PERFECT RADICALS AND HOMOLOGY OF
GROUP EXTENSIONS

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A group epimorphism \( Q: G \to Q \) preserves perfect radicals if \( \phi PG = PQ \), where \( PG, PQ \) is maximal perfect in \( G, Q \) respectively. The following two questions are considered.

Question 1. If \( P\pi_1(B) \) acts trivially on \( H_\ell(F; Z) \), does the fibration \( F \to E \to B \) then have \( \pi_1(p)(P\pi_1(E)) = P\pi_1(B) \)?

Question 2. If \( f: X \to Y \) is a map of spaces which induces an isomorphism of homology groups, is \( \pi_1(X) \to \pi_1(Y) \to \pi_1(Y)/P\pi_1(Y) \) then an epimorphism?

It is shown that each question is equivalent to its group-theoretic counterpart obtained when the spaces involved are classifying spaces of discrete groups. It is also shown that an affirmative answer to the second question implies an affirmative answer to Question 1. By means of a direct product construction on finite nilpotent groups a continuum of examples is exhibited to resolve these question in the negative. This leaves to an example of an inclusion map of a locally finite \( p \)-group in a countable hypoabelian group which induces an isomorphism of all homology groups.

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1. Introduction and review

Perfect groups (those generated by their commutator elements) have claimed the attention of topologists in recent years from their relation to Quillen’s plus-construction as well as other matters such as J.H.C. Whitehead’s aspherical 2-complex question and laminations of manifolds. Since images and joins of perfect subgroups are also perfect, each group \( G \) contains a maximal such, its perfect radical \( PG \) (normal in \( G \)), and any epimorphism \( \phi: G \to Q \) must have \( \phi PG \leq PQ \). A fundamental question studied here is: When does \( \phi PG = PQ \)? In this event \( \phi \) (or the extension \( \text{Ker} \phi \to G \to Q \)) is said to be an epimorphism (or extension) preserving perfect radicals, or, more briefly, EP^2R. Before investigating the question itself we mention some results indicating its significance.
First recall that the plus-construction consists of an acyclic map \( q : X \to X^+ \) that induces an extension \( P\pi_1(X) \to \pi_1(X) \xrightarrow{\pi_1(q)} \pi_1(X^+) \). By saying that \( q \) is acyclic we mean that the homotopy fibre \( F_q \) has \( H_*(F_q; \mathbb{Z}) = 0 \) or, equally [1, (4.3)], both \( H_*(q) : H_*(X; \mathbb{Z}) \to H_*(X^+; \mathbb{Z}) \) and \( \pi_1(X) \) acts trivially on \( H_*(F_q; \mathbb{Z}) \). Then a major reason for the importance of the EP\(^R\) property is the following characterization of plus-constructive fibrations \( F \to E \xrightarrow{p} B \) (that is, such that, after the application of the plus-construction, the induced map from \( F^+ \) to the homotopy fibre \( F^+ \) of \( p^+ : E^+ \to B^+ \) is a homotopy equivalence). (We always assume \( F, E, B \) to have the homotopy type of connected CW-complexes.)

1.1. **Theorem** [3]. \( F \to E \xrightarrow{p} B \) is plus-constructive if and only if both

(a) \( \pi_1(p) \) is EP\(^R\), and

(b) \( P\pi_1(B) \) acts trivially on \( \pi_*(F^+, \ast) \).

In the case where the fibration comes from a group extension \( N \times G \to Q \) via the classifying space functor, condition (a) reduces to \( \phi \) being EP\(^R\). Applications of Theorem 1.1 to group-extension-theoretic problems may be found in [3, 4, 5]. Sufficient conditions on \( \phi \) (principally on \( N, G \)) for it to be EP\(^R\) include the following, additional to Theorem 1.1 above.

1.2. **Proposition** [4]. The extension \( N \times G \to Q \) is EP\(^R\) provided either

(i) it is split;

(ii) \( G^{(m)} \leq N \cdot PG \) for some finite \( m \); or

(iii) \( N^{(n)} \leq PG \) for some finite \( n \).

