STABLE CLASSICAL GROUPS AND STRONGLY TORSION GENERATED GROUPS

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Abstract. Strongly torsion generated groups are those with a single normal generator, of arbitrary finite order. They are useful for realizing sequences of abelian groups as homology groups. Known examples include stable alternating groups and stable groups generated by elementary matrices. Here the class of such groups is extended, by consideration of other stable classical groups, including orthogonal and symplectic groups. Discussion of other "classical" groups includes a similar result for the stable special automorphism group of a free group. Failure of such a result for mapping class and braid groups is analyzed. It is also shown that the product of finitely many strongly torsion generated groups is strongly torsion generated.

1. Introduction

For $n \geq 2$, a group G is *strongly n-torsion generated* if there is an element $g_n \in G$ of order n such that the conjugates of g_n generate G; that is, the normal closure of g_n is all of G. A group G is *strongly torsion generated* if it is strongly n-torsion generated for every $n \geq 2$ [2], [7]. Surprisingly, such groups are sufficiently common to generate all possible sequences of homology groups of perfect groups [7]. The constructions of [7] are in the realm of combinatorial group theory; this poses the question of finding "natural" examples of strongly torsion generated groups, which we address here.

For example, since any finite simple group G is strongly p-torsion generated for every prime p dividing the order of G, the stable alternating group A_∞ of even finitary permutations of a countable set is strongly torsion generated [2]; for, A_∞ is the direct limit of the alternating groups A_k.

Other historically important examples of strongly torsion generated groups include the subgroup $E(R)$ of the stable general linear group $\mathrm{GL}(R) = \operatorname{dirlim} \mathrm{GL}_k(R)$ generated by all elementary matrices and the Steinberg groups $\mathrm{St}(R)$, where R is an associative ring with 1. These examples featured in the homology and K-theory realization results

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of [2], [5], [6]. The historic examples typically arise from a sequence of like groups G_k endowed with natural homomorphisms $G_k \to G_{k+1}$ permitting *stabilization* (that often respects the formation of homology groups, among other properties) and the construction of the *stable* group $\varinjlim G_k$.

Now, stable alternating and general linear groups form part of a web of stable "classical" linear and geometric groups that also involves braid and mapping class groups, automorphism groups of free groups, and orthogonal and symplectic groups (related as in the diagrams of [4] §1). They have topological features in common too (see Remark 2.5 below). This all suggests that study of these other objects might yield further examples. We explore each of these classes below – indeed, because of recent work on torsion in these much-studied groups, new examples do result.

For automorphism groups of free groups and linear groups, the arguments proceed smoothly. There are well-known stabilizations (see Sections 2 and 3 for the definitions), about which we obtain the following results.

Theorem 2.4. *The special stable automorphism group* $\mathrm{SAut}(F_\infty)$ *of the countably infinite free group, and the special stable outer automorphism group* $\mathrm{SOut}(F_\infty)$*, are strongly torsion generated.*

Theorem 3.2. *For every form ring* (R,Λ)*, every perfect central extension of the perfect commutator subgroup* $\mathrm{EU}(R,\Lambda)$ *of* $U(R,\Lambda)$ *is strongly torsion generated.*

On the other hand, our analysis reveals a breakdown of the usual analogy between automorphism groups of free groups and mapping class groups. Whereas the former stabilize to yield a strongly torsion generated group, in the case of mapping class groups we observe two contrasting phenomena. Certain mapping class groups $\Gamma_{g,1}$ are well-known to stabilize with respect to increasing genus g (and, as with the classes above, the stability is respected by the passage to homology groups); however, such mapping class groups are torsion-free and therefore are not amenable to strong torsion generation. For the mapping class groups Γ_g that do contain torsion, we note some unstable instances of strong p-torsion generators. By combining these with known results on the existence of prime torsion in relation to genus, we deduce *rigidity* results that forbid stabilization. These take the following form.

Corollary 4.6. *Let* $m \geq 1$*, and let* p *be the largest prime factor of* $g(g-1)(2g+1)$*. If*

$$p - \sqrt{2g} > 2m + 2$$

(and $m \geq 2$ *if* $p|g$*), then the only homomorphism* $\Gamma_g \to \Gamma_{g+m}$ *is the trivial homomorphism.*

Similar considerations apply to braid groups. In the cases of torsion, namely the braid groups of the sphere and projective plane, there are vestigial, unstable, results pertaining to strongly n-torsion generated groups. Because n depends on the number of strands, these results on strong torsion generators lead to rigidity results that preclude the possibility of stabilization. Hence, as with mapping class groups, strongly torsion generated groups cannot arise, because where stable groups exist, they must be torsion-free.

A final section shows that the product of (a finite number of) strongly torsion generated groups is also strongly torsion generated, a fact applied in [6].

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2. Automorphism groups of free groups

For $n = 1, 2, \ldots$, write F_n for the free group of rank n, with automorphism group abbreviated to $\mathrm{Aut}(F_n)$. The action of $\mathrm{Aut}(F_n)$ on the abelianization of F_n may be regarded as a homomorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{GL}_n(\mathbb{Z})$ that factors through the outer automorphism group $\mathrm{Out}(F_n)$. The inverse images of the special linear group $\mathrm{SL}_n(\mathbb{Z})$ under these maps are normal subgroups denoted here by the *special automorphism group* $\mathrm{SAut}(F_n)$ and $\mathrm{SOut}(F_n)$ respectively. (Other notations for $\mathrm{SAut}(F_n)$ in the literature include $\mathrm{Aut}^+(F_n)$, although that notation invites confusion when the plus-construction is applied to classifying spaces; and SA_n, although here we work with A_n too.) We thus have the following diagram of pullbacks and vertical inclusions.

$$\begin{array}{ccccc}
\mathrm{SAut}(F_n) & \longrightarrow & \mathrm{SOut}(F_n) & \longrightarrow & \mathrm{SL}_n(\mathbb{Z}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{Aut}(F_n) & \longrightarrow & \mathrm{Out}(F_n) & \longrightarrow & \mathrm{GL}_n(\mathbb{Z})
\end{array}$$

Because inner automorphisms of F_n are special, the canonical inclusion $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_{n+1})$, given by trivial action on the final generator of F_{n+1}, induces inclusions $\mathrm{SAut}(F_n) \hookrightarrow \mathrm{SAut}(F_{n+1})$ and $\mathrm{SOut}(F_n) \hookrightarrow \mathrm{SOut}(F_{n+1})$. The colimits of these sequences of inclusions are called $\mathrm{Aut}(F_\infty)$, $\mathrm{SAut}(F_\infty)$ and $\mathrm{SOut}(F_\infty)$, the *stable* (or finitary, or finite type) automorphism groups of the free group on a countably infinite set of generators.

