Journal of Pure and Applied Algebra 88 (1993) 3–22
North-Holland

Imperfect groups

A.J. Berrick

Department of Mathematics, National University of Singapore, Kent Ridge, 0511, Singapore

Derek J.S. Robinson*

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Communicated by P.H. Kropholler
Received 27 November 1992

To K.W. Gruenberg on his sixty-fifth birthday

Abstract

Berrick, A.J. and D.J.S. Robinson, Imperfect groups, Journal of Pure and Applied Algebra 88 (1993) 3–22.

A group is said to be imperfect if it has no non-trivial perfect quotient groups. A detailed study of imperfect groups is carried out; topics include the normal and subnormal structure of imperfect groups, characterizations of the imperfect radical and residual, and connections with linear groups.

1. Introduction

Perfect groups, or groups that coincide with their derived subgroups, have featured quite extensively in the literature of group theory, and have even been the subject of a recent monograph [6]. In this work we shall study a type of group that is far removed from the domain of perfect groups. A group is said to be *imperfect* if it has no non-trivial perfect quotient groups. It is our object here to draw attention to a range of interesting phenomena associated with groups of this type.

Obvious examples of imperfect groups include soluble groups and finite symmetric groups. Equally clear is the observation that a finite group is imperfect if and only if it has no non-cyclic simple quotients. On the other hand, the situation

Correspondence to: Professor D.J.S. Robinson, Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA.

* The second author was a visitor at the National University of Singapore while this research was carried out.

is more complicated for infinite groups, as is shown by the well-known characteristically simple group of McLain [11]; this group is perfect, yet it has no simple quotients.

The first mention of imperfect groups in the literature appears to be in [16, Lemma 9.22]. Quite recently imperfect groups have turned out to be relevant to work on the homology of torsion generated groups [3]; for an application to the homology of torsion generated imperfect groups see Theorem 2.8 below. Imperfect groups also play a role in a paper of Berrick and Menal [4] in connection with the so-called μ-problem: when does $\mu(G)=\mu(G_{ab})$ where $\mu(G)$ is the smallest cardinality of a subset whose normal closure equals the group G? An application to this problem is given in Theorem 4.7. It is our belief that imperfect groups form a natural class of groups that deserves closer attention.

We begin our study by establishing the fundamental closure properties of the class of imperfect groups, and by characterizing these in terms of quasicentral chief factors, that is, chief factors in which conjugation by any group element is an inner automorphism (Proposition 2.6).

It turns out that the subnormal structure of an imperfect group can be arbitrarily complicated, in the sense that every group can be embedded as 2-step subnormal subgroup of some imperfect group (Theorem 3.2). It is a more delicate matter to describe the groups that embed as normal subgroups of imperfect groups. A necessary and sufficient condition for a group to be so embeddable is given in Theorem 3.3. Essentially the condition requires that the group be rich in outer automorphisms. For example, McLain's characteristically simple group is normally embeddable in an imperfect group, but the Mathieu group M_{11} is not.

In Section 4 we characterize the imperfect radical and imperfect residual of a group which has finite composition length. For example, it is shown that the imperfect radical coincides with the purely non-abelian residual (Corollary 4.3); here a group is called *purely non-abelian* if every subnormal composition factor is non-abelian (cf. [23]). This result, which seem to be new even in the finite case, has a nice application to the μ-problem (Theorem 4.7).

Our object in Section 5 is to construct a perfect group which satisfies min-sn, (the minimal condition on subnormal subgroups), and which is the union of a chain of imperfect normal subgroups. This group has a number of notable properties; for example, normality is transitive and the multiplicator is trivial. We feel that the group is of independent interest since it illustrates the complexity of groups with min-sn. By rights this group should have appeared when the theory of groups with min-sn was being worked out by Roseblade [17] and the second author [15] over twenty-five years ago.

The final section of the paper explores the relationship between imperfect groups and linearity. The group $GL_n(\mathbb{Z})$ is never imperfect if $n>1$; we illustrate this by determining the imperfect radical and residual (Theorems 6.1 and 6.4). Then sufficient conditions on a ring with identity R are found for $GL_n(R)$ to be imperfect. Our results are most complete when R is a commutative semilocal ring (see Theorem 6.9). Here essential use is made of fundamental results on the normal structure of $\mathrm{GL}_n(R)$ due to Bass, Vaserstein and others.

2. Elementary properties

We begin with the basic closure properties of the class of imperfect groups. It is obvious that a quotient of an imperfect group is imperfect. Closure under extensions is almost as easy.

Lemma 2.1. *If N is a normal subgroup of a group G, and both N and G/N are imperfect, then G is imperfect.*

Proof. Assume that G/L is a perfect quotient. Then $G = LN$, and thus $N/L \cap N$ is perfect. Hence $N \leq L$ and $G = L$. □

Corollary 2.2. *If M and N are imperfect normal subgroups of a group, then MN is imperfect.* □

The corollary motivates us to introduce the *imperfect radical* of a group G,

$$\mathrm{Imp}(G) .$$

This is the subgroup generated by all the imperfect normal subgroups of G; it is locally imperfect, but not necessarily imperfect, as we see from McLain's locally nilpotent group.

Dual to Corollary 2.2 is the following:

Lemma 2.3. *If M and N are normal subgroups of a group G, and G/M and G/N are imperfect, then $G/M \cap N$ is imperfect.*

Proof. We may assume that $M \cap N = 1$. Let G/L be a perfect quotient group. Then $G = LM = LN$. Since we can factor out by $L \cap M$, there is nothing to be lost in assuming that $L \cap M = 1$. Then $[L, M] = 1 = [N, M]$, so that $M \leq Z(G)$. Since $G = LM$, it follows that $L = G$. □

Next we introduce the *imperfect residual* of a group G,

$$\mathrm{Imp}^*(G);$$

this is the intersection of all $N \lhd G$ such that G/N is imperfect. Thus $G/\mathrm{Imp}^*(G)$ is residually imperfect, but not necessarily imperfect, since free groups are residually nilpotent.

The next closure property is a little more unexpected.

Lemma 2.4. *A direct product of imperfect groups is imperfect.*

Proof. Let $G = \mathrm{Dr}_{\lambda \in \Lambda} G_\lambda$ where G_λ is an imperfect group. Assume that G/L is perfect. Factoring out by the product of the $L \cap G_\lambda$, we may assume that $L \cap G_\lambda = 1$ for every λ. Therefore, $L \leq Z(G) = \mathrm{Dr}_{\lambda \in \Lambda} Z(G_\lambda)$, which shows that $G/Z(G)$ is perfect. This implies that $G_\lambda = Z(G_\lambda)$ for all λ, so G is abelian and $L = G$. □

Quasicentrality

A normal factor H/K of a group G is said to be *quasicentral* if each element of G induces by conjugation an inner automorphism in H/K. Thus a quasicentral factor is central precisely when it is abelian. Quasicentrality is a useful tool in the study of imperfect groups.

