Math. Nachr. 121 (1985) 25—31

Comparison Theorems in Homotopy Theory Via Operations on Homotopy Sets

A. J. BERRICK of Singapore and K. H. KAMPS of Hagen

(Received July 26, 1983)

In this paper we present comparison theorems in semicubical homotopy theory. Topological versions are more or less well known and have been obtained amongst others by A. DOLD ([6], [7]), I. M. JAMES ([11]), P. R. HEATH ([9]), S. P. LAM ([15]), and the first author ([2], [3]). A partially abstract approach has been given by the first author in [3]. We use operations on homotopy sets and the notion of CHEP (covering homotopy extension property).

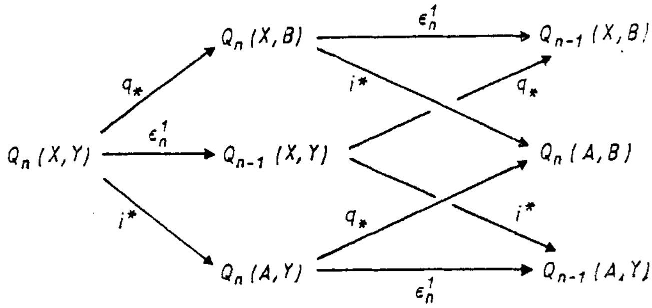
1. Preliminaries

We assume that the reader is familiar with the basic notions of homotopy theory (homotopic, homotopic under A, over B, under A and over B and the corresponding notions of homotopy equivalence) (see [5], § 0) in the framework of semicubical homotopy theory ([12], 1, 2, 3, [13], 1, 2, [14], 1). Notation is as in [13] and [14]. Thus $\mathcal{S}$ stands for the category of sets, $\mathcal{K}$ for the category of semicubical sets, ε_n^i, ζ_n^j denote the face and degeneracy operators of a semicubical set. Recall the definition of a semicubical homotopy system $Q : \mathcal{C} \times \mathcal{C} \to \mathcal{K}$ in a category $\mathcal{C}$ ([13], 2.1). Every cylinder functor in a category induces a semicubical homotopy system (see [13], 2.2). Thus semicubical homotopy theory includes homotopy theory in the categories $\mathcal{T}op$ (topological spaces), $\mathcal{CG}$ (compactly generated HAUSDORFF spaces), $\partial\mathcal{A}$ (chain complexes over an abelian category $\mathcal{A}$ ([15], 2)), and $\mathcal{G}d$ (groupoids). It also includes equivariant homotopy theory, that is homotopy theory in the category G-$\mathcal{T}op$ of G-spaces where G is a topological group.

2. The covering homotopy extension property

Let $Q : \mathcal{C} \times \mathcal{C} \to \mathcal{K}$ be a semicubical homotopy system in a category $\mathcal{C}$.

2.1. Definition ([14], 1.3). Let $i : A \to X$, $q : Y \to B$ be morphisms of $\mathcal{C}$, and $n \geqq 1$ a natural number. We say that (i, q) *has* CHEP (n) (CHEP = *covering homotopy extension* *property*), if for $\varepsilon = 0, 1$



is a weak limit diagram in $\mathscr{S}$. This means that for

$$\beta \in Q_n(X, B), \qquad f \in Q_{n-1}(X, Y), \qquad \alpha \in Q_n(A, Y)$$

with $\varepsilon_n^1\beta = q_* f$, $\varepsilon_n^1\alpha = i^* f$, $i^*\beta = q_*\alpha$ there exists $\Phi \in Q_n(X, Y)$ with $q_*\Phi = \beta$, $\varepsilon_n^1\Phi = f$, $i^*\Phi = \alpha$.

Remark. CHEP (n) is selfdual. CHEP (n) implies CHEP (m) for $m \leqq n$.

2.2. Examples. (i, q) has CHEP (n) for all n in each of the following situations.

a) $\mathscr{C} = \mathscr{T}op$, i a closed cofibration, q a HUREWICZ-fibration ([18], Theorem 4);

b) $\mathscr{C} = G\text{-}\mathscr{T}op$, i a closed G-cofibration (a G-cofibration which is the inclusion of a closed G-subspace), q a G-fibration;

c) $\mathscr{C} = \mathscr{C}\mathscr{G}$, i a cofibration, q a HUREWICZ-fibration ([8], Theorem 2.1);

d) $\mathscr{C} = \mathscr{T}op$, i the inclusion of a relative CW-complex (X, A), q a SERRE-fibration ([15], 7.8.9);