Lemma 2.1. *For $n \geq 3$, $\mathrm{SAut}(F_n)$ is the commutator subgroup of $\mathrm{Aut}(F_n)$, and $\mathrm{SOut}(F_n)$ is the commutator subgroup of $\mathrm{Out}(F_n)$.*

Proof. By construction, $\mathrm{SAut}(F_n)$ is normal in $\mathrm{Aut}(F_n)$, and

$$\mathrm{Aut}(F_n)/\mathrm{SAut}(F_n) \cong \mathrm{Out}(F_n)/\mathrm{SOut}(F_n) \cong \mathrm{GL}_n(\mathbb{Z})/\mathrm{SL}_n(\mathbb{Z}) \cong C_2\,.$$

Therefore $\mathrm{Aut}(F_n)_{\mathrm{ab}}$ and $\mathrm{Out}(F_n)_{\mathrm{ab}}$ contain at least two elements.

In the other direction, recall that the symmetric group Σ_n embeds in $\mathrm{Aut}(F_n)$ by permuting the generators of the free group. From [8] Corollary 1, $\mathrm{Aut}(F_n)$ is the normal closure of Σ_n, whence $(\Sigma_n)_{\mathrm{ab}} \cong C_2$ maps onto $\mathrm{Aut}(F_n)_{\mathrm{ab}}$.

We conclude that the abelianization of $\mathrm{Aut}(F_n)$ has order 2, making $\mathrm{SAut}(F_n)$ the commutator subgroup of $\mathrm{Aut}(F_n)$. The epimorphism from $\mathrm{Aut}(F_n)$ to $\mathrm{Out}(F_n)$ induces an epimorphism of their abelianizations. Thus also $\mathrm{Out}(F_n)_{\mathrm{ab}} \cong C_2$, and $\mathrm{SOut}(F_n)$ is the commutator subgroup of $\mathrm{Out}(F_n)$. In effect, this is the same phenomenon as A_n and $\mathrm{SL}_n(\mathbb{Z})$ $(n \geq 3)$ being the commutator subgroups of Σ_n and $\mathrm{GL}_n(\mathbb{Z})$.□

It follows from the lemma above that $\mathrm{SAut}(F_\infty)$ is the commutator subgroup of $\mathrm{Aut}(F_\infty)$, which is labelled E_∞ in [35].

Under the embedding of Σ_n in $\mathrm{Aut}(F_n)$, the sign of the permutation corresponds to the determinant of the image of the automorphism in $\mathrm{GL}_n(\mathbb{Z})$. Therefore the alternating group A_n is correspondingly embedded in $\mathrm{SAut}(F_n)$.

Lemma 2.2. *For $n \geq 3$, A_n normally generates* $\mathrm{SAut}(F_n)$.

Proof. We use the fact (e.g. [10]) that $\mathrm{SAut}(F_n)$ is normally generated by the left Nielsen automorphisms λ_{ij}, where λ_{ij} sends the i th generator a_i of F_n to $a_i a_j$ while fixing all other generators. Let N denote the normal closure of A_n in $\mathrm{SAut}(F_n)$. Arguing as in the proof of [8] Proposition 1, we note that the 3-cycle $(i\ j\ k)$ conjugates λ_{ij} to λ_{jk}. Thus, in $\mathrm{SAut}(F_n)/N$ both λ_{ij} and λ_{jk} have the same image. Consequently, the image of

$$\lambda_{ik} = [\lambda_{ij}, \lambda_{jk}]$$

is trivial. Hence $N = \mathrm{SAut}(F_n)$. □

Lemma 2.3. *If a group G is the normal closure of a strongly m-torsion generated subgroup H, then G is also strongly m-torsion generated.*

Proof. Evidently, if H is normally generated by an element x_m of order m, then so is G. □

Since, from its simplicity for $n \geq 5$, A_n is strongly $\prod m_i$-torsion generated whenever $\sum m_i \leq n$ and $\gcd(m_i) = 1$, it follows from the lemmas above that $\mathrm{SAut}(F_n)$ is too. This gives the first of the following conclusions.

Theorem 2.4. *The groups* $\mathrm{SAut}(F_\infty)$ *and* $\mathrm{SOut}(F_\infty)$ *are strongly torsion generated.*

Proof. As noted in [7], any nontrivial quotient of a strongly torsion generated group must again be strongly torsion generated, when the kernel is torsion-free. Hence $\mathrm{SOut}(F_\infty)$, as the quotient of $\mathrm{SAut}(F_\infty)$ by the free group F_∞, is also strongly torsion generated. □

Remark 2.5. It is a curious fact that the group $G = \mathrm{SAut}(F_\infty)$ shares with $G = A_\infty$ and $G = E(R)$ the striking property that not only is it strongly torsion generated, but the plus-construction applied to its classifying space, BG^+, is an infinite loop space [28].

3. Classical linear groups

The case of the perfect commutator subgroup $E(R)$ of $\mathrm{GL}(R)$ for an associative ring R with unit was discussed in [2]. More generally, so as to embrace symplectic, unitary and orthogonal groups, one can consider a ring (R, Λ) with form parameter Λ (as, for example, in [21]). Then the natural candidate for a strongly torsion generated group in this context is the perfect commutator subgroup $\mathrm{EU}(R, \Lambda)$ of $U(R, \Lambda)$ [21] (5.4.6).