If H/K is quasicentral in G and $g \in G$, then there is an $h \in H$ such that $x^g \equiv x^h \pmod{K}$ for all $x \in H$, that is, $gh^{-1} \in C_G(H/K)$. Hence G/K is the direct product of H/K and $C_{G/K}(H/K)$ in which $Z(H/K)$ is amalgamated. The converse of this statement is obviously true. Thus we have the following lemma:

Lemma 2.5. *A normal factor H/K of a group G is quasicentral if and only if G/K is the direct product of H/K and $C_{G/K}(H/K)$ with $Z(H/K)$ amalgamated.* □

If every chief factor of a group is quasicentral, the group is called *quasinilpotent*; for the structure of finite quasinilpotent groups see [19, Chapter X, Section 13]. Here we are concerned with the opposite situation, where the only quasicentral chief factors are the central ones. For groups which satisfy max-n, the maximal condition on normal subgroups, this property characterizes imperfection.

Proposition 2.6. (a) *In an imperfect group every quasicentral chief factor is central.*

(b) *If G is a group which satisfies* max-n *and every quasicentral chief factor is central, then G is imperfect.*

Proof. Suppose that G has a non-central, quasicentral chief factor H/K. Then H/K is non-abelian, so $Z(H/K)$ is trivial and G/K is the direct product of H/K and $C_{G/K}(H/K)$ by Lemma 2.5. Since H/K is perfect, G is not imperfect.

Conversely, assume that G satisfies max-n and is not imperfect. Then G has a non-cyclic simple quotient G/K, and of course G/K is a non-central quasicentral chief factor. □

Notice that part (b) of Proposition 2.6 is not generally valid. Indeed McLain's group is perfect, yet all its chief factors are central; for in any locally nilpotent group the chief factors are central (see [16, 6.1]). In fact, (b) does not even hold for groups which satisfy min-sn, as the group of Theorem 5.1 shows.

The section concludes with a characterization of imperfect groups by mapping properties.

Lemma 2.7. *The following conditions on a group G are equivalent*:

(a) *G is imperfect*;

(b) *if $\varphi : H \to G$ is a homomorphism inducing an epimorphism $\varphi_{ab} : H_{ab} \to G_{ab}$, then $G = \langle (\mathrm{Im}\ \varphi)^G \rangle$*;

(c) *if $N \lhd G$ and the canonical map $N_{ab} \to G_{ab}$ is surjective, then $N = G$.* □

We omit the very easy proof, and pass to an application to the homology of torsion generated groups (that is, groups that are generated by elements of finite order).

Theorem 2.8. *Let $\varphi : H \to G$ be a homomorphism, where the group G is torsion generated and imperfect. Assume that $\varphi_* : H_n(H) \to H_n(G)$ is an isomorphism for all sufficiently large n. Then $G = \langle (\mathrm{Im}\ \varphi)^G \rangle$, the normal closure of* Im φ *in G.*

Proof. A result of Berrick [3] asserts that if G is torsion generated and $\varphi : H \to G$ is a homomorphism inducing an isomorphism $\varphi_* : H_n(H) \to H_n(G)$ for all sufficiently large n, then $\varphi_{ab} : H_{ab} \to G_{ab}$ is surjective. The theorem now follows via Lemma 2.7. □

3. Subnormal embedding in imperfect groups

Our intent in this section is to show that the subnormal structure of an imperfect group can be arbitrarily complicated, while the normal structure is somewhat restricted. The main embedding tool is the following theorem:

Theorem 3.1. *Let $W = H$ wr K be the wreath product of permutation groups H and K where K is transitive. Then W is imperfect if and only if either* (i) $K = 1$ *and H is imperfect, or* (ii) *K is non-trivial and imperfect.*

Proof. The conditions are obviously necessary. Assume that they are satisfied and let $K \neq 1$ act on a set X. Then the base group of W is $B = \mathrm{Dr}_{x \in X} H_x$ where $H_x \simeq H$ and $H_x^k = H_{(x)k}$ $(x \in X, k \in K)$.

Suppose that W/L is a non-trivial perfect quotient of W. Then we have $B_0 = \prod_{x \in X} (L \cap H_x)$ is normal in W. Thus we can pass to the group W/B_0, which is isomorphic to $\bar{H}$ wr K for some quotient $\bar{H}$ of H. This allows us to assume that $L \cap H_x = 1$ for all $x \in X$. It follows that $[L \cap B, H_x] = 1$ and thus $L \cap B \leq Z(B) = B_1$ say. Now $B_1 = \prod_{x \in X} Z(H_x)$ and $W/B_1 \simeq (H/Z(H))$ wr K.

Evidently $W \neq LB_1$, so we can pass to the group W/B_1. This observation enables us to assume that $L \cap B = 1$. Hence $W = LB = L \times B$, which implies that K is perfect, so $K = 1$. □

If H is an arbitrary group, then the standard wreath product $G = H \operatorname{wr} \mathbb{Z}_2$ is imperfect, according to Theorem 3.1. Therefore we have the following:

Theorem 3.2. *If H is an arbitrary group, then H is isomorphic with a 2-step subnormal subgroup of an imperfect group G such that $|G| = 2|H|^2$.* □

Turning to the question of normally embedding a group in an imperfect group, we find that for this to be possible the group must have sufficiently many outer automorphisms. This is made precise by the following technical condition:

> *a group H satisfies condition* **A** *if there is an imperfect subgroup X of* Aut H *such that, whenever H/M is a non-trivial, perfect X-admissible quotient, some element of X induces an outer automorphism in H/M.*

Theorem 3.3. *A group H is isomorphic with a normal subgroup of an imperfect group if and only if H satisfies condition* **A**.

Proof. First of all, let $H \lhd G$ where G is imperfect; we show that H satisfies **A**. Put $X = G/C_G(H)$, considered as a subgroup of Aut H. Suppose that H/M is a non-trivial, perfect X-admissible quotient of H, thus making M normal in G, and that H/M is quasicentral in G/M. Then by Lemma 2.5 the quotient group of H/M by its centre is an image of G. This leads to the contradiction $H/M = Z(H/M)$. It follows that condition **A** is satisfied by H.

Conversely, assume that H is a group satisfying **A**, and let X be the subgroup of Aut H furnished by that property. Define $G = X \ltimes H$, the semi-direct product. Suppose that G/L is a non-trivial perfect quotient of G. Since X is imperfect, $G = HL$ and $G/L \simeq H/M$ where $M = H \cap L$. Thus H/M is a non-trivial, perfect X-admissible quotient. Since $G = HL$ and $[H, L] \leq M$, every element of G (and hence of X) induces an inner automorphism of H/M, which is a contradiction. Therefore G must be imperfect. □

In principle Theorem 3.3 allows us to determine if a given group is normally embeddable in an imperfect group, although in practice it may be difficult to decide if condition **A** holds unless the normal structure of H is fairly simple.