Note that (iii) includes the cases of perfect or central \( N \).

The finiteness of the originals occurring in the derived groups \( (G^{(0)} = G, G^{(k)} = [G^{(k-1)}, G^{(k-1)}]) \) in (ii), (iii) above is crucial to the sequel. The need for this is seen in the most obvious example of a non-EP\(^R\) extension, namely a free presentation of a perfect group. This example is suggestive in that the quotient of a free presentation acts highly non-trivially, indeed faithfully, on the homology of the kernel. (Homology is here always taken to have trivial integer coefficients.) Here is a further result along these lines.

1.3. **Proposition.** Let \( N = F/R' \) for some free group \( F \) of finite rank at least 2, where \( R \leq F' \) is normal in \( F \) and \( F/R \) is residually torsion-free nilpotent. If \( PQ \) acts trivially on \( H_1(N) \), then \( N \times G \to Q \) is EP\(^R\).

**Proof.** (Cf. [6, (3.7)]). First note by [9, Theorem B(1)] that \( N \) is centreless, so that the extension \( N \times G \to Q \) is induced from \( N \times \text{Aut}(N) \to \text{Out}(N) \) by some homomorphism \( \psi : Q \to \text{Out}(N) \). Since \( PQ \) acts trivially on \( H_1(N) \), the perfect group \( \psi(PQ) \) lies in \( \text{Ker}[\text{Out}(N) \to \text{Aut}(H_1(N))] \) which by [9, Theorem C] is residually nilpotent. Thus \( \psi(PQ) \), and hence the induced extension over \( PQ \), must be trivial; this gives the result. □
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The case of Proposition 1.3 above where $R = 1$ is of course just that of a free presentation. The next result shows the need to consider the action on all homology groups.

1.4. Proposition [2]. Let $F^+$ be a nilpotent space. If $P\pi_1(B)$ acts trivially on $H_\ast(F)$, then the fibration $F \to E \xrightarrow{p} B$ is plus-constructive, and so $\pi_1(p)$ is $EP^2R$.

These considerations naturally prompted the following questions (c. 1982).

Question 1. If $P\pi_1(B)$ acts trivially on $H_\ast(F)$, does the fibration $F \to E \xrightarrow{p} B$ then have $n_\ast(p)\in EP^2R$?

In particular (it would seem), there is a group-theoretic version.

Question 1. If $PQ$ acts trivially on $H_\ast(N)$, is the extension $N \to G \to Q$ then $EP^2R$?

Somewhat surprisingly, the two questions are equivalent.

1.5. Proposition. Let $F \to E \xrightarrow{p} B$ be a fibration such that $P\pi_1(B)$ acts trivially on $H_\ast(F)$ but $\pi_1(p)$ is not $EP^2R$. Then there exists a group extension $N \to G \xrightarrow{\phi} Q$ such that $PQ$ acts trivially on $H_\ast(N)$ but $\phi$ is not $EP^2R$, where the diagram

\[ \begin{array}{ccc}
BN & \rightarrow & BG \\
\downarrow v & & \downarrow u \\
F & \rightarrow & E \rightarrow B
\end{array} \]

commutes, all vertical maps are acyclic, and $p_0 = p : E \to B$.