One way to approach this is by means of generators and relations for $\mathrm{EU}_{2n}(R, \Lambda)$ [21] (5.3B), in the spirit of [33]. The alternative treatment given here is perhaps more illuminating.

Our aim is to apply the following, slight strengthening of a lemma of [7], as observed in [5].

Lemma 3.1. *Let H be a simple group that, for each $n \geq 2$, has a superperfect subgroup L_n possessing an element of order n. Suppose that G is a group containing H in such a way that the normal closure of H in G is G itself. Then every perfect central extension of G is strongly torsion generated.*

To this end we consider, as usual, H as the infinite alternating group A_∞, with A_n hyperbolically embedded, via $E_n(R)$, in $\mathrm{EU}_{2n}(R, \Lambda)$. (This is well-known not to be the most efficient embedding, but suffices here.) Evidently, the off-diagonal entries of permutation matrices lie in no proper ideal of R. Therefore, for no proper ideal $(\mathfrak{a}, \Gamma)$ of (R, Λ) does the normal closure N of H in $G = \mathrm{EU}(R, \Lambda)$ lie in $U(\mathfrak{a}, \Gamma)$. Accordingly, the Bass Sandwich Theorem for this situation [21] (5.4.10) dictates that N has level (R, Λ), in other words that

$$\mathrm{EU}(R, \Lambda) \leq N \leq U(R, \Lambda).$$

Of course, because N lies in $\mathrm{EU}(R, \Lambda)$, we conclude that $N = \mathrm{EU}(R, \Lambda)$, and have the following result.

Theorem 3.2. *For every form ring (R, Λ), every perfect central extension of the perfect commutator subgroup* $\mathrm{EU}(R, \Lambda)$ *of* $U(R, \Lambda)$ *is strongly torsion generated.* □

Thus in particular, the two extreme central extensions, $\mathrm{EU}(R,\Lambda)$ itself and its universal central extension, the Steinberg group $\mathrm{StU}(R,\Lambda)$, are strongly torsion generated. In the case $R=\mathbb{Z}$, the integral symplectic group $\mathrm{Sp}(\mathbb{Z})$ and the commutator subgroup $O'(\mathbb{Z})$ (of index 4) of the integral orthogonal group $O(\mathbb{Z})$ are strongly torsion generated.

4. Mapping class groups

Write $S^s_{g,r}$ for (a copy of) an oriented smooth surface of genus g with s marked points (often referred to as "punctures") and r boundary components. Thus the boundary $\partial S^s_{g,r}$ of $S^s_{g,r}$ consists of the disjoint union of r circles, where $r\geq 0$. The (pure) *mapping class group* $\Gamma^s_{g,r}$ is the discrete group of components of the topological group $\mathrm{Diff}_+(S^s_{g,r})$ of orientation-preserving diffeomorphisms of $S^s_{g,r}$ that fix the marked points and the boundary pointwise. Conventionally, the suffices r and s are often omitted when zero.

For $r\geq 1$, there are some standard operations that enable one to vary the suffices. For example, to pass from g to $g+1$, glue a two-holed torus $S^0_{1,2}$ to $S^s_{g,r}$, yielding $S^s_{g+1,r}$. Extending diffeomorphisms by the identity produces a homomorphism $\psi:\Gamma^s_{g,r}\longrightarrow\Gamma^s_{g+1,r}$. In this way, we are able to obtain the stable mapping class group $\lim_{g\to\infty}\Gamma^s_{g,r}$, so long as $r\geq 1$ [22]. The restriction $r\geq 1$ here is important for two reasons, as follows.

First, as is well known, each group $\Gamma^s_{g,r}$ is torsion-free when $r\geq 1$. It follows that the stable group is also torsion-free, and hence cannot be a candidate for a strongly torsion generated group.

Second, when $r=0$ the group $\Gamma^s_{g,0}$ does indeed contain torsion. For example, it is known that every finite group embeds in some $\Gamma_g=\Gamma^0_{g,0}$ [20]. However, the stabilization process described above requires $r>0$. This prompts the question of rigidity (considered for $\mathrm{Aut}(F_m)\to\mathrm{Aut}(F_n)$ $(m>n)$ in [8], and for $\Gamma_g\to\Gamma_h$ $(g>h)$ in [25]): whether for suitable g,h there can exist a nontrivial map from Γ_g to Γ_h. Of course, with stabilization in mind, the interesting cases have $g<h$. To examine this issue, we begin by reworking a result of Glover and Mislin [12]. (Our statement is easily seen to be equivalent to that of Harvey [24].) In this section, we find it convenient to use the notation, for an integer k,

$$\varepsilon_k=\begin{cases}1 & k \text{ even},\\ 0 & k \text{ odd}.\end{cases}$$

Lemma 4.1. *Γ_h contains an element of prime order $2a+1$ if and only if for some $k\geq 1$*

$$ka\leq h\leq k(a+\frac{1}{2})+\varepsilon_k.$$

Proof. First, recall that the only odd prime torsion in $\Gamma_1=\mathrm{SL}_2(\mathbb{Z})\cong C_4*_{C_2}C_6$ is 3-torsion, so that we may safely assume that $h\geq 2$. Then,

[12] (3.3) asserts that Γ_h contains an element of prime order $2a+1$ if and only if for some $u \geq 0$ and $v \in \{-2, 0, 1, 2, 3, \ldots\}$

$$h = u(2a+1) + va.$$

We write this last expression as $ka + u$ where $k = 2u + v$. When k is even, this corresponds to all pairs $(u, k-2u)$ with $u = 0, \ldots, \frac{k}{2} + 1$; while for k odd we have all pairs $(u, k-2u)$ with $u = 0, \ldots, \frac{k-1}{2}$. The result follows. □

It follows immediately from this result that there is a bound on the prime orders of torsion elements, and hence that no Γ_g can be strongly torsion generated. However, partial strong torsion generation results are usually possible. Throughout, we draw upon the observation that if an element x normally generates a group G, then its image $\varphi(x)$ must normally generate any homomorphic image $\varphi(G)$ of G. For the case of genus 2, we shall employ the following lemma (doubtless well-known).

