HERMITIAN PERIODICITY AND COHOMOLOGY OF INFINITE ORTHOGONAL GROUPS

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Abstract. As an application of our papers in hermitian $K$-theory, in favourable cases, we prove the periodicity of hermitian $K$-groups with a shorter period than previously obtained. We also compute the homology and cohomology with field coefficients of infinite orthogonal and symplectic groups of specific rings of integers in a number field.

1. Introduction

We provide applications of our previous papers [2] and [4] in two directions. Firstly, we prove a refinement of our periodicity theorem proved in [4] which leads to a shorter period for hermitian $K$-groups of specific rings $A$. This is the case if $A$ is an algebra over the real subfield $R$ of a cyclotomic field. The computation of the hermitian $K$-theory of $R$ is detailed in [2] and enables us to define explicit “Bott elements” for $R$. An important subcase is when $A$ is an $F$-algebra, where $F$ is algebraically closed. This leads to the classical 8-periodicity of the hermitian $K$-groups of $A$: see Theorem 2.2.

The second application is the computation of the cohomology and homology with fields coefficients of the infinite orthogonal and symplectic groups associated to specific rings of 2-integers in a number field. This computation is quite explicit and relies partly on computations made in [6] for finite coefficients. For rational coefficients, these results are particular cases of those of Borel [5].

2. Refinements of the periodicity theorem in hermitian $K$-theory

In this section we establish a more refined periodicity theorem for a class of rings introduced in [4], and there called “hermitian regular”. As seen in [3], this class includes many rings (and more generally schemes) of geometric nature. First, we have a lemma that clarifies the definition in the setting of $K$-theoretic Bott periodicity. For this, we use the “positive Bott element” in $1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p)$ constructed in [4, Theorem 1.1] for a 2-power $p \geq 8$.

Lemma 2.1. Let $A$ be a ring with involution such that $1/2 \in A$, and let $m = 2^\nu$ and $p = \sup\{8, 2^{\nu - 1}\}$. We assume the existence of an integer $d$, such that the cap-product with the Bott element in $K_p(\mathbb{Z}; \mathbb{Z}/m)$ induces an isomorphism

$$K_n(A; \mathbb{Z}/m) \xrightarrow{\approx} K_{n+p}(A; \mathbb{Z}/m).$$

for $n \geq d$. Then the following are equivalent.

Date: January 13, 2012.

2000 Mathematics Subject Classification. 19D50.

Key words and phrases. Periodicity, hermitian $K$-theory, group cohomology.

First and second authors partially supported by the National University of Singapore R-146-000-097-112, and first author by Singapore Ministry of Education grant MOE2010-T2-2-12. Third author partially supported by RCN 185335/V30.
implies (iii). Without loss of generality, we may assume that 

\[ \text{Example 4.3} \] that (ii) is equivalent to (iii), it remains to show that (i) 

Since (i) is evidently a special case of (iii), and it is shown in [4, Theorems 

abbreviated] that arise from cup-products with Bott elements.

There exists an integer \( n \) such that all iterated cup-products with the positive Bott element in \( 1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p) \) induce isomorphisms 

\[ K_n(A; \mathbb{Z}/m) \xrightarrow{\sim} K_{n+p}(A; \mathbb{Z}/m) \]

and 

\[ K_{n+1}(A; \mathbb{Z}/m) \xrightarrow{\sim} K_{n+1+p}(A; \mathbb{Z}/m). \]

(ii) \( A \) is hermitian regular, as in Definition 0.5 of [4].

(iii) Whenever \( n \geq d \), cup-product with the positive Bott element in \( 1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p) \) induces an isomorphism 

\[ \varepsilon KQ_n(A; \mathbb{Z}/m) \xrightarrow{\sim} \varepsilon KQ_{n+p}(A; \mathbb{Z}/m). \]

**Proof.** Since (i) is evidently a special case of (iii), and it is shown in [4, Theorems 0.7 and 4.2, Example 4.3] that (ii) is equivalent to (iii), it remains to show that (i) implies (iii). Without loss of generality, we may assume that \( n \geq d \).