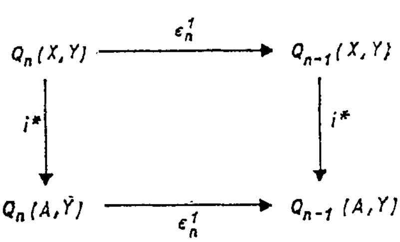
e) $\mathscr{C} = \mathscr{T}op$, i the inclusion of a closed subspace A of a metric space X, Y a metric ANR ([10], III, 6.), q a regular HUREWICZ-fibration ([1], Theorem (2.4));

f) $\mathscr{C} = \partial\mathscr{A}$, i a normal monomorphism, q a normal epimorphism ([15], 3);

g) $\mathscr{C} = \mathscr{G}d$, i the inclusion of a subgroupoid A of a groupoid X, q a fibration of groupoids ([4], Proposition 2.11).

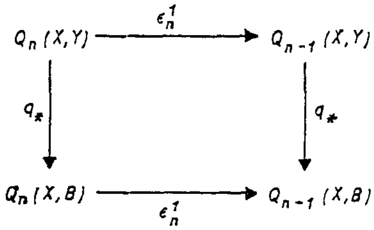
Note that Example b) is due to the fact that the formulae of [18] also work in the equivariant case, while f) can be proved by elementary computations.

2.3. Definitions. a) Let $i : A \to X$ be a morphism of $\mathscr{C}$, Y an object of $\mathscr{C}$, and $n \geqq 1$ a natural number. We say that *i has* HEP (n) *for Y* (HEP $=$ *homotopy extension property*), if for $\varepsilon = 0, 1$



is a weak limit in $\mathscr{S}$.

b) Let $q : Y \to B$ be a morphism of $\mathscr{C}$, X an object of $\mathscr{C}$, and $n \geqq 1$ a natural number. We say that *q has* CHP (n) *for* X (CHP = *covering homotopy property*), if for $\varepsilon = 0, 1$



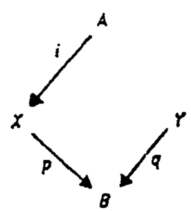
is a weak limit in $\mathscr{S}$.

2.4. Remark. If $*$ is a terminal object of $\mathscr{C}$ and if $Q_n(X, -)$, $Q_n(A, -)$, $Q_{n-1}(X, -)$ preserve terminal objects, then $i : A \to X$ has HEP (n) for Y, if and only if (i, c_Y) has CHEP (n), where $c_Y : Y \to *$ denotes the unique morphism from Y to $*$. Dually, if ϕ is an initial object of $\mathscr{C}$ and if $Q_n(-, Y)$, $Q_n(-, B)$, $Q_{n-1}(-, Y)$ preserve initial objects, then $q : Y \to B$ has CHP (n) for X, if and only if (c^X, q) has CHEP (n), where $c^X : \phi \to X$ denotes the unique morphism from ϕ to X.

3. An operation on homotopy sets

Let $Q : \mathscr{C} \times \mathscr{C} \to \mathscr{K}$ be a semicubical homotopy system in a category $\mathscr{C}$, let

(3.1)



be a diagram in $\mathscr{C}$. Let Q' be the semicubical subset of $Q(A, Y)$ with

$$Q'_0 = \{j \in \mathscr{C}(A, Y) \mid qj = pi\},$$
$$Q'_n = \{\psi \in Q_n(A, Y) \mid q_*\psi = \zeta^1_{n-1} \cdot \ldots \cdot \zeta^1_0(pi)\}, \qquad n \geqq 1.$$

We assume that Q satisfies the Kan-conditions DNE (2) and DNE (3) ([12], 3, [13], 2). Then the Kan-conditions E (2) and E (3) hold for Q'. Let $\mathscr{G}'$ be the fundamental groupoid $\Pi Q'$ of Q' ([13], 1). Thus the objects of $\mathscr{G}'$ are the morphisms $j : A \to Y$ of $\mathscr{C}$ with $qj = pi$, while the morphisms $u : j \to j'$ of $\mathscr{G}'$ are equivalence classes $u = [G]$ of elements $G \in Q_1(A, Y)$ with $G_0 = j$, $G_1 = j'$ and $q_*G = \zeta^1_0(pi)$.

We are now in a position to define a groupoid operation.