Lemma 4.2. *For $n \geq 1$, the principal congruence subgroup*

$$\mathrm{Ker}[\mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{F}_p)]$$

contains no torsion coprime to p.

Proof. $I_n = A^k = (I_n + p^\nu M)^k$, with ν the maximal p-exponent of $A - I_n$, yields that

$$kM = -p^\nu \sum_{i=2}^{k} \binom{k}{i} p^{\nu(i-2)} M^i,$$

which contradicts the choice of ν when $p \nmid k$. □

Theorem 4.3. (a) *If both $g \geq 3$ and d divides g or $g-1$ or $4g+2$, then Γ_g is strongly d-torsion generated.*

(b) *Conversely, if Γ_g is strongly d-torsion generated, then both $g \geq 3$ and $d \leq 4g+2$, and the prime divisors p of d satisfy*

$$-\frac{2\varepsilon_k}{k} \leq p - \frac{2g}{k} \leq 1$$

for some integer k.

Proof. We first exclude the cases of low genus. Γ_0 is trivial. For the cases $g = 1, 2$, we recall that the greatest order of any torsion element of Γ_g is precisely $4g+2$. However, $\Gamma_1 \cong \mathrm{SL}_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6$ has abelianization cyclic of order 12, so that it cannot be normally generated by an element of smaller order.

The case of Γ_2 is more subtle, and indeed is the subject of [29], where Mumford shows that $(\Gamma_2)_{\mathrm{ab}}$ is cyclic of order 10 (whence $d = 10$ is the only possibility), and that Γ_2 is normally generated by a single, torsion-free element. [25] identifies an element of order 10 in Γ_2 that maps to the fourth power of Mumford's generator of $(\Gamma_2)_{\mathrm{ab}}$, and therefore normally generates an index 2 subgroup of Γ_2.

To contradict the supposition that Γ_2 is normally generated by an element ε of order 10, we consider the canonical epimorphisms

$$\Gamma_2 \overset{\eta}{\twoheadrightarrow} \mathrm{Sp}_4(\mathbb{Z}) \overset{\pi}{\twoheadrightarrow} \mathrm{Sp}_4(\mathbb{F}_2),$$

and note that each of these groups must be normally generated by the image of ε. Now $\mathrm{Sp}_4(\mathbb{F}_2) \cong S_6$, and by consideration of cycle lengths one sees that S_6 contains no element of order 10. Therefore the image of ε has order 2 or 5. Now order 5 is not possible, since any element of that order in S_6 must be an even permutation, with normal closure A_6. Thus, $\pi\eta(\varepsilon)$ is an involution. Hence, in $\mathrm{Sp}_4(\mathbb{Z})$ $\eta(\varepsilon)$ has square lying in $\mathrm{Ker}\pi$. Because the kernel of η is the Torelli group, which is known to be torsion-free, $\eta(\varepsilon^2)$ is an element of order 5 in the congruence subgroup $\mathrm{Ker}\pi$. Such a possibility is denied by the lemma above.

When $g \geq 3$, the first assertion is a restatement of Theorems 4, 11 of [25] in our language. (Note that the case $d = 2$ follows from Theorem 11 rather than from Theorem 4.)

The converse claim uses our low-genus discussion. The condition $d \leq 4g+2$ is due to [23], while the inequalities for prime divisors follow from Lemma 4.1 above. □

Since for $g \geq 3$ this result affords distinct prime orders of elements that normally generate Γ_g, and whose images therefore generate $(\Gamma_g)_{\mathrm{ab}}$, an elementary consequence is the famous result of [31] that Γ_g is a perfect group. However, that result is already used in the reasoning of [25] cited in our argument above. The first cases left open by the theorem above are whether Γ_3 is strongly 6-torsion generated, and (for d prime) whether Γ_5 is strongly 3-torsion generated.

Theorem 4.4. *Consider the set of homomorphisms $\Gamma_g \to \Gamma_{g+m}$ where $g \in \mathbb{N}$, $m \in \mathbb{Z}$.*

(a) [25] *If $m < 0$, then there is only the trivial homomorphism.*
(b) *If $m = 0$ or $g \leq 2$, then there are nontrivial homomorphisms.*
(c) *Suppose that $g \geq 3$, $m \geq 1$. Then the only homomorphism $\Gamma_g \to \Gamma_{g+m}$ is the trivial homomorphism, provided that at least one of the following holds.*
(i) $g - 1 = pu$ *with prime* $p > 2(m + u + 2)$;
(ii) $g = pv$ *with prime* $p > 2(m + v + 1)$, *and* $m \geq 2$;
(iii) $2g + 1 = pw$ *with prime* $p > m + w + 1$.

Proof. (a) is the rigidity result of Harvey and Korkmaz [25] Theorem 7. For (b), when $m = 0$ the identity homomorphism is always available, while both Γ_1 and Γ_2 have quotients of order 2. Since for all $h \geq 1$ Γ_h contains an involution, there are always nontrivial homomorphisms $\Gamma_g \to \Gamma_h$ when the domain is Γ_1 or Γ_2.

To obtain (c), we apply Theorem 4.3 above. Here is the argument in case (i) (with (ii) and (iii) similar). By (4.3)(a), Γ_g is strongly p-torsion generated. It therefore suffices to use (4.3)(b) to show that

Γ_{g+m} contains no element of order p. This means contradicting the possibility that for some integer k we have

$$-\frac{2\varepsilon_k}{k} \leq p - \frac{2(g+m)}{k} \leq 1.$$

On multiplying by k, substituting $g-1 = pu$, and adding 2 throughout, we can rewrite this as

$$2 - 2\varepsilon_k \leq (k-2u)p - 2m \leq k+2.$$

The left-hand inequality gives $k \geq 2u+1$. In conjunction with the right-hand inequality, this yields that

$$p - 1 \leq (k-2u)(p-1) = (k-2u)p - k + 2u \leq 2m + 2 + 2u,$$

in defiance of our further hypothesis. □

Evidently, for a given positive m, the theorem shows that there are infinitely many values of g forcing triviality of $\Gamma_g \to \Gamma_{g+m}$. This prompts the following conjecture.