This follows by the argument of “downward Karoubi induction” as in [1, Theorem 3.1(b)], applied to the commuting diagrams of exact sequences (notation obviously abbreviated)

\[
\begin{align*}
-\varepsilon KQ_{n+1} & \rightarrow -\varepsilon U_n \rightarrow K_n & -\varepsilon KQ_n & \rightarrow -\varepsilon KQ_{n+p} \\
\downarrow \epsilon & \downarrow \epsilon & \downarrow \epsilon & \downarrow \epsilon \\
K_{n+1+p} & \rightarrow \varepsilon U_{n+p} \rightarrow K_n & -\varepsilon KQ_{n+p} & \\
\end{align*}
\]

and

\[
\begin{align*}
K_n & \rightarrow \varepsilon V_{n-1} \rightarrow \varepsilon KQ_{n-1} \rightarrow K_{n-1} \\
\downarrow \epsilon & \downarrow \epsilon & \downarrow \epsilon & \downarrow \epsilon \\
K_{n+p} & \rightarrow \varepsilon V_{n-1+p} \rightarrow \varepsilon KQ_{n-1+p} \rightarrow K_{n-1+p} \\
\end{align*}
\]

that arise from cup-products with Bott elements.

The most interesting result in this section is the following generalization of the theorem proved in [8]. It is a consequence of more general results which are proved in this paper.

**Theorem 2.2.** Let \( F \) be an algebraically closed field of characteristic \( \neq 2 \) with the trivial involution, and let \( A \) be an involutive \( F \)-algebra which is hermitian regular as above. Suppose that, with \( m \) prime to the characteristic of \( F \), for \( n \geq d \) the classical periodicity map 

\[ K_n(A; \mathbb{Z}/m) \rightarrow K_{n+2}(A; \mathbb{Z}/m) \]

is an isomorphism. Then, for \( n \geq d \) the hermitian \( K \)-groups \( \varepsilon KQ_n(A; \mathbb{Z}/m) \) are periodic of period \( 8 \) with respect to \( n \), the isomorphism being given by the cup-product with a “Bott element” in \( 1KQ_8(F; \mathbb{Z}/m) \).

In order to prove this theorem and more general ones below, we combine our general periodicity theorems [4, Theorem 0.7] with the more detailed information in [4, Theorem 2.6] for totally real 2-regular number fields, so as to obtain sharper periodicity results for certain algebras \( A \) with involution.

Writing \( \mathbb{Z}' = \mathbb{Z}[1/2] \), for \( \alpha > 2 \) define \( R = R_\alpha^+ \) to be the ring \( \mathbb{Z}'[\zeta_{2^\alpha} + \zeta_{2^\alpha}^{-1}] \) of 2-integers in the maximal real subfield \( F = \mathbb{Q}(\zeta_{2^\alpha} + \zeta_{2^\alpha}^{-1}) \) of the cyclotomic field \( \mathbb{Q}(\zeta_{2^\alpha}) \), provided with the trivial involution. Then, from e.g. [2], \( F \) is a totally real 2-regular number field to which Proposition 2.3 of [4] applies, as follows. Let \( \nu \geq 4 \), \( p = 2^{\nu-1} \) and \( M = 2^{\nu+a-2} = 2p \cdot 2^{a-2} \). Then the image of the Bott element in \( 1KQ_p(R; \mathbb{Z}/2p) \) by the canonical map 

\[ 1KQ_p(\mathbb{Z}'; \mathbb{Z}/2p) \rightarrow 1KQ_p(R; \mathbb{Z}/2p) \]

is the reduction mod \( 2p \) of an “exotic Bott element” \( \bar{b} \in 1KQ_p(R; \mathbb{Z}/M) \).
Theorem 2.4 of [4] now takes the following form. The theorem stated afterwards is a direct consequence.

Lemma 2.3. [4] For \( n \geq 0 \), the cup-product with \( b \) in \( KQ_p(R; \mathbb{Z}/M) \) induces an isomorphism
\[
\overline{b} : \varepsilon KQ_n(R; \mathbb{Z}/M') \xrightarrow{\cong} \varepsilon KQ_{n+p}(R; \mathbb{Z}/M')
\]
whenever \( M' \) divides \( M \).

\[\square\]

Theorem 2.4. Let \( A \) be an \( R \)-algebra with an involution that is trivial\(^1\) on \( R \), and suppose that \( A \) is hermitian regular. Let \( \nu \geq 4 \) and \( p = 2^{\nu-1} \), and assume the existence of an integer \( d \), such that, for \( n \geq d \), the cup-product with the Bott element in \( K_p(Z; \mathbb{Z}/2p) \) induces an isomorphism
\[
K_n(A; \mathbb{Z}/m) \xrightarrow{\cong} K_{n+p}(A; \mathbb{Z}/m)
\]
whenever \( m \) divides \( 2p \). Then for \( n \geq d \), the cup-product with the exotic Bott element \( b \) defined above induces an isomorphism
\[
\varepsilon KQ_n(A; \mathbb{Z}/M') \xrightarrow{\cong} \varepsilon KQ_{n+p}(A; \mathbb{Z}/M')
\]
whenever \( M' \) divides \( 2^{\nu+a-2} = 2p \cdot 2^{a-2} \).