For example, if H is a non-cyclic simple group, then it is clear that H satisfies **A** if and only if Out $H \neq 1$. Hence the following corollary:

Corollary 3.4. *A simple group is normally embeddable in an imperfect group if and only if it is not complete.* □

As a second example, consider McLain's characteristically simple group H, with the usual generators $x_{\lambda\mu} = 1 + e_{\lambda\mu}$, where $\lambda < \mu \in \mathbb{Q}$. If $\nu \in \mathbb{Q}$, there is an automorphism α_ν of H such that

$$(x_{\lambda\mu})^{\alpha_\nu} = x_{\lambda+\nu,\mu+\nu} .$$

If ξ in $\mathbb{Q}$ is positive, there is also an automorphism β_ξ such that

$$(x_{\lambda\mu})^{\beta_\xi} = x_{\lambda\xi,\mu\xi} .$$

Now define

$$X = \langle \alpha_\nu, \beta_\xi \mid \nu \in \mathbb{Q}, 0 < \xi \in \mathbb{Q} \rangle .$$

Evidently X is isomorphic with the semidirect product $\mathbb{Q}^+ \ltimes \mathbb{Q}$, where $\mathbb{Q}^+$ is the multiplicative group of positive rationals. Thus X is metabelian. Now let H/M be a non-trivial X-admissible quotient. If $M \neq 1$, then by [11] M contains a generator $x_{\lambda\mu}$. Applying elements of X, we quickly see that M must contain every generator, that is, $M = H$. It follows that $M = 1$. Since X contains outer automorphisms, H satisfies **A**, and we have proved the following corollary:

Corollary 3.5. *McLain's characteristically simple group is normally embeddable in an imperfect group.* □

4. Imperfect radicals and residuals

Our aim in this section is to characterize the imperfect radical and imperfect residual of a group with finite composition length. The first characterization involves the concept of a purely non-abelian group. Here a group G is termed *purely non-abelian* if each subnormal composition factor—that is, each simple factor H/K with H subnormal in G—is non-abelian. An alternative way of defining purely non-abelian groups is indicated by the following lemma:

Lemma 4.1. *A group G is purely non-abelian if and only if every subnormal subgroup of G is perfect.*

Proof. Suppose that H is a non-perfect subnormal subgroup of G. Then H/H' has a composition series (of some order type), so there is an abelian composition factor L/M with $H' \leq M < L \leq H$. Thus G is not purely non-abelian. Conversely, if G is not purely non-abelian, it is obvious that it possesses non-perfect subnormal subgroups. □

The close relationship between imperfect normal subgroups and purely non-abelian quotient groups is demonstrated by the following theorem:

Theorem 4.2. *Let G be a group which satisfies the minimal condition on subnormal subgroups. Then $G/\mathrm{Imp}(G)$ is the largest purely non-abelian quotient group of G.*

Proof. We prove first that $G/\mathrm{Imp}(G)$ is purely non-abelian, for which purpose we can assume $\mathrm{Imp}(G)$ to be trivial. If G is not purely non-abelian, then by Lemma 4.1 there is a subnormal subgroup H of G which is minimal subject to being non-perfect. Let $H = H_0 \lhd \cdots \lhd H_n = G$. Suppose that $\mathrm{Imp}(H_i) = 1$, and let L be an imperfect normal subgroup of H_{i-1}. Then L is subnormal in G. Since G satisfies min-sn, it follows from [15] or [17] that L has only finitely many conjugates in G. Therefore $\langle L^{H_i} \rangle$ is imperfect, and $L \leq \mathrm{Imp}(H_i) = 1$. Consequently $\mathrm{Imp}(H_{i-1}) = 1$. Since $\mathrm{Imp}(G) = 1$, induction on $n - i$ yields $\mathrm{Imp}(H) = 1$, so that H is not imperfect. It follows that there is a non-trivial perfect quotient H/K. By minimality of H we can conclude that K is perfect, whence so is H, a contradiction.

So far we have shown that $G/\mathrm{Imp}(G)$ is purely non-abelian. On the other hand, if G/J is purely non-abelian and $M \lhd G$ is imperfect, then Lemma 4.1 shows that $M/M \cap J$ is perfect, which implies that $M \leq J$. It follows that $\mathrm{Imp}(G) \leq J$. □

Specializing to groups with finite composition length (that is, satisfying min-sn and max-sn, the maximal condition on subnormal subgroups), we obtain the following:

Corollary 4.3. *Let G be a group with finite composition length. Then* $\mathrm{Imp}(G)$ *is simultaneously the maximum imperfect normal subgroup of G, and the minimum normal subgroup of G with purely non-abelian quotient.* □

Remarks. In Theorem 4.2 one cannot assert that $\mathrm{Imp}(G)$ is imperfect, as is shown by Theorem 5.1. Also Theorem 4.2 is not valid for groups with max-sn. For $G = \mathrm{GL}_3(\mathbb{Z})$ satisfies max-sn by Wilson [22], and by Theorem 6.1 $\mathrm{Imp}(G) = \langle -I_3 \rangle$, so that $G/\mathrm{Imp}(G) \simeq \mathrm{SL}_3(\mathbb{Z})$, which is not purely non-abelian.

In order to describe the imperfect residual, we introduce a new characteristic subgroup which may be formed in any group.

Lemma 4.4. *Let G be an arbitrary group. Then there is a unique normal subgroup which is maximal with respect to being perfect and having every G-simple quotient quasicentral in G.*

Proof. Let **P** denote the following property of normal subgroups N of G: the subgroup N is perfect and each G-simple quotient of N is quasicentral in G. Consider a chain $\{N_i \mid i \in I\}$ of normal subgroups with **P**, and let U be the union of the chain. Clearly U is perfect. If U/V is a G-simple quotient of U, then $N_i \not\leq V$ for some i, and then $U = VN_i$. Hence $U/V \overset{G}{\cong} N_i/V \cap N_i$, which shows that U/V is quasicentral. Thus U has **P**, and Zorn's Lemma implies that there exists an $M \lhd G$ which is maximal with **P**.

Now let $N \lhd G$ have **P**. We claim that MN has **P**, from which it will follow that $N \leq M$ and M is the required subgroup. Of course MN is perfect. Also a G-simple quotient of MN is G-isomorphic with a quotient of M or of N and so is quasicentral. Thus MN has **P**. □

We shall write

$$\rho(G)$$

for the maximum normal subgroup of G with **P**; this is characteristic in G. Our characterization of $\mathrm{Imp}^*(G)$ can now be formulated. Recall that a group has a chief series of finite length precisely when it satisfies both max-n and min-n (the minimal condition on normal subgroups).