3.2. Definitions. Assume that (i, q) has CHEP (2).

a) Let $j : A \to Y$ be an object of $\mathscr{G}'$. Then define

$$T(j) := [(i, p), (j, q)]^A_B$$

to be the set of homotopy classes $[f]^A_B$ under A and over B from (i, p) to (j, q).

b) Let $u = [G] : j \to j'$ be a morphism of $\mathscr{G}'$, and $[f]_B^A \in [(i, p), (j, q)]_B^A$. Since (i, q) has CHEP (1), there exists $L \in Q_1(X, Y)$ such that $L_0 = f$, $q_* L = \zeta_0^1(p)$, $i^* L = G$. For $f' := L_1$ we have $f'i = j'$, $qf' = p$. Then define

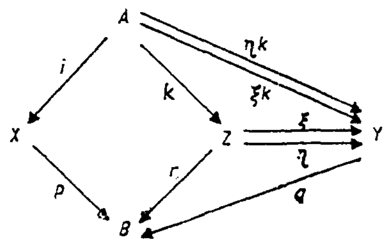
$$(Tu)\,[f]_B^A := [f']_B^A .$$

Using the KAN-conditions, CHEP (2) for (i, q) and Lemma 2.1 of [14], it is not hard to show that T is well defined on morphisms and gives a functor $T : \mathscr{G}' \to \mathscr{S}$. It follows that Tu is a bijection.

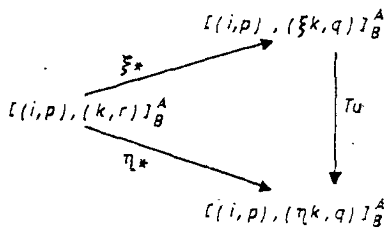
Remark. The construction of T is a modification of a special case of [14], 2.2.

The following naturality condition is crucial for the applications.

3.3. Lemma. *Let*



be a diagram in $\mathscr{E}$ such that the left square commutes, $\xi, \eta : r \to q$ are morphisms over B and (i, q) has CHEP (2). *Let $\psi : \xi \underset{B}{\simeq} \eta$ be a homotopy over B. Then $u := [k^*\psi] : \xi k \to \eta k$ is a morphism of $\mathscr{G}'$. We claim that the diagram*



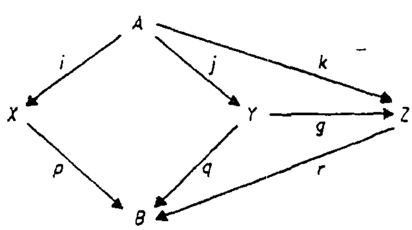
where $\xi_*[f]_B^A = [\xi f]_B^A$, is commutative.

Proof. If $[f]_B^A \in [(i, p), (k, r)]_B^A$, then $L := f^*\psi$ satisfies $L_0 = \xi k$, $q_* L = \zeta_0^1(p)$, $i^* L = k^*\psi$. Thus

$$(Tu)\,\xi_*[f]_B^A = [L_1]_B^A = [\eta f]_B^A = \eta_*[f]_B^A . \quad \square$$

By standard arguments (see [5], (10.5)) we obtain

3.4. Proposition. *Let*



be a commutative diagram in $\mathscr{C}$ such that (i, q) and (i, r) have CHEP (2). *Then, if g is a homotopy equivalence over B,*

$$g_* : [(i, p), (j, q)]_B^A \to [(i, p), (k, r)]_B^A$$

is bijective.

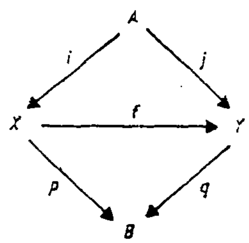
4. Comparison theorems

We apply the results of section 3 to deduce comparison theorems in homotopy theory.

Let Q be a semicubical homotopy system in $\mathscr{C}$ such that Q satisfies DNE (2) and DNE (3).

4.1. Proposition. *If in the commutative diagram in $\mathscr{C}$*

(4.2)



each of (i, p), (i, q), (j, p), (j, q) has CHEP (2), *then f is a homotopy equivalence under A and over B, provided that either*

(1) *f is a homotopy equivalence over B*

or

(1*) *f is a homotopy equivalence under A.*

Proof. By duality, we may restrict ourselves to condition (1). The assertion is equivalent to the fact that

$$f_* : [(i, p), (i, p)]_B^A \to [(i, p), (j, q)]_B^A$$

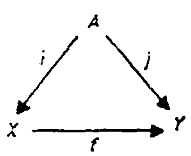
and

$$f_* : [(j, q), (i, p)]_B^A \to [(j, q), (j, q)]_B^A$$

are bijective. But this follows from 3.4. □

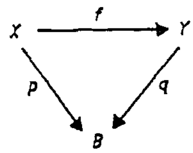
By specialization according to 2.4, we get the following results due to Dold ([7], 3.6, [6], 6.1) in the topological case.

4.3. Proposition. *Let*



be a commutative diagram in $\mathscr{C}$ such that i, j have HEP (2) *for X and Y. Then, if f is a homotopy equivalence, f is a homotopy equivalence under A.*

4.4. Proposition. *Let*



be a commutative diagram in $\mathscr{C}$ *such that* p, q *have* CHP (2) *for* X *and* Y. *Then, if* f *is a homotopy equivalence,* f *is a homotopy equivalence over* B.

