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THE K-THEORY OF TRIANGULAR MATRIX RINGS

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ABSTRACT. It is shown that the K-theory of a triangular matrix ring is just that of the direct sum of its diagonal components. This answers in the affirmative a conjecture of Dennis and Geller. The proof involves an application of an interesting result on the homology of direct limit groups.

Let R and S be rings (associative, with unit) and let U be an R-S bimodule. We form the matrix ring

$$T = \begin{pmatrix} R & U \\ 0 & S \end{pmatrix} ,$$

with the evident addition and multiplication. There is an obvious split surjection π of T onto $R \oplus S$. The object of this note is to show that the following result holds _without any further condition on_ R,S _or_ U.

THEOREM. _The induced homomorphism_

$$\pi_i : K_i(T) \to K_i(R) \oplus K_i(S)$$

is an isomorphism for all integers i.

This provides an affirmative solution to a conjecture of Dennis and Geller, who established the assertion in [4] for $i = 0, 1, 2$. The result has greatest interest when $i > 1$; the argument here is modelled on that given in [2] in the case $R = S = U$. Although that particular case had previously been established in [8], the argument presented there was rather indirect, making its generalization to the current situation non-obvious. It is to be hoped that this note will have the effect of drawing attention to the alternative, more direct, proof found

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in [2]. When $i = 0$, the result is classical (see [1, Chapter IX, Proposition 1.3, p 449] as in [4]) and for negative indices we shall derive it from the zero case by using iterated suspensions. (An alternative argument which utilizes homological conditions on the bimodule U is given in [7].)

We also note (§6) that these results provide further examples of fibrations which are not preserved by the plus-construction.

1. We show first that π induces an isomorphism of homology with trivial coefficients

$$\pi_* : H_*(GL(T)) \to H_*(GL(R \oplus S)).$$

Let $M_n(U)$ be the set of n×n matrices over U, and embed $M_n(U)$ in $M_{n+1}(U)$ by $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. For each $n \geq 1$ there is an isomorphism

$$r_n: GL_n(T) \longrightarrow \begin{pmatrix} GL_n(R) & M_n(U) \\ 0 & GL_n(S) \end{pmatrix} ;$$

that is, $GL_n(T)$ is the semidirect product

$$(GL_n(R) \times GL_n(S)) \ltimes M_n(U).$$

The diagram

$$\begin{array}{ccc} GL_n(T) & \xrightarrow{r_n} & \begin{pmatrix} GL_n(R) & M_n(U) \\ 0 & GL_n(S) \end{pmatrix} \\ \downarrow & & \downarrow \\ GL_{n+1}(T) & \xrightarrow{r_{n+1}} & \begin{pmatrix} GL_{n+1}(R) & M_{n+1}(U) \\ 0 & GL_{n+1}(S) \end{pmatrix} \end{array}$$

commutes, where the vertical arrows are the usual inclusions. In the limit, we obtain an isomorphism

$$r : GL(T) \longrightarrow \begin{pmatrix} GL(R) & M(U) \\ 0 & GL(S) \end{pmatrix} .$$

The argument of [2, (3.10)] can be applied mutatis mutandis to show that

$$H_*\left(\begin{pmatrix} GL(R) & M(U) \\ 0 & GL(S) \end{pmatrix}\right) \cong H_*(GL(R \oplus S)),$$

which is to say,

$$H_*((GL(R) \times GL(S)) \ltimes M(U)) \cong H_*(GL(R) \times GL(S)).$$

This is perhaps an appropriate place to make two remarks concerning [2, (3.11)] and its applications both in [2] and above. On pp 30, 31 of [2] it is incorrectly assumed that the identity map on a filtration induces the identity map. In the case of the Künneth sequence this fails because its splitting is non-natural. To avoid the pitfall one should first work with field coefficients to remove Tor terms in the Künneth formula. So the proof of (3.11) of [2] goes through for field coefficients; the universal coefficient theorem then makes ρ_* an isomorphism over integral coefficients. One should then conclude on p 30 of [2] that the Lyndon-Hochschild-Serre spectral sequence leads to an isomorphism on the homology of the semi-direct product (as above). In the applications the homology map is already known to be idempotent, so that it is the identity map after all. Thus the applications are unaffected. (The authors would like to thank the referee for drawing their attention to these matters.)