Theorem 4.5. (a) *If G is a group with* min-n, *then* $\mathrm{Imp}^*(G)$ *has the property* **P**, *and so is contained in* $\rho(G)$.

(b) *If G has finite chief length, then* $\mathrm{Imp}^*(G) = \rho(G)$.

Corollary 4.6. *Let G be a group with finite chief length. Then* $\mathrm{Imp}^*(G)$ *is simultaneously the maximum normal subgroup with property* **P** *and the minimum normal subgroup with imperfect quotient group.* □

Proof of Theorem 4.5. (a) Let $R = \mathrm{Imp}^*(G)$; then G/R is imperfect by min-n and Lemma 2.3. Thus G/R' is imperfect by Lemma 2.1, so $R = R'$. Next suppose that R/N is a G-simple quotient of R. Since G/N is not imperfect, there is a non-trivial perfect quotient G/L where $N \leq L$. Now R/N is G-simple and $R \not\leq L$, so $L \cap R = N$. Since $G = LR$, we have $G/N = L/N \times R/N$, which shows that R/N is quasicentral in G. Therefore R has **P**.

(b) Now G has finite chief length. Suppose that $R \neq J = \rho(G)$. Then by max-n there is a G-simple quotient J/K where $R \leq K$. Since J has **P**, it follows that J/K is quasicentral in G. But G/R is imperfect, so Proposition 2.6 shows that J/K is central, in contradiction to the fact that J is perfect. □

It is not easy to relax the finiteness conditions in Theorem 4.5. In Corollary 5.4 it is shown that $\mathrm{Imp}^*(G)$ need not be contained in $\rho(G)$ when G satisfies max-sn. Also $\mathrm{Imp}^*(G)$ and $\rho(G)$ can be different when G satisfies min-sn (Corollary 5.6).

Application to the μ-problem

We now present an application of our main result on the imperfect radical, Theorem 4.2, to what we call the μ-problem.

If G is an arbitrary group, define

$$\mu(G)$$

to be the smallest cardinal of a non-empty subset X whose normal closure $\langle X^G \rangle$ equals G. Then it is obvious that $\mu(G) \geq \mu(G_{ab})$. The μ-problem asks when equality holds:

$$\mu(G) = \mu(G_{ab}) .$$

Naturally $\mu(G_{ab})$ is just the (Prüfer) rank of G_{ab}. That equality does not always hold, even for finitely generated groups, is shown by the group $G = \mathbb{Z} * P$ where P is a non-trivial finite perfect group. For $\mu(G_{ab}) = 1$, but $\mu(G) > 1$ since the Kervaire conjecture is true in this case (see [9, p. 50]). (In fact, $\mu(G) = 2$.) In addition, if G is either McLain's locally nilpotent group or the locally finite, perfect group with min-sn of Theorem 5.1, then $\mu(G_{ab}) = 1$ and $\mu(G) = \aleph_0$.

Despite these examples, the μ-equality holds for a surprisingly wide class of groups, as we shall show.

Theorem 4.7. *If the group G is an extension of an imperfect group by a group with finite composition length, then $\mu(G) = \mu(G_{ab})$.*

Thus in particular $\mu(G) = \mu(G_{ab})$ if G has finite composition length. The proof depends on two auxiliary results.

Lemma 4.8. *Let G be a purely non-abelian group which has finite chief length. Then $\mu(G) = 1$, that is, $G = \langle x^G \rangle$ for some element x.*

Proof. Assume that G is non-trivial and choose a minimal normal subgroup N. By induction on the chief length $G = N\langle x^G \rangle$ for some element x. Since G is purely non-abelian, $Z(N) = 1$. If $[N, x]$ happens to be trivial, we choose $a \in N \backslash 1$ and set $x' = xa$; then $G = N\langle x'^G \rangle$ and $[N, x'] \neq 1$. By this argument we may suppose that $[N, x] \neq 1$. Hence $N \cap \langle x^G \rangle \neq 1$, whence $N \leq \langle x^G \rangle$ and $G = \langle x^G \rangle$. □

A special case of the next result may be found in [4].

Proposition 4.9. *Let $I \lhd G$ where I is imperfect and G/I is perfect. Then $\mu(G) \leq \mu(G/I)\mu(G_{ab})$.*

Proof. We can write $G = I\langle X^G \rangle$ where $|X| = \mu(G/I)$. Since $G = G'I$, for each x in X there is an element x^* in I such that $xx^* \in G'$. For the same reason we can write $G = G'\langle Z \rangle$ where $Z \subseteq I$ and $|Z| = \mu(G_{ab})$. Now define N to be the normal closure in G of the subset

$$S = \{xx^*z \mid x \in X, z \in Z\}.$$

Then $xx^*z \in N$ and $xx^* \in G'$ for all $x \in X$, $z \in Z$; therefore $Z \subseteq G'N$ and $G = G'N$. Also, if $x \in X$ and $z \in Z$, then NI contains xx^*z and x^*z. Thus $X \subseteq NI$ and $G = I\langle X^G \rangle = NI$. But $G/N \simeq I/N \cap I$, and G/N is perfect, while I is imperfect. Hence $G = N$. It follows that $\mu(G) \leq |S| \leq \mu(G/I)\mu(G_{ab})$. □

Proof of Theorem 4.7. Let $I = \operatorname{Imp}(G)$. The hypothesis shows that I is imperfect, whence we see that $\operatorname{Imp}(G/I)$ is trivial. Thus G/I is purely non-abelian by Corollary 4.3; it also has finite composition length. We can now apply Lemma 4.8 and Proposition 4.9 to deduce that $\mu(G) \leq \mu(G_{ab})$. □

5. A perfect group with min-sn

Here we shall construct a perfect group with min-sn which is the union of a countably infinite chain of imperfect normal subgroups.

Theorem 5.1. *There is countably infinite, locally finite group G with the following properties*:

(a) *G satisfies* min-sn;

(b) *G is the union of a chain of imperfect normal subgroups each of which has finite composition length*;

(c) *G is perfect and its Schur multiplicator is* 0 (*so G is superperfect*);

(d) *G has no proper subgroups of finite index*;

(e) *every subnormal subgroup of G is normal, that is, G is a* **T**-*group*.

The construction

In what follows S_∞ and S denote the symmetric group and the restricted symmetric group respectively on a countably infinite set. The alternating group of even finitary permutations is denoted by A.

We begin with two elementary lemmas.

Lemma 5.2. *Let H be any countably infinite group. Then there is an embedding of H in S_∞ as a regular subgroup*; *in particular*, $H \cap S = 1$. □

The embedding is simply the regular representation of H. Since $C_{S_\infty}(A) = 1$, there follows:

Corollary 5.3. *If H is a countably infinite group, there is an embedding of H in* Aut A *such that* $H \cap \operatorname{Inn} A = 1$. □

This corollary allows us to settle a point raised by Theorem 4.5.