Example. $\mathscr{C} = \mathscr{T}op$, X, Y CW-complexes, p, q Serre-fibrations.

Combination of 4.1, 4.3, 4.4 yields our main result.

4.5. Theorem (Comparison theorem). *If in the commutative diagram* (4.2) f *is a homotopy equivalence, then* f *is a homotopy equivalence under* A *and over* B, *provided that both* (1) *and either* (2) *or* (2*) *hold:*

(1) (i, p), (i, q), (j, p), (j, q) *satisfy* CHEP (2)

(2) i, j *satisfy* HEP (2) *for* X *and* Y,

(2*) p, q *satisfy* CHP (2) *for* X *and* Y.

4.6. Examples. The following table contains a series of examples of Theorem 4.5 (see [11], (6.1), [2], 2.3, [3], 2.3, [9], 2.3).

$\mathscr{C}$	i, j	p, q
$\mathscr{T}op$	closed cofibrations	Hurewicz-fibrations
G-$\mathscr{T}op$	closed G-cofibrations	G-fibrations
$\mathscr{C}\mathscr{S}$	cofibrations	Hurewicz-fibrations
$\mathscr{T}op$	inclusions of relative CW-complexes	Serre-fibrations
$\mathscr{T}op$	inclusions of A as a closed subspace of a metric ANR	regular Hurewicz-fibrations
$\partial\mathscr{A}$	normal monomorphisms	normal epimorphisms
$\mathscr{G}d$	inclusions of A as a subgroupoid	fibrations of groupoids

4.7. Remark. As one expects from the topological case ([5], (2.18), (6.21)), Proposition 4.3 and, dually, 4.4 hold, if the homotopy extension property HEP (2) is weakened to the weak homotopy extension property WHEP (2) ([13], 3.8) and, dually, the covering homotopy property CHP (2) is replaced by the weak covering homotopy property WCHP (2) ([13], 3.1 (2)) (see also [12], 6).

Thus the generalized version of 4.4 applied to the category $\mathscr{T}op$, together with Proposition 4.3 applied to the category $\mathscr{T}op_B$ of topological spaces over B, gives Lam's comparison theorem ([15], 1.3) which assumes i and j to be cofibrations over B and p and q to be weak fibrations ($=h$-fibrations in the sense of [5], (6.4)).

References

[1] G. Allaud, E. Fadell, A fiber homotopy extension theorem. Trans. Amer. Math. Soc. **104** (1962) 239–251
[2] A. J. Berrick, The Samelson ex-product. Quart. J. Math. Oxford (2) **27** (1976) 173–180
[3] —, Ex-homotopy comparison theorems. J. Singapore Nat. Acad. Sci. **10–12** (1983) 52–55
[4] R. Brown, Fibrations of groupoids. J. Algebra **15** (1970) 103–132
[5] T. tom Dieck, K. H. Kamps, D. Puppe, Homotopietheorie. Lecture Notes in Math. **157**, Berlin 1970
[6] A. Dold, Partitions of unity in the theory of fibrations. Ann. of Math. **78** (1963) 223–255
[7] —, Halbexakte Homotopiefunktoren. Lecture Notes in Math. **12**, Berlin 1966
[8] H. M. Hastings, Fibrations of compactly generated spaces. Michigan Math. J. **21** (1974) 243–251
[9] P. R. Heath, Homotopy equivalence of a cofibre fibre composite. Canad. J. Math. **29** (1977) 1152–1156
[10] S.-T. Hu, Theory of retracts. Detroit 1965
[11] I. M. James, Bundles with special structure I. Ann. of Math. **69** (1969) 359–390
[12] K. H. Kamps, Kan-Bedingungen und abstrakte Homotopietheorie. Math. Z. **124** (1972) 215–236
[13] —, Zur Homotopietheorie von Gruppoiden. Arch. Math. **23** (1972) 610–618
[14] —, Operationen auf Homotopiemengen. Math. Nachr. **81** (1978) 233–241
[15] —, Note on normal sequences of chain complexes. Colloq. Math. **39** (1978) 225–227
[16] S. P. Lam, A note on ex-homotopy equivalence. Indag. Math. **42** (1980) 33–37
[17] E. H. Spanier, Algebraic Topology. New York 1966
[18] A. Strøm, Note on cofibrations. Math. Scand. **19** (1966) 11–14

National University of Singapore
Department of Mathematics
Kent Ridge
Singapore 0511

Fernuniversität
Fachbereich Mathematik und Informatorik
Postfach 940
D-5800 Hagen