Conjecture 4.5. *Given a positive integer m, then for all sufficiently large g the only homomorphism $\Gamma_g \to \Gamma_{g+m}$ is the trivial homomorphism.*

From the previous theorem, we may deduce a number-theoretic condition that is sufficient for affirmation.

Corollary 4.6. *Let $m \geq 1$, and let p be the largest prime factor of $g(g-1)(2g+1)$. If*

$$p - \sqrt{2g} > 2m + 2$$

(and $m \geq 2$ if $p|g$), then the only homomorphism $\Gamma_g \to \Gamma_{g+m}$ is the trivial homomorphism.

Proof. Again we give details only for the case where $g - 1 = pu$, the other two possibilities being similar. From the inequality above,

$$(p - (2m+2))^2 > (\sqrt{2g})^2 = 2pu + 2\,;$$

so that

$$\begin{aligned}(p - (2m+2+u))^2 &> 2 + 2u(2m+2) + u^2 \\ &> (u+2)^2\,,\end{aligned}$$

which implies that $p > 2(m+u+2)$ and thus we are in case (i) of the previous theorem. □

We therefore turn the following question over to the number-theorists...

Question 4.7. Let $A(n)$ denote the largest prime factor of $n(n-1)(2n+1)$, and let $B(n) = A(n) - \sqrt{2n}$. Given $k > 0$, what is the density of the set $\{n \in \mathbb{N} \mid B(n) < k\}$?

In view of the preceding corollary, were this set finite, it would imply Conjecture 4.5. However, the following argument shows that such is not the case.

Proposition 4.8. *For any $k \in \mathbb{R}$, the set $\{n \in \mathbb{N} \mid B(n) < k\}$ is infinite.*

Proof. For $r \geq 0$, define

$$\begin{bmatrix} u_r \\ v_r \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}^r \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since $\begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$ commutes with $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, we have

$$\begin{bmatrix} u_r & 2v_r \\ v_r & 3u_r \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}^r \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

(In fact, $(2v_r, u_r)$ is the $2r$th convergent in the continued fraction representation for $\sqrt{6}$.) Then $A(v_r^2)$ is the largest prime factor of $v_r^2(v_r - 1)(v_r + 1)3u_r^2$ and so bounded above by $v_r + 1$. This leaves

$$B(v_r^2) \leq 1 - v_r(\sqrt{2} - 1) \to -\infty \quad \text{as } r \to \infty.$$

□

This assertion also follows from [1]. Moreover, the conjecture (1.4) of [27] indicates that the likely answer to Question 4.7 is $(1 - \log 2)^3 = 0.0288928\ldots$.

The contrast with the situation for mapping class groups of surfaces with boundary is stark, as follows.

Lemma 4.9. *For $g \geq 2$ and $r = 1$, the homomorphism $\psi : \Gamma_{g,r} \longrightarrow \Gamma_{g+1,r}$ has $\Gamma_{g+1,r}$ as the normal closure of the image of ψ.*

Proof. The result follows readily from the presentation of $\Gamma_{g,r}$ found in [36] Theorem 1′. Specifically, all generators there are mapped under ψ to their namesakes in $\Gamma_{g+1,r}$, whose presentation contains the further generators e_g, a_{g+1}. They commute with all other generators except for the braid relations

$$a_g e_g a_g = e_g a_g e_g \qquad\qquad a_{g+1} e_g a_{g+1} = e_g a_{g+1} e_g$$

The former shows that e_g lies in the normal closure of a_g and so of $\mathrm{Im}\psi$, while the latter in turn forces a_{g+1} also to lie in the normal closure. □

Note that we have deliberately phrased the assertion of the lemma above so as to provoke the question as to whether it is also valid when $r > 1$.

These results enable us to show that the obvious candidate for stabilization also fails for the subgroup K_g of $\Gamma_{g,1}$ comprising the kernel of the capping homomorphism $\Gamma_{g,1} \to \Gamma_g$.

Lemma 4.10. *Given a map of group extensions*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow & & \downarrow & & \downarrow \\ N_1 & \rightarrowtail & G_1 & \twoheadrightarrow & Q_1 \end{array} ;$$

if the image of G in G_1 normally generates G_1, then the image of Q in Q_1 normally generates Q_1. The converse holds provided also that the image of N in N_1 normally generates N_1. □

Now, suppose that $\psi : \Gamma_{g,1} \longrightarrow \Gamma_{g+1,1}$ were to restrict to a homomorphism $\psi|$ from K_g to K_{g+1}. Consequently, ψ would induce a homomorphism

$$\bar{\psi} : \Gamma_g \cong \Gamma_{g,1}/K_g \longrightarrow \Gamma_{g+1}$$

making commutative the diagram

$$\begin{array}{ccccc} K_g & \rightarrowtail & \Gamma_{g,1} & \twoheadrightarrow & \Gamma_g \\ \downarrow^{\psi|} & & \downarrow^{\psi} & & \downarrow^{\bar{\psi}} \\ K_{g+1} & \rightarrowtail & \Gamma_{g+1,1} & \twoheadrightarrow & \Gamma_{g+1} \end{array} .$$

Now suppose that $g \geq 2$. Then, by the last lemma, the image of ψ cannot lie in K_{g+1}. Thus, $\bar{\psi}$ is nontrivial. However, under the further assumption that $2g+1$ is a prime at least 7, this is contradicted by the theorem above. We conclude that there are infinitely many values of g for which a restriction $\psi| : K_g \to K_{g+1}$ fails to exist.