Proof. By using cofinality in direct systems as in [4, §4] and applying the lemma above, we observe that the group \( \lim_{a} KQ_{n+p}(A; \mathbb{Z}/M') \) may be computed by taking cup-product either with the usual Bott element in \( 1KQ_p(Z; \mathbb{Z}/2p) \) or with an exotic Bott element in \( 1KQ_p(Z; \mathbb{Z}/(2p \cdot 2^{a-2})) \). Therefore, the theorem is a consequence of Theorem 4.5 in [4, Section 4].\[\square\]

When \( \ell \) is an odd prime the periodicity statement is quite different as we shall see, in contrast with the situation in algebraic \( K \)-theory (see also [11] and [3] for very similar arguments). We consider the algebra \( R_\alpha = \mathbb{Z}[[\zeta_\alpha]] \) and the group \( K_2(R_\alpha; \mathbb{Z}/\ell^\alpha) \), \( \alpha > 2 \). From the exact sequence
\[
K_2(R_\alpha) \rightarrow K_2(R_\alpha) \rightarrow K_2(R_\alpha; \mathbb{Z}/\ell^\alpha) \rightarrow K_1(R_\alpha) \rightarrow K_1(R_\alpha)
\]
we can define a “Bott element” \( u \) in the group \( K_2(R_\alpha; \mathbb{Z}/\ell^\alpha) \) which maps to a generator of the kernel of the map between the \( K_1 \)-groups. Now let \( \sigma \) be the involution on the \( K \)-groups induced by the duality. Then \( c = u \cdot \sigma(u) \) in \( K_4(R_\alpha; \mathbb{Z}/\ell^\alpha) \) is invariant by \( \sigma \), and therefore belongs to the group\(^2\)
\[
1KQ_4(R_\alpha; \mathbb{Z}/\ell^\alpha) \cong 1K_4(R_\alpha; \mathbb{Z}/\ell^\alpha)_+ \cong 1K_4(R_\alpha^+; \mathbb{Z}/\ell^\alpha)_+ \cong 1KQ_4(R_\alpha^+; \mathbb{Z}/\ell^\alpha)_+,
\]
where the last isomorphism is a transfer map between \( K \)-groups and \( KQ \)-groups. Moreover, if \( \nu > 0 \), a classical Bockstein argument shows that \( \ell^{\nu-1} \) may also be lifted to an “exotic Bott element” \( b_+ \) in the group \( 1KQ_{4\ell^{\nu-1}}(R_\alpha^+; \mathbb{Z}/\ell^\alpha+\nu-1)_+ \). On the other hand, as in Section 5 of [4], we may define a “negative” Bott element \( b_- \) in \( 1KQ_{4\ell^{\nu-1}}(R_\alpha^+; \mathbb{Z}/\ell^{\alpha+\nu-1}-) \) that is the reduction mod \( \ell^{\alpha+\nu-1} \) of an integral class in \( 1KQ_{4\ell^{\nu-1}}(R_\alpha^+) \), as detailed in [7, p. 278]. Thus, following the spirit of [4, §5] we define a “mixed Bott element”
\[
b_+ + b_- \in 1KQ_{4\ell^{\nu-1}}(R_\alpha^+; \mathbb{Z}/\ell^\alpha+\nu-1).
\]

The following theorem is a consequence of the analogous one in \( K \)-theory and the periodicity theorem for the higher Witt groups. More precisely, Theorem 4.3, p. 278 in [7] implies that the higher Witt groups are 4-periodic mod 2-torsion. On

\[\footnotesize{\begin{enumerate}
\item[]\text{1}\text{If } \sigma \text{ is the involution, this precisely means that } \sigma(\lambda a) = \lambda \sigma(a) \text{ for } \lambda \in R \text{ and } a \in A.
\item[]\text{2In general, we indicate by } G_+ \text{ the invariant part of an abelian group } G \text{ (with 2 invertible) by an involution. We also indicate by } G_- \text{ its anti-invariant part. Finally, we note that the groups } KQ_\nu(\lambda; \mathbb{Z}/\ell^\alpha)_- \text{ are periodic of period 4 according to the periodicity theorem proved in } [7].
\end{enumerate}}\]
the other hand, the symmetric part of hermitian $K$-theory is isomorphic to the symmetric part of $K$-theory modulo 2-torsion: see [4, § 5] for more details.