Proof. The proof constructs the following diagram whose vertical maps are acyclic (that is, have homologically trivial fibre) and whose non-vertical sequences represent fibrations.

\[ \begin{array}{ccc}
BN & \rightarrow & BG \\
\downarrow v & \downarrow s & \downarrow B\phi \\
F & \rightarrow & T \rightarrow E \rightarrow B
\end{array} \]

By [12], for a suitable group $Q$ there exists an acyclic map $t : BQ \to B$. Form the pull-back $T = E \times_B BQ$; then the induced map $t' : T \to E$ is also acyclic [1, (4.1)]. By [12] again, there is a group $G$ and acyclic map $s : BG \to T$. Note that both $\pi_1(p')$ and $\pi_1(s)$ are surjective, so that the (homotopy) fibre of $p' \circ s$ is also the classifying space, $BN$ say, of a group. Let $\phi = \pi_1(p' \circ s)$; thus $B\phi = p' \circ s$ and $N = \text{Ker} \phi$. Then by [1, (4.2)] the map $v : BN \to F$ induced from $s$ on the fibres is also acyclic. Now let $u = t' \circ s$; because $t'$ and $s$ are acyclic, so is $u$. Arguing as in [1, p. 46], we may replace $p$ (unique up to homotopy and such that $p \circ u = t \circ B\phi$) by $p_0$ (unique up to homotopy under $BG$) such that $p_0 \circ u = t \circ B\phi$. 
Since $P\tau_1(B)$ acts trivially on $H_*(F)$ and contains the image of $P\tau_1(BQ)$ under $\tau_1(t)$, and $p': T \to BQ$ is the pull-back of $p: E \to B$, we deduce that $PQ$ acts trivially on $H_*(F)$. Then the fact that $\nu$ is acyclic implies that $PQ$ also acts trivially on $H_*(N)$. On the other hand, because $t$ is acyclic $\tau_1(t)$ has perfect kernel and is therefore EP$^2R$ (Proposition 1.2), so that $\tau_1(p) \circ \tau_1(u) = \tau_1(t) \circ \phi$, hence $\tau_1(p)$, must be EP$^2R$ whenever $\phi$ is. (This deduction uses the obvious facts that the composite of EP$^2R$ maps is EP$^2R$ and, conversely, that if two maps have EP$^2R$ composite then the latter is also EP$^2R$.)

We turn now to a second pair of questions, of quite independent interest.

**Question 2.** If $f: X \to Y$ is a map of spaces which induces an isomorphism of homology groups, is $\tau_1(X) \xrightarrow{\tau_1(f)} \tau_1(Y) \twoheadrightarrow \tau_1(Y)/P\tau_1(Y)$ then an epimorphism?

**Question 2$_Gp$.** If $\phi: G_1 \to G_2$ is a group homomorphism which induces an isomorphism on homology groups, is $G_1 \xrightarrow{\phi} G_2 \twoheadrightarrow G_2/PG_2$ then an epimorphism?

The passage to the quotients by $P\tau_1(Y)$ and $PG_2$ is so as to avoid uninteresting counterexamples obtained by embedding the image of $f$ or $\phi$ in its product with an acyclic space or group. Question 2 appears to be of some relevance to the study of laminations on manifolds [8]. The same arguments as in the proof of Proposition 1.5 also reveal Questions 2 and 2$_Gp$ to be equivalent.

1.6. **Proposition.** Suppose $f: X_1 \to X_2$ induces an isomorphism of homology groups but $\tau_1(X_1) \xrightarrow{\tau_1(f)} \tau_1(X_2) \twoheadrightarrow \tau_1(X_2)/P\tau_1(X_2)$ is not an epimorphism. Then there exist $f_0 \cong f: X_1 \to X_2$ and a group homomorphism $\phi: G_1 \to G_2$ such that $H_*(\phi): H_*(G_1) \to H_*(G_2)$ is an isomorphism but $G_1 \xrightarrow{\phi} G_2 \twoheadrightarrow G_2/PG_2$ is not an epimorphism and such that there are acyclic maps $s_i: BG_i \to X_i (i = 1, 2)$ with

\[
\begin{array}{ccc}
BG_1 & \xrightarrow{\nu} & BG_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{f_0} & X
\end{array}
\]

commutative.

Two positive results worth recording are as follows.

1.7. **Proposition** (Hopf, e.g. [11, p. 130]). Question 2 has an affirmative answer when $X$ and $Y$ are closed, connected, orientable manifolds of the same dimension.