Corollary 5.4. *There is a group G satisfying* max-sn *such that* $\mathrm{Imp}^*(G) \not\leq \rho(G)$.

Proof. Let H denote the congruence subgroup modulo 3 in $\mathrm{GL}_3(\mathbb{Z})$. We regard H as a subgroup of $\mathrm{Aut}\, A$ such that $H \cap \mathrm{Inn}\, A = 1$, and form the semi-direct product

$$G = H \ltimes A .$$

Then G satisfies max-sn since $\mathrm{GL}_3(\mathbb{Z})$ does [22], while A is simple. Notice that H is a residually finite-3-group since it has generators $I + 3e_{ij}$. Thus $\mathrm{Imp}^*(H) = 1$ and, moreover, H has no non-trivial perfect normal subgroups. If G/L is an imperfect quotient, then $L \neq 1$ since H is not imperfect—it has $\mathrm{PSL}_3(2)$ as an image. Hence $A \leq L$, and $\mathrm{Imp}^*(G) = A$. The only non-trivial perfect normal subgroup of G is A, which is not quasicentral. Therefore $\rho(G) = 1$. □

Lemma 5.5. *If H is a regular subgroup of S_∞, there is an embedding $\theta : \mathrm{Aut}\, H \to S_\infty$ such that for each α in* $\mathrm{Aut}\, H$, *the element α^θ induces the automorphism α of H by conjugation in S_∞. Thus the embedding of H in S_∞ extends to an embedding of the holomorph of H.*

Proof. Let S_∞ act on the set $X = \{1, 2, \ldots\}$. Since H is regular, each x in X is uniquely expressible in the form $x = (1)h$ where $h \in H$. If $\alpha \in \mathrm{Aut}\, H$, we define α^θ in S_∞ by the rule

$$((1)h)\alpha^\theta = (1)h^\alpha \quad (h \in H) .$$

If $h_1 \in H$, then

$$((1)h_1)(\alpha^\theta)^{-1}h\alpha^\theta = ((1)h_1^{\alpha^{-1}})h\alpha^\theta = (1)(h_1^{\alpha^{-1}}h)^\alpha = ((1)h_1)h^\alpha ,$$

showing that $h^\alpha = (\alpha^\theta)^{-1}h\alpha^\theta$. Clearly θ is an injective homomorphism. □

We now begin the construction of the group of Theorem 5.1. It is known that $M(A) \simeq \mathbb{Z}_2$ (cf. [13] and [7, V, 25.12]); also A has a unique universal covering group

$$C ,$$

where of course $Z(C) \simeq \mathbb{Z}_2$ and $C/Z(C) \simeq A$. In addition $H^2(A, \mathbb{Z}_2) \simeq \mathbb{Z}_2$, so each automorphism of A lifts uniquely to C. It follows that the natural map

$$\mathrm{Aut}\, C \to \mathrm{Aut}\, A$$

is an isomorphism.

We now take copies $C_1, C_2, \ldots$ of the group C. Assume that we have already constructed groups $1 = U_0, U_1, U_2, \ldots, U_i$ subject to the following rules:

(a) C_i embeds in $\operatorname{Aut} U_{i-1}$ in such a way that $C_i \cap \operatorname{Inn} U_{i-1} = 1$;

(b) $U_i = C_i \ltimes U_{i-1}$;

(c) if $\operatorname{Aut}^* U_i$ is the group of automorphisms of U_i that leave $C_1, C_2, \ldots, C_i$ fixed set-wise, there is a splitting $\operatorname{Aut} C_i \to \operatorname{Aut}^* U_i$ of the natural map $\operatorname{Aut}^* U_i \to \operatorname{Aut} C_i$.

We must show how to construct the next group in the sequence U_{i+1}.

By Lemma 5.2 there is an embedding of C_{i+1} in S_∞ as a regular subgroup. Keeping in mind that $\operatorname{Aut} C \simeq \operatorname{Aut} A$, we have

$$C_{i+1} \hookrightarrow S_\infty \hookrightarrow \operatorname{Aut} C_i \hookrightarrow \operatorname{Aut}^* U_i .$$

Use the embedding $C_{i+1} \hookrightarrow \operatorname{Aut}^* U_i$ to form

$$U_{i+1} = C_{i+1} \ltimes U_i .$$

Then $C_{i+1} \cap \operatorname{Inn} U_i = 1$. For, if $c \in C_{i+1}$ were induced by an element of U_i, conjugation by c in C_i, and hence A, would be inner, contradicting $C_{i+1} \cap A = 1$.

Next from the embedding $C_{i+1} \hookrightarrow S_\infty$ and Lemma 5.5 we obtain embeddings

$$\operatorname{Aut} C_{i+1} \hookrightarrow S_\infty \hookrightarrow \operatorname{Aut} C_i \hookrightarrow \operatorname{Aut}^* U_i$$

such that each automorphism of C_{i+1} lifts to an automorphism of $U_{i+1} = C_{i+1} \ltimes U_i$ inducing the corresponding automorphism in U_i. Thus we have a splitting $\operatorname{Aut} C_{i+1} \to \operatorname{Aut}^* U_{i+1}$ of $\operatorname{Aut}^* U_{i+1} \to \operatorname{Aut} C_{i+1}$ and the construction has been effected.

Now form the direct limit of the direct system $U_1 \hookrightarrow U_2 \hookrightarrow U_3 \hookrightarrow \cdots$ to obtain a group G. Thus

$$G = \bigcup_{i=1,2,\ldots} U_i \quad \text{and} \quad U_i = \langle C_1, \ldots, C_i \rangle .$$

By construction C_j normalizes each C_i for $i \leq j$, so that $C_i \lhd \langle C_i, C_{i+1,\ldots} \rangle$ and $U_i \lhd G$.

Proof of Theorem 5.1. (i) *G is countably infinite and locally finite.* For $U_{i+1}/U_i \simeq C$, which has these properties.

(ii) *G is perfect.* For G is generated by copies of the perfect group C.

(iii) *G is a* **T**-*group.* First observe that U_i has no proper normal subgroups of finite index. Indeed suppose that N is such a subgroup. Then $U_{i-1}/U_{i-1} \cap N$ is finite, whence $U_{i-1} \cap N = U_{i-1}$ and $U_{i-1} \leq N$ by induction on $i > 0$. But C has no proper normal subgroups of finite index, so $N = U_i$. Note that U_i satisfies min-sn since it has finite composition length. It follows from [15] or [17] that U_i is a **T**-group. Hence G is a **T**-group [14, Corollary 2].