**Theorem 2.5.** With the previous notations, let $A$ be an $R^\times_{\alpha}$-algebra, $\alpha > 2$, with an involution that is trivial on $R^\times_{\alpha}$. Writing $p = 2(\ell - 1)\nu - 1$, $\nu > 0$, we assume the existence of an integer $d$, such that, for $n \geq d$, the cup-product with the Bott element in $K_p(Z; Z/m)$ induces an isomorphism

$$K_n(A; Z/m) \xrightarrow{\sim} K_{n+p}(A; Z/m)$$

whenever $m$ divides $\ell\nu$. Then, for $n \geq d$ and $p' = 4\ell\nu - 1$, the cup-product with the mixed Bott element $b_+ + b_-$ defined above induces an isomorphism

$$\varepsilon K_{n}(A; Z/m') \xrightarrow{\sim} \varepsilon K_{n+p'}(A; Z/m')$$

whenever $m'$ divides $\ell\nu + \nu - 1$. In particular, if $A$ is an $R^\times_{\alpha}$-algebra for all $\alpha$, we may choose $\nu = 1$ so that the groups $\varepsilon K_n(A; Z/m')$ are periodic of period 4 with respect to $n$ for all powers $m'$ of $\ell$.

We may now combine this theorem (for $\ell$ odd) with the previous one for 2-primary coefficients to prove Theorem 2.2. It is a generalization of the theorem proved in [8] when $A = F$ (see below).

**Proof of Theorem 2.2.** We remark that $F$ contains all the rings $R^\times_{\alpha}$ considered before. We now decompose the integer $m$ into primary powers. According to the previous theorem, when $m$ is odd the groups $\varepsilon K_{n}(A; Z/m)$ are periodic of period 4 with respect to $n$. For 2-primary powers, we have 8-periodicity according to Theorem 2.4 (choose $\nu = 4$). Note that the “exotic” Bott element in $\varepsilon K_{8}(F; Z/m)$ was already defined in [8]. \(\square\)

3. **Cohomology of orthogonal and symplectic groups of rings of 2-integers in 2-regular totally real number fields**

Let $A$ be a commutative ring and let $\epsilon = \pm 1$. The group $\varepsilon O_{n,n}(A) \subseteq GL_{2n}(A)$ is the subgroup of automorphisms of $A^n \oplus A^n$ that leave invariant the $\epsilon$-quadratic form

$$\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}.$$ 

Passing to the colimit by componentwise inclusion, we obtain the group $\varepsilon O(A)$ and its classifying space $B_\varepsilon O(A)$. The group cohomology of $\varepsilon O(A)$ coincides with the singular cohomology of $B_\varepsilon O(A)$, and therefore with the singular cohomology of Quillen’s $+$-construction $B_\varepsilon O(A)^+$. For a number field $F$ with ring of 2-integers $R_F$, the rational group cohomology of $\varepsilon O(R_F)$ was computed by Borel [5] almost forty years ago. When $F$ is a totally real 2-regular number field, our results in [2] identify the homotopy type of $B_\varepsilon O(R_F)^+$ in terms of classical topological invariants. In the following, we use these results to compute explicitly the group cohomology of $\varepsilon O(R_F)$ with $F_2$-coefficients.

In this section, all displayed spectra are implicitly 2-completed and connective. For example, we let $\varepsilon KQ(R_F)$ denote the 2-completion of the connective cover of the hermitian $K$-theory spectrum of $R_F$. Let $r$ denote the number of real embeddings of the number field $F$. We refer to [2] for the choice of the residue field $F_q$ of $R_F$.

We recall from [2] the homotopy cartesian square:

$$\begin{array}{ccc}
\varepsilon KQ(R_F) & \rightarrow & \varepsilon KQ(R)^r \\
\downarrow & & \downarrow \\
\varepsilon KQ(F_q) & \rightarrow & \varepsilon KQ(C)^r
\end{array}$$
When \( \varepsilon = 1 \), \( _1KQ(\mathbb{R}) \simeq K(\mathbb{R}) \vee K(\mathbb{R}) \) and \( _1KQ(\mathbb{C}) \simeq K(\mathbb{R}) \), where we consider the real and complex numbers with their usual topologies. The map from \( _1KQ(\mathbb{R}) \) to \( _1KQ(\mathbb{C}) \) is induced by the Whitney sum of real vector bundles. Since it has a splitting [2, Appendix B], we deduce the following theorem.