1.8. **Theorem** [14]. Question 2 (resp. Question 2$_Gp$) has an affirmative answer when $\tau_1(Y)$ (resp. $G_2$) is a perfect-by-nilpotent group.

Actually, Proposition 1.7 may be generalised to other situations where a well-behaved degree map is defined. Here is the link between the two pairs of questions.
1.9. Proposition [4]. Given a fibration \( F \to E \xrightarrow{p} B \) with \( F_{p^+} \) the homotopy fibre of \( p^+: E^+ \to B^+ \), then the induced map of fibres \( f: F \to F_{p^+} \) has

(a) \( \pi_1(f) \) an epimorphism if and only if \( \pi_1(p) \) is \( EP^{R} \), and

(b) \( H_*(f) \) an isomorphism if and only if \( P\pi_1(B) \) acts trivially on \( H_*(F) \).

To relate this result to Question 2 observe that the exact sequence \( \pi_2(B^+) \to \pi_1(F_{p^+}) \to \pi_1(E^+) \) ensures that \( P\pi_1(F_{p^+}) = 1 \). Thus an affirmative answer to the second pair of questions implies an affirmative answer to the first. In fact, we proceed to answer all questions in the negative by exhibiting a counterexample to Question 1_{GP}.

2. The examples

As indicated in the introduction, we shall be interested in automorphisms of groups that act trivially on the homology. Examples of such automorphisms are locally inner automorphisms. By definition, an automorphism \( \alpha \) of a group \( H \) is locally inner, if given any finite set \( h_1, h_2, \ldots, h_n \) of elements of \( H \), there exists an element \( x \in H \) (in general depending on the finite set chosen) such that \( \alpha(h_i) = h_i^x = x^{-1}h_ix \) for \( 1 \leq i \leq n \). Since the homology of \( H \) is the direct limit of the homologies of the finitely generated subgroups of \( H \) [7, p. 121] and since inner automorphisms operate trivially on homology, we have the following lemma.

2.1. Lemma. Each locally inner automorphism of a group \( H \) operates trivially on \( H_*(H) \).

In this section, our aim will be to construct examples of groups \( G \) with normal subgroups \( N \) satisfying the following conditions.

2.2. Conditions. (a) \( PG = 1 \);

(b) \( Q = G/N \) is non-trivial and perfect;

(c) each element of \( G \) operates (by conjugation) on \( N \) as a locally inner automorphism.

These will answer Question \( Q_{CP} \) negatively.

By [4, (2.6)] these conditions exclude the possibility that \( N \) is finitely generated, since then Condition 2.2(c) says that the induced map \( Q \to \text{Out } N \) is trivial. The following question appears interesting.

Question 3. Among group extensions \( N \to G \to G/N \) satisfying Conditions 2.2(a) and (b) and such that \( G \) operates trivially on \( H_*(N) \), do there exist examples with \( N \) or \( G \) finitely generated or with \( G/N \) finite?

We will give our basic construction first and then indicate how it can be modified to produce other examples with various special properties. We write \( d(H) \) for the
minimum number of generators of a group $H$, and refer to the length of its derived series as the derived length of $H$.

2.3. Theorem. Let $G_1, G_2, \ldots$ be nilpotent groups with the following properties:

(a) There exists a fixed integer $m$ such that $d(K) \leq m$ for all $i = 1, 2, \ldots$ and subgroups $K$ of $G_i$.

(b) If $l_i$ is the derived length of $G_i$, then $l_i \to \infty$ as $i \to \infty$.

Let $D$ be the unrestricted direct product $\prod_{i=1}^{\infty} G_i$, and $N$ the corresponding restricted direct product. Let

$$G = \{ x \in D : \text{given } j \geq 1, \exists n(j) \text{ such that } x_i \in G_i^{(j)} \text{ for all } i \geq n(j) \}.$$

Then $G$ and $N$ satisfy Conditions 2.2(a)-(c).