(iv) $C_G(C_i) \le U_i$. Assuming this to be false, we choose $g \in C_G(C_i) \backslash U_i$, and write $g = u_{i-1} c_i c_{i+1} \ldots c_k$, where $u_{i-1} \in U_{i-1}$, $c_j \in C_j$, $c_k \ne 1$ and $i < k$. For any x in C_i

$$1 = [x, g] = [x, c_i c_{i+1} \ldots c_k][x, u_{i-1}]^{c_i c_{i+1} \cdots c_k} .$$

Since $[x_i, c_i c_{i+1} \ldots c_k] \in C_i$ and $[x, u_{i-1}] \in U_{i-1}$, it follows that $[x, c_i c_{i+1} \ldots c_k] = 1$ for all $x \in C_i$. So we may as well take g to be $c_i c_{i+1} \ldots c_k$. Furthermore, let g be chosen with $k - i$ minimal subject to $[C_i, g] = 1$ and $g \notin U_i$.

If $k - i = 1$, then c_{i+1} induces an inner automorphism in C_i, which means that $c_{i+1} = 1$ since $C_{i+1} \cap \operatorname{Inn} C_i = 1$. Hence $k - i > 1$.

Next let $x \in C_i$ and $y \in C_{i+1}$. Then $x^y = (x^y)^g = x^{y^g}$, and here $y^g \in U_{i+1}$. But $C_{U_{i+1}}(C_i) \le U_i$ by the case $k - i = 1$; hence $u = y(y^g)^{-1} \in U_i$. Since $y = uy^g$, we have

$$y = u(y^{c_i})^{c_{i+1} \cdots c_k} = u([c_i, y^{-1}]y)^{c_{i+1} \cdots c_k} = u' y^{c_{i+1} \cdots c_k} ,$$

where $u' \in U_i$. Since $y^{c_{i+1} \cdots c_k} \in C_{i+1}$, it follows that $y = y^{c_{i+1} \cdots c_k}$ for all y in C_{i+1}. Thus $c_{i+1} \ldots c_k \in C_G(C_{i+1})$, contradicting the minimality of $k - i$.

(v) *The only proper subnormal subgroups of G are the U_i and K_i where* $Z(U_i/U_{i-1}) = K_i/U_{i-1}$.

By (iii) we need only consider a proper normal subgroup N of G. Now there is a largest i such that $U_{i-1} \le N$. Then $N \cap U_i = U_{i-1}$ or K_i, so that N centralizes U_i/K_i. Also N centralizes K_i/U_{i-1}. Since $\operatorname{Hom}(U_i/K_i, K_i/U_{i-1}) = 0$, it follows that N centralizes U_i/U_{i-1}, and therefore $[C_i, N] \le U_{i-1}$. Let $g \in N$, and write $g = u_{i-1} c_i c_{i+1} \ldots c_k$ where $u_{i-1} \in U_{i-1}$, $c_j \in C_j$, $i \le k$. Now $[C_i, g] \le U_{i-1}$, and it is easily seen from this that $[C_i, c_i c_{i+1} \ldots c_k] = 1$, which by (iv) implies that $c_i c_{i+1} \ldots c_k \in U_i$ and $g \in U_i$. Hence $N \le U_i$ and $N = U_{i-1}$ or K_i.

(vi) *G satisfies* min-sn. This follows from (v).

(vii) *K_i is imperfect with finite composition length and* $G = \bigcup_{i=1,2,\ldots} K_i$.

By (v) the only proper non-trivial normal subgroup of K_i/K_{i-1} is U_{i-1}/K_{i-1}. It follows that K_i/K_{i-1} is imperfect for all i, and thus each K_i is imperfect. Clearly K_i has finite composition length.

(viii) $M(G) = 0$. It is enough to prove that $M(U_i) = 0$ for all i. Consider the homology spectral sequence associated with the group extension $1 \to U_{i-1} \to U_i \to C \to 1$. Now $E^2_{02} = 0$ by induction hypothesis on i, and $E^2_{11} = 0$ since U_{i-1} is perfect. Finally $E^2_{20} = 0$ since $M(C) = 0$ (see [18, p. 123]). Therefore $M(U_i) = 0$.

The proof of Theorem 5.1 is now complete. □

Corollary 5.6. *There is a group H satisfying* min-sn *such that* $\operatorname{Imp}^*(H) \ne \rho(H)$.

Proof. Let $H = G \operatorname{wr} \langle x \rangle$ where G is the group of Theorem 5.1 and x has order 2.

Then H is imperfect by Theorem 3.1, so $\mathrm{Imp}^*(H) = 1$. However we shall show that $\rho(H) = B$, the base group of the wreath product.

In the first place B is certainly perfect since G is. Suppose that B/N is an H-simple quotient of B. Then $D = (G_1 \cap N)(G_x \cap N) \lhd H$ and $N/D \leq Z(B/D)$. Therefore

$$N/D = Z(B/D) = Z(G_1D/D) \times Z(G_xD/D)$$

and $B/N \simeq \bar{G} \times \bar{G}$, where $\bar{G}$ is a non-trivial quotient of G. However the normal structure of the group G indicates that B/N cannot be H-simple; thus B has the property **P** and $\rho(H) = B$. □

6. Linear groups

In this section we study the general linear group $\mathrm{GL}_n(R)$, where R is a ring with identity, from the point of view of imperfection. We begin with the group $G = \mathrm{GL}_n(\mathbb{Z})$. Since G is never imperfect when $n > 1$, it is of interest to determine the imperfect radical and residual. In what follows

$$G(m)$$

denotes the congruence subgroup of G modulo m, comprising those matrices which reduce modulo m to the identity matrix I_n.

Theorem 6.1. *Let* $G = \mathrm{GL}_n(\mathbb{Z})$.

(a) *If* $n = 2$ *or* n *is odd, then* $\mathrm{Imp}(G) = \langle -I_n \rangle$.

(b) *If* n *is even and* $n \geq 4$, *then* $\mathrm{Imp}(G) = G(2)$.

The following lemma will prove useful in the proof of the theorem, and also in locating the imperfect residual.

Lemma 6.2. *Let* $G = \mathrm{GL}_n(\mathbb{Z})$ *where* $n > 1$. *If* $m > 1$, *then* $G/G(m)$ *is imperfect if and only if* $n = 2$ *or* n *is even and* m *is odd.*

Proof. Assume that $G/G(m)$ is imperfect and $n > 2$. If m is even, then $G/G(m)$ has $\mathrm{PSL}_n(2)$ as an image; this is impossible for $n > 2$, so m must be odd, and $G(m) \leq S = \mathrm{SL}_n(\mathbb{Z})$. If n is odd, then $G = S \times \langle -I_n \rangle$ and $\mathrm{PSL}_n(m)$ is an image of $G/G(m)$, again impossible.

