbb{R}^{2n+1} \to \mathbb{P}^n \) carries each point \( x \neq 0 \) to the line through \( 0 \) and \( x \). The induced homomorphism \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.


We consider smooth embeddings of real projective n-space \( \mathbb{P}^n \) into \( \mathbb{R}^{2n+1} \) and denote the natural projection map \( \mathbb{R}^{2n+1} \to \mathbb{P}^n \) by \( \pi \).

Theorem 1.1. Let \( \mathbb{P}^n \) be smoothly embedded in \( \mathbb{R}^{2n+1} \) and let \( \pi : \mathbb{R}^{2n+1} \to \mathbb{P}^n \) be the natural projection. Then \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.

The proof of Theorem 1.1 is based on the fact that the natural projection map \( \mathbb{R}^{2n+1} \to \mathbb{P}^n \) carries each point \( x \neq 0 \) to the line through \( 0 \) and \( x \). The induced homomorphism \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.

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Theorem 1.2. Suppose \( 2k > (n+1) \). Then for each embedding \( \varphi : \mathbb{P}^n \to \mathbb{R}^{2n+1} \), \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.

Remarks

(i) In view of (i), the condition \( 2k > (n+1) \) is necessary for the conclusion of Theorem 1.2 to hold.

(ii) The condition \( 2k > (n+1) \) is also sufficient for the conclusion of Theorem 1.2 to hold.

Theorem 1.3. Let \( \mathbb{P}^n \) be smoothly embedded in \( \mathbb{R}^{2n+1} \) and let \( \pi : \mathbb{R}^{2n+1} \to \mathbb{P}^n \) be the natural projection. Then \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.

The proof of Theorem 1.3 is based on the fact that the natural projection map \( \mathbb{R}^{2n+1} \to \mathbb{P}^n \) carries each point \( x \neq 0 \) to the line through \( 0 \) and \( x \). The induced homomorphism \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.

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Theorem 1.4. Suppose \( 2k > (n+1) \). Then for each embedding \( \varphi : \mathbb{P}^n \to \mathbb{R}^{2n+1} \), \( \pi_* : \pi_1(S^{2n+1}) \to \pi_1(\mathbb{P}^n) \) is an isomorphism if \( n \) is even and an epimorphism if \( n \) is odd.
2. Embeddings and Stiefel manifolds

Let $D$ denote the dihedral group of order 8, with subgroup $H$ the elementary Abelian group of order 4. $H$ and $D$ act on the $(n + k - 1)$-sphere $S^{n+k-1}$ and on $V_n = V_{n+1,2}$ (the Stiefel manifold of ordered pairs of orthonormal vectors in $R^{n+1}$), as follows. $D$ acts faithfully on $V_n$ by sending the pair $(x, y) \in V_n$ to the pairs $(\pm x, \pm y)$, $(\pm y, \pm x)$, while $H$ maps $(x, y)$ to $(\pm x, \pm y)$. Then the generators of $D$ corresponding to $(x, y) \to (y, -x)$ and $(x, y) \to (-x, y)$ act antipodally on $S^{n+k-1}$.

**Proposition 2.1.** a) There is a function from the set of isotopy classes of embeddings $P^r \subset R^{n+k}$ to the set of $D$-equivariant homotopy classes of $D$-equivariant maps from $V_n$ to $S^{n+k-1}$, which is surjective if $2k \geq (n + 3)$ and bijective if $2k \geq (n + 4)$.

b) There is a function from the set of regular homotopy classes of embeddings $P^r \subset R^{n+k}$ to the set of $H$-equivariant homotopy classes of $D$-equivariant maps from $V_n$ to $S^{n+k-1}$, which is injective if $2k \geq (n + 2)$ and bijective if $2k \geq (n + 3)$.

c) There is a function from the set of regular homotopy classes of immersions $P^r \subset R^{n+k}$ to the set of $H$-equivariant homotopy classes of $D$-equivariant maps from $V_n$ to $S^{n+k-1}$, which is surjective if $2k \geq (n + 1)$ and bijective if $2k \geq (n + 2)$.