To obtain examples of such groups we can take $G_i$ to be the subgroup of $GL(2, \mathbb{Z}/p^i\mathbb{Z})$ consisting of all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $b = c = 0$ and $a = d = 1 \pmod{p}$; its order is clearly $p^4(i-1)$. (See [13, Part 2, p. 179] for the argument that $l_i \to \infty$ with $i$ and [13, Part 2, p. 72] for the fact that here we can take $m = 9$.)

Proof. Since $G$ is a subgroup of the residually nilpotent group $D$, $G$ is residually nilpotent and so certainly $PG = 1$. Each element of $G$ is a sequence $g = (g_1, g_2, \ldots)$ with $g_i \in G_i$, and each element of $N$ is such a sequence in which only a finite number of terms differ from 1. If $x_1, \ldots, x_n$ is a finite set of elements of $N$, then there is a finite subset $A$ of $\{1, 2, \ldots\}$ such that all the components of all the $x_i$ outside $A$ are 1. With $g$ as above, let $g'$ be obtained by replacing each component of $g$ outside $A$ by 1. Then $g' \in N$, and $x_i^g = x_i^{g'}$ for $1 \leq i \leq n$. Thus, Condition 2.2(c) holds.

To establish Condition 2.2(b), let $y = (y_1, y_2, \ldots) \in G$. We must show that there exist finitely many elements $u_k, v_k \in G$ such that $y = \prod_{k} [u_k, v_k] \mod N$. Let $n(1), n(2), \ldots$ be integers corresponding to $y$ as given in the definition of $G$, where we may assume without loss of generality that $n(1) < n(2) < \ldots$.

For $j \geq 1$ and $n(j) \leq i < n(j + 1)$, we have $y_i \in G_i^{(j)}$ and so $y_i$ can be expressed as a product of commutators of elements of $G_i^{(j-1)}$. The point is that Theorem 2.3(a) and the nilpotence of $G_i^{(j)}$ ensure, by a well-known elementary result [10], that the number of commutators needed is at most $m$, and so bounded independently of $i$.

Thus for $j \geq 1$ and $n(j) \leq i < n(j + 1)$, we have

$$y_i = \prod_{k=1}^{m} [u_{ik}, v_{ik}]$$

where $u_{ik}, v_{ik} \in G_i^{(j-1)}$. Let $u_{ik} = v_{ik} = 1$ for $1 \leq i < n(1)$ and $k = 1, \ldots, m$. Now let $u_k = (u_{1k}, u_{2k}, \ldots)$ and $v_k = (v_{1k}, v_{2k}, \ldots) (1 \leq k \leq m)$. Clearly $u_k$ and $v_k$ belong to
$H$, and $\prod_{k=1}^{n-1} [u_k, v_k] \equiv y \mod N$, since the two sides agree except in the first $n(1) - 1$ components.

Finally, Theorem 2.3(b) shows that $G \not\cong N$, and hence $Q = G/N$ is nontrivial. □

Suppose now that in (2.3), $G_i$ is a finite $p_i$-group, where $p_1$, $p_2$, . . . are primes. If $p_i = p$ for infinitely many values of $i$, then clearly $l_i \to \infty$ as $i \to \infty$ through the sequence consisting of those values, and we see that $G$ will contain an infinite number of elements of order $p$, obtained by choosing components of order $p$ in appropriate groups $G_{i}^{(j)}$. On the other hand, if $p$ occurs only finitely often among the $p_i$, clearly $G$ has only a finite number of elements of order $p$. Thus we can vary the construction to ensure that any given non-empty subset of the set of all primes occurs as the set of primes $p$ such that $G$ has infinitely many elements of order $p$. This gives $2^\aleph_0$ pairwise non-isomorphic groups satisfying Conditions 2.2(a)–(c). The groups constructed are uncountable. However every non-trivial perfect group has a non-trivial countable perfect subgroup. To see this, let $P_i$ be any non-trivial finitely generated subgroup of $P$, and inductively let $P_{i+1}$ be a finitely generated subgroup of $P$ such that $P_i \leq P_{i+1}$. Then $\hat{P} = \bigcup_{i=1}^{\infty} P_i$ is clearly perfect and countable. Taking the inverse image $\hat{G}$ in $G$ of a non-trivial countable perfect subgroup of $G/N$ gives a countable group satisfying Conditions 2.2(a)–(c). Furthermore, for each prime $p$, $G$ has infinitely many elements of order $p$ if and only if $\hat{G}$ has. For if $G$ has infinitely many such elements, then $p_i = p$ for infinitely many $i$, and so $N$ already has infinitely many elements of order $p$.