Conversely, assume that the condition holds. If $m = p_1^{e_1} \cdots p_k^{e_k}$ with distinct primes p_i, then $G(m) = G(p_1^{e_1}) \cap \cdots \cap G(p_k^{e_k})$, while $G(p_i)/(G(p_i^{e_i})$ is a finite p_i-group. Thus we can suppose that $m = p$, a prime, by Lemmas 2.1 and 2.3. Since $G/G(2) \simeq S_3$ when $n = 2$, we can further assume that p is odd.

Let $D/G(p) = Z(S/G(p))$. Then conjugation in $F = S/D$ by the diagonal matrix with entries $-1, 1, \dots, 1$ produces an outer automorphism. For otherwise this diagonal matrix would be in S since $C_{\mathrm{GL}_n(p)}(\mathrm{SL}_n(p))$ consists of scalar matrices ([1, p. 240], or [21]) and n is even. It follows that F, the only non-central chief factor of $G/G(p)$, is not quasicentral. Proposition 2.6 now shows that $G/G(p)$ is imperfect. □

Proof of Theorem 6.1. *Case* $n > 2$. Let $I = \mathrm{Imp}(G)$; then I is imperfect since G satisfies max-n. Assume first that n is odd; then $G = \mathrm{SL}_n(\mathbb{Z}) \times \langle -I_n \rangle$. If p is an odd prime, then $G/G(p) \simeq \mathrm{SL}_n(p) \times \langle -I_n \rangle$ and the quotient group of $G/G(p)$ by its centre is non-cyclic and simple. Therefore $[I, G] \leq G(p)$ for all odd primes p, and so $[I, G] = 1$. Hence $I = \langle -I_n \rangle$.

Now let n be even. Then $I \leq G(2)$ since $G/G(2)$ is non-cyclic simple. To complete this part of the proof we must show that $G(2)$ is imperfect. Assuming this to be false, we can find by max-sn a non-cyclic simple quotient $G(2)/N$. Now N has finitely many conjugates in G since $G/G(2)$ is finite. Let M be the normal core of N in G. If $M \cap \mathrm{SL}_n(\mathbb{Z}) \leq Z(G)$, then G would have finite composition length, which is certainly not true. Therefore the Congruence Subgroup Theorem [2, 12] shows that M contains some $G(m)$ where $m > 2$. Here we can suppose m to be chosen minimal subject to $G(m) \leq M$. Notice that m must be even because $G(m) \leq G(2)$.

If m is divisible by 4 and $m = 4l$, then $G(2l)/G(m)$ is a finite 2-group and, because $G(2)/N$ is non-cyclic simple, it follows that $G(2l) \leq N$ and hence $G(2l) \leq M$, in contradiction to the choice of m. Thus $m = 2d$ where d is odd. Now $G(m) = G(2) \cap G(d)$, and $G = G(2)G(d)$ since $G/G(2)$ is simple. Also $G/G(d)$ is imperfect by Lemma 6.2. Therefore $G(2)/G(m)$ is imperfect, and hence $G(2)/N$ cannot be perfect.

Case $n = 2$. Here a different strategy must be adopted because the Congruence Subgroup Theorem fails for $\mathrm{SL}_2(\mathbb{Z})$. The conclusions will follow at once if we can prove the following lemma:

Lemma 6.3. $\mathrm{Imp}(\mathrm{PGL}_2(\mathbb{Z})) = 1$.

Proof. Write $G = \mathrm{PGL}_2(\mathbb{Z})$; the first step is to show that G is not imperfect. Recall from [5, p. 86], that G has the presentation

$$G = \langle u, v, w \mid u^2, v^2, w^2, (uv)^3, (uw)^2 \rangle .$$

If $\bar{u} = (12)(34)$, $\bar{v} = (25)(34)$, and $\bar{w} = (13)(24)$, then $1 = \bar{u}^2 = \bar{v}^2 = \bar{w}^2 = (\bar{u}\bar{v})^3 = (\bar{u}\bar{w})^2$, and $\langle \bar{u}, \bar{v}, \bar{w} \rangle = A_5$. Therefore G is not imperfect.

Define groups H and K by

$$H = \langle u, v \mid u^2, v^2, (uv)^3 \rangle \quad \text{and} \quad K = \langle u', w \mid u'^2, w^2, (u'w)^2 \rangle ;$$

thus $H \simeq S_3$ and K is a Klein 4-group. Moreover, G is the generalized free product

$$G = H *_{u=u'} K .$$

Let N be an imperfect normal subgroup of G; we shall argue that $N = 1$. Suppose first that $N \cap H \neq 1$. Then $uv \in N$ and G/N is abelian, and it follows that G is imperfect by Lemma 2.1. Thus $N \cap H = 1$. Next consider $N \cap K$. If $u' = u \in N \cap K$, then $H \leq N$ and again G/N is abelian. If $w \in N \cap K$ or $u'w \in N \cap K$, then G/N is an image of S_3. It follows that $N \cap K = 1$. We can now apply the Subgroup Theorem for generalized free products [8] to conclude that N is a free group. Since N is imperfect, it must be trivial or infinite cyclic. In the second case N^2 is an infinite cyclic normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Now if $p \geq 5$, we have $\mathrm{PSL}_2(p) \simeq \mathrm{PSL}_2(\mathbb{Z})/\bar{G}(p)$ where $\bigcap_p \bar{G}(p) = 1$; also $\mathrm{PSL}_2(p)$ is non-cyclic and simple, so we reach the contradiction $N^2 = 1$. This completes the proof of Lemma 6.3. □

It is considerably easier to locate the imperfect residual of $\mathrm{GL}_n(\mathbb{Z})$.

Theorem 6.4. *Let* $G = \mathrm{GL}_n(\mathbb{Z})$ *and* $S = \mathrm{SL}_n(\mathbb{Z})$.
(a) *If n is even, then* $\mathrm{Imp}^*(G) = 1$.
(b) *If n is odd, then* $\mathrm{Imp}^*(G) = S$.

Proof. If n is even and p is any odd prime, then $G/G(p)$ is imperfect by Lemma 6.2. Therefore $\mathrm{Imp}^*(G) \leq G(p)$, and thus $\mathrm{Imp}^*(G) = 1$.

Now assume that $n > 1$ is odd; thus $G = S \times Z$ where $Z = \langle -I_n \rangle$. Suppose that G/J is a non-trivial imperfect quotient of G. Then $G/S \cap J$ is imperfect, so we can assume that $S \geq J$. Hence $G/JZ \simeq S/J$ is imperfect. However S is perfect since $n > 2$, so $J = S$. We conclude that $\mathrm{Imp}^*(G) = S$. □

A comparison of Theorems 6.1 and 6.4 indicates that for $n > 2$ the group $\mathrm{GL}_n(\mathbb{Z})$ comes closest to being imperfect when n is even. However the case $n = 2$ is seen to be less clear-cut, as one might expect.