5. Braid groups

The braid groups of interest here are of course those that contain some torsion, namely, after [34], those of the 2-sphere and the real projective plane. Presentations of these groups are displayed in [30].

Sphere.

The n-braid group $B_n(S^2) = \Gamma^n_{0,0}$ is a quotient of the classical Artin braid group, of braids on the disc. It is well-known to be torsion-generated, and in fact generated by just two torsion elements [14], namely α_0, α_1 as defined in Murasugi's theorem ((b) below). (Here, the cycle type of a braid refers to that of its corresponding induced permutation of its nodes, and σ_i is the usual generator passing the i th strand over the $(i+1)$ st.)

Theorem 5.1. (a) [9] *$B_n(S^2)$ has the presentation with generators σ_i $(1 \leq i \leq n-1)$ and relations*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad 1 < i+1 < j < n$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad 1 \leq i \leq n-2$$

$$\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = 1$$

(b) [30] *For $n \geq 3$, each torsion element of $B_n(S^2)$ is conjugate to a power of one of the following:*

- $\alpha_0 = \sigma_1 \cdots \sigma_{n-1}$ *of order* $2n$*, with cycle type* (n)*;*
- $\alpha_1 = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2$ *of order* $2(n-1)$*, with cycle type* $(n-1,1)$*;*
- $\alpha_2 = \sigma_1 \cdots \sigma_{n-3}\sigma_{n-2}^2$ *of order* $2(n-2)$*, with cycle type* $(n-2,1,1)$*.*

A simple, no doubt known, consequence is the following, reminiscent of Theorem 4.4 above.

Corollary 5.2. *Let* $n \geq 4$*.*
(a) *Adjoining a free strand does not induce a homomorphism* $B_n(S^2) \to B_{n+1}(S^2)$*.*
(b) *Doubling a strand does not induce a homomorphism* $B_n(S^2) \to B_{n+1}(S^2)$*.*

Proof. **(a)** By the above result, if such a homomorphism existed, then it would send α_2 to an element of finite order having cycle type $(n-2,1,1,1)$. However, by the theorem again (or [11] (4.8)), any element with such a cycle type must have infinite order.

(b) Similarly, there is some conjugate of α_2, necessarily of the same cycle type $(n-2,1,1)$, such that the doubled strand is one of the two that is stable under the associated permutation. In that case, we again have the contradiction that its image has finite order dividing $2(n-2)$, but is of cycle type $(n-2,1,1,1)$ and so of infinite order in $B_{n+1}(S^2)$. (For a fuller discussion of the interaction between doubling, deleting and permutations, see [3].) □

Thus, as with mapping class groups, the motto is that torsion prohibits stabilization. On the other hand, at the non-stable level we can salvage the following information. For this result and its corollary, we use the obvious fact that in $B_n(S^2)$

$$\sigma_{n-1} = \alpha_0^{-1}\alpha_1$$

and, from the defining relator $\sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1$,

$$\alpha_0 = \sigma_1^{-1} \cdots \sigma_{n-1}^{-1} .$$

A quick induction then shows that, for all i with $2 \leq i \leq n-1$,

$$(*) \qquad \sigma_{i-1} = \alpha_0^{-1}\sigma_i\alpha_0 .$$

Proposition 5.3. *The normal closure of* α_1 *in* $B_n(S^2)$ *has index* $\gcd(n,2)$*. In particular, for* n *odd,* $B_n(S^2)$ *is strongly* $2(n-1)$*-torsion generated.*

Proof. Write $\bar{\sigma}_i$ to indicate the image of σ_i in the quotient of $B_n(S^2)$ by the normal closure of α_1. From the relations above, every $\bar{\sigma}_i = \alpha_0^{-1}$. On the one hand, this makes the quotient group cyclic (generated by α_0), and thus a quotient of $(B_n(S^2))_{\mathrm{ab}} \cong C_{2(n-1)}$ (the isomorphism being evident from the defining relations for $B_n(S^2)$). On the other, the defining expression for α_0 reduces to $\alpha_0 = \alpha_0^{-(n-1)}$. The result follows. □

This leads to another counterpart of the rigidity result Theorem 4.4. Note that the permutation associated to any n-braid gives rise to the nontrivial homomorphism, for any r,

$$B_n(S^2) \twoheadrightarrow \Sigma_n \twoheadrightarrow C_2 \xrightarrow{\cong} \mathcal{Z}(B_{n+r}(S^2)) \hookrightarrow B_{n+r}(S^2)\,.$$

In certain circumstances, this is the unique nontrivial homomorphism from $B_n(S^2)$ to $B_{n+r}(S^2)$. Here is an example.

Corollary 5.4. *Let k, r be integers with $k, k+r \geq 1$, and suppose that k is coprime to each of $r-1$, r and $r+1$ (see Remark below). Then the image of any homomorphism $B_{k+1}(S^2) \to B_{k+r+1}(S^2)$ lies in $\mathcal{Z}(B_{k+r+1}(S^2)) \cong C_2$.*

Remark. To have k coprime to r and $r \pm 1$, a necessary condition is that $k \equiv \pm 1 \pmod 6$; while a sufficient condition is that also $r \equiv \pm 2$ or $\pm 3 \pmod k$.

Proof. Since the only torsion in $B_m(S^2)$ divides $2m$, $2(m-1)$ or $2(m-2)$ [11], the order of the image $\bar{\alpha}_1$ of α_1 is at most 2. That forces $\bar{\alpha}_1$ to lie in $\mathcal{Z}(B_{k+r+1}(S^2))$ (the unique subgroup of order 2 [11]). Since $\alpha_0 = \alpha_1 \sigma_k$, we obtain by induction from equation (*) that, for all i with $2 \leq i \leq k$, $\bar{\sigma}_i = \bar{\sigma}_k$. Again, the fact that the image is thereby cyclic makes it a quotient of C_{2k}. The defining expression for α_1 now yields that $\bar{\alpha}_1 = \bar{\sigma}_1^{k+1}$, so that $\bar{\sigma}_1^{2(k+1)} = 1$. Combining these two facts gives $\bar{\sigma}_1^2 = 1$; and hence the image of $B_{k+1}(S^2)$ is of order at most 2.□

Projective plane.