2.4. Corollary. There exist $2^\aleph_0$ pairwise non-isomorphic countable groups satisfying Conditions 2.2(a)–(c).

We are now in a position to construct directly counterexamples to Question 2Gp. To the sequence of finitely generated subgroups $P_i$ as above we associated a sequence of finitely generated free groups $R_i$ and maps $R_i \to P_i$ and $R_i \to R_{i+1}'$ so that each square

\[
\begin{array}{ccc}
R_i & \longrightarrow & R_{i+1}' \\
\downarrow & & \downarrow \\
P_i & \longrightarrow & P_{i+1}'
\end{array}
\]

commutes. Let $R = \lim_{\longrightarrow} R_i$. Then not only is $R$ perfect, but it shares with free groups the property that $H_n(R) = 0$, $n \geq 2$. So $R$ is a countable acyclic group. Now let $N \rightarrowtail M \twoheadrightarrow R$ be the induced extension over $R$; the conjugation action of $M$ on $N$ is again locally inner. Thus by Proposition 1.9(b) or the Serre homology sequence applied to the fibration $BN \rightarrow BM \rightarrow BR$, $H_k(\theta)$ is an isomorphism. Now the epimorphism $R \twoheadrightarrow \hat{G}/N$ induces an epimorphism $M \twoheadrightarrow \hat{G}$; since $P\hat{G} = 1$ this factors through an epimorphism $M/PM \rightarrow \hat{G}$. However, $N \rightarrowtail M \rightarrow M/PM$ being surjective would then have the contradictory consequence that $N \rightarrow \hat{G}$ is also surjective.
Hence $\theta : N \to M$ is a monomorphism of countable groups which provides a negative answer to Question 2$_{\text{gp}}$. (In fact, if the requirement of countability is relaxed, then this method constructs a counterexample to Question 2$_{\text{gp}}$ from any counterexample to Question 1$_{\text{gp}}$.)

At this juncture we note the following alternative pair of questions to Questions 2 and 2$_{\text{gp}}$, again designed to avoid uninteresting counterexamples.

**Question 2'.** If $f : X \to Y$ is a map of spaces which induces an isomorphism of homology groups, and if $\pi_1(Y)$ is hypoabelian ($\mathcal{P}\pi_1(Y) = 1$), is $\pi_1(f)$ then an epimorphism?

**Question 2$_{\text{gp}}$.** If $\phi : G_1 \to G_2$ is a group homomorphism which induces an isomorphism of homology groups, and $G_2$ is hypoabelian, is $\phi$ then an epimorphism?

Concerning the latter question, a theorem of Evens asserts that if $G_1$ and $G_2$ are both finite, then (without the assumption $\mathcal{P}G_2 = 1$) $\phi$ is an isomorphism. So an affirmative answer to Question 2$_{\text{gp}}$ when $G_1$ is finite would allow this conclusion with the assumption $\mathcal{P}G_2 = 1$ replacing that of finiteness of $G_2$. It is apparent from the remarks following Proposition 1.9 that Question 2' has already been answered negatively. However, unlike previously, Question 2$_{\text{gp}}$ does not appear to be generally accessible from Question 2' through topological arguments involving [12]. It is therefore noteworthy that we are now able to set up a counterexample by using instead the group-theoretic constructions above.