**Theorem 3.1.** There is a homotopy equivalence of 2-completed connective spectra
\[
_1KQ(R_F) \simeq _1KQ(\mathbb{F}_q) \vee K(\mathbb{R})^*.
\]

By considering the underlying infinite loop spaces of the spectra in Theorem 3.1 we obtain the following group cohomology computation.

**Corollary 3.2.** Let \( H^* \) denote cohomology with \( \mathbb{F}_2 \)-coefficients. Then there is an isomorphism of Hopf algebras and modules over the mod 2 Steenrod algebra
\[
H^*(O(R_F)) \cong H^*(O(\mathbb{F}_q)) \otimes H^*(BO)^{\otimes \varepsilon}.
\]

Here \( H^*(BO) \cong \mathbb{F}_2[w_1, w_2, \cdots] \) is a polynomial algebra generated by the Stiefel-Whitney classes \( w_i \), \( i \geq 1 \), and \( H^*(O(\mathbb{F}_q)) \) is a polynomial algebra on generators \( x_1, x_{2i-1}, i \geq 1 \).

The cohomology of the classifying space \( BO \) is computed in e.g. [12, Corollary 16.11], and that of \( O(\mathbb{F}_q) \) in [6, IV, Corollary 4.3].

When \( \varepsilon = -1 \), \( -1KQ(\mathbb{R}) \simeq K(\mathbb{C}) \) and \( -1KQ(\mathbb{C}) \simeq K(\mathbb{H}) \), where we consider the quaternions with the usual topology. Hence, there is a homotopy cartesian square:
\[
\begin{align*}
_1KQ(R_F) & \longrightarrow K(\mathbb{C})^* \\
\downarrow & \\
_1KQ(\mathbb{F}_q) & \longrightarrow K(\mathbb{H})^*
\end{align*}
\]

We note that there is a naturally induced isomorphism
\[
-1KQ_0(\mathbb{R}) \longrightarrow -1KQ_0(\mathbb{C}) \cong \mathbb{Z}
\]

Corresponding to the (even) rank of the free symplectic \( A \)-inner product space [10, p.7]. Thus, by considering the underlying infinite loop spaces of these spectra, we find the homotopy cartesian square:
\[
\begin{align*}
BSp(R_F)^+ & \longrightarrow BU^r \\
\downarrow & \\
BSp(\mathbb{F}_q)^+ & \longrightarrow BSp^r
\end{align*}
\]

We note that the classifying space \( BSp \) of the symplectic group is simply-connected. Moreover, see e.g. [12, Corollary 16.11], \( H^*(BU) \) is a polynomial algebra generated by Chern classes \( c_i \), \( i \geq 1 \), and \( H^*(BSp) \) is the subring generated by \( p_i = c_2i \). Thus \( H^*(BU) \) is a free module over the subalgebra \( H^*(BSp) \). It follows that the Eilenberg-Moore spectral sequence in cohomology [9, §7,8]
\[
\operatorname{Tor}_H^*(BSp)^*(H^*(BSp(\mathbb{F}_q)), H^*(BU^r)) \Longrightarrow H^*(BSp(R_F))
\]
collapses to its zeroth column, and we conclude the following result.

**Theorem 3.3.** Let \( H^* \) denote cohomology with \( \mathbb{F}_2 \)-coefficients. Then there is an isomorphism of Hopf algebras and modules over the mod 2 Steenrod algebra
\[
H^*(BU)^{\otimes \varepsilon} \otimes_{H^*(BSp)^{\otimes \varepsilon}} H^*(Sp(\mathbb{F}_q)) \xrightarrow{\cong} H^*(Sp(R_F)).
\]

The cohomology of \( Sp(\mathbb{F}_q) \) is the tensor product of a polynomial algebra on generators \( g_i \), \( i \geq 1 \), and an exterior algebra on generators \( h_j \), \( j \geq 1 \) [6, IV, §6].

Dually, we may compute the homology of \( BSp(R_F) \) by using the cotensor product \( \Box \) instead of the usual tensor product over \( \mathbb{F}_2 \). This gives an isomorphism
\[
H_*(Sp(R_F)) \xrightarrow{\cong} H_*(BU^r \Box_{H^*(BSp^r)} H_*(Sp(\mathbb{F}_q))).
\]
In particular, there is a naturally induced injective map

\[ H_\ast(\text{Sp}(R)) \to H_\ast(\text{BU}) \otimes H_\ast(\text{Sp}(F_q)). \]

The \( F_2 \)-homology of \( \text{Sp}(F_q) \) is the tensor product of a polynomial algebra on generators \( \sigma_i, i \geq 0 \), and an exterior algebra on generators \( \tau_j, j \geq 1 \) [6, IV, §5].

References


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