As one might expect, despite similarities with the spherical case, now there is a complication according to parity. We thus find it convenient to write n' for the greatest odd integer not exceeding n. Here is a compilation of the facts that we use.

Theorem 5.5. *Let $n \geq 2$.*

(a) [30] *$B_n(P^2)$ has the presentation with generators σ_i, ρ_i ($1 \leq i \leq n-1$) and ρ_n, and relations*

$$\begin{aligned}
\sigma_i\sigma_j &= \sigma_j\sigma_i \qquad 1 < i+1 < j < n\\
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad 1 \leq i \leq n-2\\
\sigma_i\rho_j &= \rho_j\sigma_i \qquad 1 < i+1 < j \leq n \quad \text{or} \quad 1 \leq j < i < n\\
\rho_i &= \sigma_i\rho_{i+1}\sigma_i \qquad 1 \leq i \leq n-1\\
\rho_{i+1}^{-1}\rho_i^{-1}\rho_{i+1}\rho_i &= \sigma_i^2 \qquad 1 \leq i \leq n-1\\
\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1 &= \rho_1^2
\end{aligned}$$

(b) (i) [30], [13] *The element*

$$\alpha_n = \begin{cases} \sigma_1\cdots\sigma_{n-1} & n \text{ odd}\\ \sigma_1\cdots\sigma_{n-1}\sigma_1 & n \text{ even}\end{cases}$$

has order $2n'$.

(ii) [13] *The centre* $\mathcal{Z}(B_n(P^2))$ *is generated by the unique element* $(\alpha_n)^{n'}$ *of* $B_n(P^2)$ *of order* 2.

(iii) [13] *A natural number* k *is the order of an element of* $B_n(P^2)$ *if and only if* $k|4n$ *or* $k|4(n-1)$.

(c) [18] *The commutator subgroup of* $B_n(P^2)$ *is the normal closure in* $B_n(P^2)$ *of* α_n , *and* $B_n(P^2)_{\rm ab} \cong C_2 \times C_2$.

Although we do not claim that $B_n(P^2)$ is strongly k-torsion generated for any value of k, nevertheless the above information suffices to establish the following rigidity result, comparable to (5.4) above.

Proposition 5.6. *For* $n \geq 3$, *the only nontrivial homomorphism from* $B_n(P^2)$ *to* $B_m(P^2)$ *is*

$$B_n(P^2) \twoheadrightarrow \Sigma_n \twoheadrightarrow C_2 \xrightarrow{\cong} \mathcal{Z}(B_m(P^2)) \hookrightarrow B_m(P^2)\,,$$

provided $\gcd(n', m) = \gcd(n', m-1) = 1$.

Proof. The numerical conditions guarantee that, from (b) in the previous result, α_n must have image of order at most 2, and so be central in $B_m(P^2)$. We now use the relations of (a) above. Because the images of σ_1 and α_n commute, while σ_1 commutes with all σ_i with $i \neq 2$, it follows that the images of σ_1 and σ_2 commute. The braid relation between σ_1 and σ_2 then forces that $\bar\sigma_1 = \bar\sigma_2$ (again using the overline to denote images), and thereby

$$\bar\sigma_1 = \cdots = \bar\sigma_{n-1}.$$

Now $\bar\rho_n$ commutes with $\bar\sigma_1$ (since $n \geq 3$), and therefore with each $\bar\sigma_i$, and so with $\bar\sigma_{n-1}\bar\rho_n\bar\sigma_{n-1} = \bar\rho_{n-1}$. Hence,

$$\begin{aligned}\bar\sigma_{n-1}^2 &= [\bar\rho_n^{-1}, \bar\rho_{n-1}^{-1}]\\ &= 1.\end{aligned}$$

Therefore, $\bar\alpha_n = 1$. So, from (c) above, the image of the homomorphism must be a nontrivial quotient of $C_2 \times C_2$. However, since $B_m(P^2)$ contains a unique element of order 2, the displayed homomorphism is the only possibility. □

Observe that the proposition above fails for $n = 2$, since the numerical conditions allow the identity map on $B_2(P^2)$.

The same argument as in the proof above may be applied to any finite 2-group quotient of $B_n(P^2)$ $(n \geq 3)$.

Corollary 5.7. *For* $n \geq 3$, *the only nontrivial finite* 2*-group quotients of* $B_n(P^2)$ *are* C_2 *and* $C_2 \times C_2$.

Proof. First, observe that if the quotient fails to be abelian, and so a quotient of $B_n(P^2)_{\rm ab} \cong C_2 \times C_2$, then the image $\bar\alpha_n$ of α_n has order 2. Therefore, $\bar\alpha_n = (\bar\alpha_n)^{n'}$, which is the image of the unique element of

$B_n(P^2)$ of order 2, and so central in the quotient. However, as in the proof above, this leads to a contradiction. □

In contrast, note that $B_2(P^2)$ is the generalized quaternion group of order 16 [34] p. 87, and so also stands in contrast to the next two consequences. The results above compare interestingly with recent investigations of Goncalves and Guaschi concerning quaternion subgroups of braid groups [16], [17].

Corollary 5.8. *For $n \geq 3$, the only nontrivial nilpotent quotients of $B_n(P^2)$ are C_2 and $C_2 \times C_2$.*

Proof. Since $B_n(P^2)$ is finitely generated with finite abelianization, any nilpotent quotient must be finite [32] p.127. Since $B_n(P^2)_{\mathrm{ab}} \cong C_2 \times C_2$, such a finite quotient can only be a 2-group, whence the previous corollary applies. □

In particular, there is no nilpotent quotient of class 2; so, the following is immediate.