Let $K$ denote the kernel of the epimorphism $R \to \bigcup P_i$ obtained above, and consider time extension $N \to M_0 \to R/PK$ induced from $\rho : R/PK \to R/K = \bigcup P_i$. Since $\text{Ker} \rho \cong K/PK$ is hypoabelian (Proposition 1.2(iii)), as is $\hat{G}$, it follows from the induced extension $\text{Ker} \rho \to M_0$ that $\mathcal{P}M_0 = 1$. On the other hand, because $N \to M_0 \to R/PK$ is induced from the original extension $N \to G \to Q$, we again have trivial action of the quotient group on $H_*(N)$. So, as before, $H_*(\theta_0)$ will be an isomorphism once it is known that $R/PK$ is acyclic. Now $R$ is by construction locally free. So the subgroup $PK$ will also be locally free, which implies that it shares with free groups the property that, for $n \geq 2$, its $n$th homology group is zero; hence, being perfect, $PK$ is acyclic. Therefore the map $BR \to B(R/PK)$ has acyclic fibre $BPK$ and so is acyclic. This makes $\tilde{H}_*(R/PK) = \tilde{H}_*(R) = 0$, so that $R/PK$ is indeed acyclic, and the monomorphism $\theta_0 : N \to M_0$ of countable groups is the example we seek.

A further variation leads to groups with trivial centre. The groups of Theorem 2.3 are hypercentral, and in fact if $Z_n(G)$ is the $n$th term of the upper central series of $G$, we have $G = \bigcup_{n=1}^{\infty} Z_n(G)$.

However, suppose for simplicity that each $G_i$ is a finite $p$-group, where $p$ is a fixed prime. Let $F_i$ be a free group of rank equal to the order of $G_i$, and let $G_i$ act on $F_i$ by permuting a set of free generators according to the regular representation of $G_i$. Then the semidirect product $L_i = F_iG_i$ is residually a finite $p$-group. For $F_i$ is residually a finite $p$-group [13, Part 2, p. 117], that is, if $1 \neq x \in F_i$, then there
exists a normal subgroup $K$ of finite $p$-power index in $F_i$ such that $x \notin K$. Clearly $\bigcap_{i \in G_i} K^x_i$ is a normal subgroup of finite $p$-power index in $L_i$, not containing $x$.

Now in the unrestricted direct product $L = \prod L_i$, let $F$ be the restricted direct product of the $F_i$, let $G^* = F G$ and $N^* = FN$. By the above, $L$ is residually a finite $p$-group. Therefore so is $G^*$, and in particular $PG^* = 1$. As previously, $G^*/N^*$ is a non-trivial perfect group, and every element of $G^*$ acts by locally inner automorphisms on $N^*$. Now $L_i$ is clearly a centreless group, and hence so is $N^*$, which is the restricted direct product of the $L_i$. In summary, we have the following corollary.

2.5. Corollary. There exists a residually finite $p$-group $G^*$ with a normal subgroup $N^*$ such that Conditions 2.2(a)-(c) hold, and further $Z(N^*) = 1$.

Note that $N^*$ centreless implies that the extension $N^* \twoheadrightarrow G^* \to G^*/N^*$ is induced from $N^* \to Aut(N^*) \to Out(N^*)$ by some homomorphism $\psi : G^*/N^* \to Out(N^*)$. If $Aut_b(N^*)$ comprises those automorphisms which act trivially on $H_b(N^*)$, and $Out_b(N^*) = Aut_b(N^*)/N^*$, then Condition 2.2(c) forces the image of $\psi$ to lie in $Out_b(N^*)$. So $POut_b(N^*) \neq 1$. However, $Out_b(N^*)$ is residually nilpotent because $N^*$ is [13, Part 1, p. 56]. Hence the extension $N^* \to Aut_b(N^*) \to Out_b(N^*)$ provides a universal counterexample to Question 1 among those with kernel $N^*$.

References