**Proof** a) Haefliger [3] described the function from the set of isotopy classes as far as to the set of $Z_2$-homotopy classes of $Z_2$-maps from $P^r \times P^r - \Delta$ to $S^{n+k-1}$. Equivalently, in view of [5] (2.6), its range is a set of $D$-homotopy classes $D$-maps from $V_n$ to $S^{n+k-1}$, a set equivalent to ours via the $D$-involution $\Psi$ on $V_n$ given by $\Psi(x, y) = ((x + y)/\sqrt{2}, (x - y)/\sqrt{2})$.

b) This is a consequence of a) and c), together with [4] (2.2). Note that the construction $\Phi$ of [4] §2 corresponds here to composition with $\Psi$.

c) Since $V_n/H$ is just the projective tangent bundle of $P^r$, c) is [4] (4.2).

3. Proofs of theorems

**Proof** of (1.1). In view of (2.1) it suffices to associate to each $D$-homotopy class $D$-maps from $V_n$ to $S^{n+k-1}$ a symmetric homotopy (resp. $H$-homotopy) class to $D$-maps from $V_n$ to $S^{n+k-1}$ a symmetric homotopy (resp. $H$-homotopy) class of $\text{SAMs}$ of type $(n, k)$.

Let $\xi$ be the real line bundle over $V_n/D$ comprising triples $[x, y, t]$, $(x, y) \in V_n$, $t \in R$, with $[x, y, t] = [y, -x, t] = [-x, y, -t]$; let $\eta$ be the canonical line bundle over $P^{n+k-1}$, comprising pairs $[z, t]$, $z \in S^{n+k-1}$, $t \in R$, with $[z, t] = [-z, -t]$. Then a $D$-map $f: V_n \to S^{n+k-1}$ induces a bundle map $f: \xi \to \eta$ given by $f([x, y, t]) = [f(x, y), t]$. Likewise a $D$-homotopy induces a linear homotopy of bundle maps. However, if $\pi: V_n \to V_n/D$ is the double covering projection, then an $H$-homotopy induces a linear homomorphism of bundle maps only from $\pi^* \xi \to \eta$.

Now observe the following relative homeomorphisms of sphere and disc bundles. $(B^r \xi, S^r \xi) \to \langle P^r \times P^r, \Delta \rangle$; $(B^r \xi, S^r \xi) \to \langle (P^r \times P^r)/Z_2, \Delta/Z_2 \rangle$ where the involution is that of interchanging the factors; $(B^r \xi, S^r \xi) \to \langle (P^r \times P^r, \pm e), e = (0, \ldots, 0, 1) \in S^{n+k} \rangle$. The first two maps are given by: $[x, y, t] \to ((x + y), t)$.
\(v(y + tx)\) for \(v: R^N \rightarrow \{0\} \rightarrow S^{n-1}, w \rightarrow w/\|w\|\); and the last by \([z,t] \rightarrow (le + (1 - l^2)^{1/2}i(z))\). Where \(i\) is the map from \(S^{n+k-1} \rightarrow S^{n+k}\), defined by \((z_0, \ldots, z_{n+k}) \rightarrow (z_0, \ldots, z_{n+k}, 0)\).

It follows that the induced Thom space maps and homotopies \(P^a \times P^a \rightarrow T^{a+1} \rightarrow T^a\) are SAMs and symmetric homotopies (resp. homotopies), as required.

**Remark.** Let \(g: P^{a} \rightarrow R^{n+k}\) be an embedding. Then it is not difficult to see that a representative of the symmetric homotopy class obtained from \(g\) by the above construction is covered by \(h:S^a \times S^a \rightarrow S^{n+k}\), where

\[
h(x,y) = \begin{cases} 
v((x,y)e + i(||x + y||,||x - y||,g(\pm r(x + y))) \\ - g(\pm r(x - y))) \end{cases}
\]

if \(x \neq \pm y\),

\[
(x,y)e, \text{ if } x = \pm y.
\]

**Proof of (1.2).** Translation from axial maps of type \((n,k)\) to equivariant maps from \(V_n\) to \(S^{n+k}\) is given by restriction to \(V_n \subset S^a \times S^a\) of the map \(S^a \times S^a \rightarrow S^{n+k}\) covering the axial map. The remaining passage to embeddings of \(P^{a}\) in \(R^{n+k+1}\) is achieved by (2.1).

**References**