2. The plus-construction induces a natural isomorphism on homology [2, (4.3), (5.1)ff], giving rise to a commutative diagram of isomorphisms

$$\begin{array}{ccccc} H_*(GL(T)) & \cong & H_*(BGL(T)) & \cong & H_*(BGL(T)^+) \\ \downarrow \pi_* & & \downarrow (B\pi)_* & & \downarrow (B\pi^+)_* \\ H_*(GL(R \oplus S)) & \cong & H_*(BGL(R \oplus S)) & \cong & H_*(BGL(R \oplus S)^+). \end{array}$$

By [2, (11.6)], both the spaces $BGL(T)^+$ and $BGL((R \oplus S)^+)$ are nilpotent and so [2, (4.18)] $B\pi^+$ is a homotopy equivalence. From the definition of K-theory for positive indices,

$$K_i(T) \cong \pi_i(BGL(T)^+)$$

and

$$K_i(R) \oplus K_i(S) \cong \pi_i(BGL(R)^+) \oplus \pi_i(BGL(S)^+)$$
$$\cong \pi_i(BGL(R \oplus S)^+).$$

Theorem 1 follows for $i \geq 1$.

3. To deal with negative indices, first recall the definition of the suspension SA of the ring A [6, p 22]. The cone **CZ** of **Z** is the ring consisting of all those infinite integer matrices such that each row and column of the matrix has only finitely many non-zero entries. In **CZ** there is a two-sided ideal m**Z** which consists of the matrices whose non-zero entries occur in only a finite set of rows and columns. The suspension of **Z** is

$$S\mathbf{Z} = C\mathbf{Z}/m\mathbf{Z},$$

and in general

$$SA = S\mathbf{Z} \otimes_{\mathbf{Z}} A.$$

We then have for all integers i

$$K_i(A) \cong K_{i+1}(SA),$$

which can be taken as the definition of K-theory for negative indices. It is clear that $SU = S\mathbf{Z} \otimes_{\mathbf{Z}} U$ is an SR-SS-bimodule and that

$$ST \cong \begin{pmatrix} SR & SU \\ 0 & SS \end{pmatrix}.$$

The proof of the theorem can now be completed by induction. (Note that this device also provides an alternative proof of the classical K_0 result.)

4. Iteration of the argument above leads readily to the following result on a system of rings $R_0, R_1, \ldots, R_{n-1}$ and R_j-R_k-bimodules U_{jk} $(j < k)$, which, given the obvious matrix multiplication data, form a triangular matrix ring T_n with "diagonal" $R_0 \oplus R_1 \oplus \ldots \oplus R_{n-1}$.

COROLLARY.

$$K_i(T_n) \cong \bigoplus_{j<n} K_i(R_j)$$

for all integers i.

In particular, this provides a computation of the K-theory of the Artinian hereditary algebras constructed in [9].

5. A further generalization of the considerations in §1 above consists in varying the size of the general linear group discussed. While the general linear group whose homology was investigated above was the direct limit dir lim $GL_n(A)$, for a given ring A it is also of interest to study the larger group of all row-finite (or column-finite) matrices. It will be noted that the key lemma here in the homology of groups [2, (3.11)] is applicable only to groups of direct limit type. Alternative methods are therefore required. Such were initiated by Quillen [8, p 203], whose arguments imply homology isomorphisms for "very" general linear groups associated to the rings

$$\begin{pmatrix} R & U \\ 0 & S \end{pmatrix}$$

and R + S, so long as R or S is an algebra over the rationals (cf [5]). Suslin [10, §1] has also extended Quillen's argument to deal with algebras over other infinite fields. (Of course, Quillen and Suslin introduced their techniques in order to tackle problems different from those discussed in this paragraph.)

6. A final observation is that all the above situations involve split ring epimorphisms and thereby split group epimorphisms of their general linear groups. It is easily checked that split surjections preserve the perfect radicals (groups of elementary matrices) of the general linear groups involved. Here the kernel of the split surjection is an abelian group. On the other hand, we have seen that application of the plus-construction to the classifying space fibration results in a simply-connected (indeed, contractible) fibre, which thus cannot be the plus-construction of the classifying space fibre. In terms of [3, §3] these considerations provide further examples of group epimorphisms preserving perfect radicals whose induced classifying space fibration nevertheless fails to be plus-constructive.

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