Corollary 5.9. *For $n \geq 3$, the lower central series of $B_n(P^2)$ terminates at the commutator subgroup $\gamma_2(B_n(P^2))$.* □

By means of routine computations along the lines of the proof of Theorem 1.4 of [15], one can strengthen this result, for $n \geq 5$, as follows.

Proposition 5.10. *For $n \geq 5$, the derived series of $B_n(P^2)$ terminates at the commutator subgroup $(B_n(P^2))'$.* □

The same result was proved for the braid groups of the disc in [19] and of the sphere in [15].

6. Products of strongly torsion generated groups

In this section we further enlarge the class of known strongly torsion generated groups by establishing that a finite product of strongly torsion generated groups is strongly torsion generated. We also highlight the problems with attempts to extend to infinite products.

We begin with a more general result.

Proposition 6.1. *Let I be a finite, nonempty indexing set and let $n \geq 2$ be an integer. Suppose given for each $i \in I$ an integer $m_i \geq 2$ which is prime to n, and a group G_i which is strongly m_i-torsion generated and strongly n-torsion generated. Then the cartesian product group $\prod_{i\in I} G_i$ is strongly n-torsion generated.*

Proof. For each i, let $g_i, h_i \in G_i$ be normal generators of order n and m_i respectively, and consider the element $g := (g_i)_{i\in I}$ of the product group $G := \prod_{i\in I} G_i$. Clearly, g is of order n and we intend to show that it is a normal generator of G. For the sake of readability, we consider each G_i as a subgroup of G, in the obvious way. Now, we fix $i \in I$ and we claim that

$$\langle\!\langle\, g \,\rangle\!\rangle \supseteq \{\, x^n \mid x \in G_i \} \ .$$

Fix an element $x \in G_i$. Since g_i is a normal generator of G_i, there exist elements $y_1, \ldots, y_s \in G_i$ (for some $s \geq 1$) such that

$$x = y_1 g_i y_1^{-1} \cdots y_s g_i y_s^{-1}$$

in G_i, hence in G. It follows that the element

$$y_1 g y_1^{-1} \cdots y_s g y_s^{-1}$$

of G has x as i th component, and g_j^s as j th component for every $j \in I$ distinct from i. As a consequence, using that $g_j^n = e$ for each j, we get

$$\langle\!\langle\, g \,\rangle\!\rangle \ni (y_1 g y_1^{-1} \cdots y_s g y_s^{-1})^n = x^n \,,$$

as claimed. We keep $i \in I$ fixed. From this claim, we get that $h_i^n \in \langle\!\langle\, g \,\rangle\!\rangle$. Since n is prime to m_i, it follows from Bézout's Theorem that $\langle\!\langle\, h_i^n \,\rangle\!\rangle_{G_i} = \langle\!\langle\, h_i \,\rangle\!\rangle_{G_i}$, and therefore,

$$\langle\!\langle\, g \,\rangle\!\rangle \supseteq \langle\!\langle\, h_i^n \,\rangle\!\rangle_G = \langle\!\langle\, h_i^n \,\rangle\!\rangle_{G_i} = \langle\!\langle\, h_i \,\rangle\!\rangle_{G_i} = G_i \,.$$

Since this holds for every $i \in I$, and I is finite, we infer that $\langle\!\langle\, g \,\rangle\!\rangle = G$, completing the proof. □

The next corollary is an immediate consequence of the proposition.

Corollary 6.2. *A finite cartesian product of strongly torsion generated groups is strongly torsion generated.* □

The above result is applied in [6].

Remark 6.3. If $\{G_i\}_{i \in I}$ is an infinite collection of groups, then, their *restricted product* $G := \prod'_{i \in I} G_i$ is the subgroup of the cartesian product $\prod_{i \in I} G_i$ consisting of the elements with finite support, *i.e.* those elements $(g_i)_{i \in I}$ with $g_i = e$ for almost all $i \in I$. Note that G is the filtered colimit of the finite products $\prod_{i \in J} G_i$, with J running over the finite subsets of I. Now, suppose that each G_i is nontrivial. Plainly, any finite subset F of G is contained in some finite subproduct $H := \prod_{i \in J} G_i$ of G, *i.e.* with $J \subseteq I$ finite; as a consequence, the normal closure $\langle\!\langle\, F \,\rangle\!\rangle_G$ is also contained in H. In particular, G is *not* finitely normally generated. This shows that *an infinite restricted product of strongly torsion generated groups is* ***never*** *strongly torsion generated.* Combined with Corollary 6.2, this implies that *strongly torsion generated groups are* not *closed under filtered colimits.*

Example 6.4. We finally discuss the (unrestricted) cartesian product of strongly torsion generated groups, in particular the key example of $\prod_{i \in \mathbb{N}} A_\infty$. (As revealed in previous sections of this note, one reason for the importance of A_∞ as a strongly torsion generated group is that it normally generates a number of other classical groups, and so, by Lemma 2.3, makes them also strongly torsion generated.)

We show that $\prod_{i\in\mathbb{N}} A_\infty$ fails to be strongly 2-torsion generated (the argument clearly generalizes). To see this, recall the well-known fact that the involutions in A_∞ are precisely the conjugates in A_∞ of products of disjoint transpositions, each of which is conjugate in A_∞ to some

$$a_k = (1\ 2)(3\ 4)\cdots(4k-3\ 4k-2)(4k-1\ 4k) \in A_{4k} \subseteq A_\infty\,.$$

Moreover, no two distinct a_k are conjugate; in fact, a_{mk} is the product of m conjugates of a_k, and cannot be expressed as the product of fewer than m conjugates of a_k.

Now consider an arbitrary involution $g = (g_i)_i \in \prod_{i\in\mathbb{N}} A_\infty$, where we may assume that each g_i is conjugate to a_{k_i} for some sequence $k_1, k_2, \ldots$ It follows that the element a_{ik_i} is the product of at least i conjugates of g_i; so the length of the product increases unboundedly as i increases. Hence, $(a_{ik_i})_i$ cannot be expressed as a product of a finite number of conjugates of g, and so lies outside the normal closure of g.

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