Some imperfect linear groups

Having seen that $\mathrm{GL}_n(\mathbb{Z})$ is not imperfect if $n > 1$, we proceed to look for general linear groups over other rings which are imperfect. It is natural to start with fields. In what follows R^* denotes the group of units of a ring R with identity.

Proposition 6.5. *Let F be a field and let $n > 1$. Then the following are equivalent*:
(a) $\mathrm{GL}_n(F)$ *is not imperfect*;

(b) $\mathrm{PGL}_n(F) \simeq \mathrm{PSL}_n(F)$ *and* $(n, |F|) \neq (2, 2)$;
(c) $(n, |F|) \neq (2, 2)$ *and* $F^* = (F^*)^n$.

Proof. Write $G = \mathrm{GL}_n(F)$ and $S = \mathrm{SL}_n(F)$. If $x, y \in F^*$, the diagonal matrix with entries $x, 1, \ldots, 1$ belongs to $(yI_n)S$ if and only if $x = y^n$, as is easily seen by taking determinants. Hence $G = Z(G)S$ holds if and only if $F^* = (F^*)^n$.

(a)$\to$(b) and (c) Let G/N be a non-trivial perfect quotient. Then $G = NS$, $(n, |F|) \neq (2, 2)$ or $(2, 3)$, and $\mathrm{PSL}_n(F)$ is non-cyclic simple. Since $S \not\leq N$, we see that $S \cap N \leq Z(S)$ and thus N centralizes $\mathrm{PSL}_n(F)$. Therefore $N \leq Z(G)$, and, by the first paragraph of the proof, $F^* = (F^*)^n$. Also $\mathrm{PGL}_n(F) = G/Z(G) \simeq S/Z(S) = \mathrm{PSL}_n(F)$.

(b)$\to$(a) The conditions imply that $(n, |F|) \neq (2, 2)$ or $(2, 3)$. Thus $\mathrm{PSL}_n(F)$ is non-cyclic simple, and G is not imperfect.

(c)$\to$(b) Since $F = (F^*)^n$, we have $G = Z(G)S$, and hence $\mathrm{PGL}_n(F) \simeq \mathrm{PSL}_n(F)$. □

For example, if F is an algebraic number field, $F^* \neq (F^*)^n$ for all $n > 1$. Thus $\mathrm{GL}_n(F)$ is always imperfect.

Proposition 6.5 can be applied to give sufficient conditions for $\mathrm{GL}_n(R)$ to be imperfect. In what follows $\mathrm{sr}(R)$ denotes the stable rank of the ring R (see [10, 11.3.4]); for other notation consult [1, Chapter V].

Proposition 6.6. *Let R be a ring and let $n > \max(2, \mathrm{sr}(R))$. If $\mathrm{GL}_n(R)/\mathrm{GL}_n(M)$ is imperfect for every maximal ideal M of R, then $\mathrm{GL}_n(R)$ is imperfect.*

Proof. Let $G = \mathrm{GL}_n(R)$, and assume that G/N is a non-trivial perfect quotient. Then $G = NE_n(R)$ since $G' = E_n(R)$ by [20, Theorem 3.2]. Next, since $n > \mathrm{sr}(R)$, there is an ideal I of R such that

$$E_n(R, I) \leq N \leq G_n(R, I)$$

by [1, p. 240]. (Here $E_n(R, I)$ is the normal subgroup of $E_n(R)$ generated by all I-elementary matrices, and $G_n(R, I)$ is the preimage of the centre of $\mathrm{GL}_n(R/I)$ under $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/I)$.) Now I cannot equal R since $E_n(R) = G'$. Hence there is a maximal ideal M containing I, and $N \leq G_n(R, M)$. Since $G/\mathrm{GL}_n(M)$ is imperfect, $G = N(\mathrm{GL}_n(M))$, and therefore $G = G_n(R, M)$. However this is impossible since $n > 1$. □

It is desirable to have a version of Proposition 6.6 involving $\mathrm{GL}_n(R/M)$. Certainly this is possible if the canonical map $\mathrm{GL}_n(R) \to \mathrm{GL}_n(R/M)$ is surjective whenever M is an ideal of R. In fact this is true if $\mathrm{sr}(R) = 1$.

To prove this known fact, first note that

$$\mathrm{GL}_n(R) = E_n(R)R^*$$

by [1, p. 240]. Now $E_n(R)$ certainly maps onto $E_n(R/M)$; thus it remains to prove that $(R/M)^* = R^* + M/M$. Let $u + M \in (R/M)^*$; then $uv + m = 1$ for some $v \in R$, $m \in M$. So, by definition of $\mathrm{sr}(R) = 1$, there is an $r \in R$ such that $u + mr \in R^*$. Moreover $u + mr$ maps to $u + M$, as required (cf. [1, 2.8, p. 87]).

Combining this observation with Proposition 6.6, we obtain the following theorem:

Theorem 6.7. *Let R be a ring such that $\mathrm{sr}(R) = 1$, and let $n \geq 3$. Then $\mathrm{GL}_n(R)$ is imperfect if and only if $\mathrm{GL}_n(R/M)$ is imperfect for every maximal ideal M of R.* □

When R is commutative, R/M is a field, and so Proposition 6.5 can be applied to determine if the conditions in Theorem 6.7 are satisfied.

Finally we consider the degree-2 case.

Proposition 6.8. *Let R be a commutative semilocal ring which does not have the field of two elements as a quotient ring. Then $\mathrm{GL}_2(R)$ is imperfect if and only if $F^* \neq (F^*)^2$ for every field F that is a quotient ring of R.*

Proof. Write $G = \mathrm{GL}_2(R)$. Assume first that the hypotheses are fulfilled, and suppose that G/N is a perfect quotient. According to a result of Vaserstein [21, Theorem 8], there is an ideal I such that

$$[E_2(R), E_2(I)] \leq N \leq G_2(R, I).$$

Suppose that $I \neq R$, and let M be a maximal ideal of R containing I. Then, because $G = NG'$, we also have $G = G_2(R, M)G'$. Put $F = R/M$, a field. Since R is semilocal, $\mathrm{sr}(R) = 1$ and hence $\mathrm{GL}_2(F) \simeq G/\mathrm{GL}_2(M)$. It follows that $\mathrm{GL}_2(F) = Z(\mathrm{GL}_2(F))\mathrm{SL}_2(F)$, which, by the proof of Proposition 6.5, implies that $F^* = (F^*)^2$, a contradiction.

It follows that $I = R$, so that $E_2(R)' \leq N$. But $G/E_2(R')$ is metabelian since $G/E_2(R) \simeq K_1(R)$ is abelian. Therefore $N = G$ and G is imperfect.

Conversely, assume that G is imperfect. If a field F is a quotient ring of R, then $\mathrm{GL}_2(F)$ is imperfect since it is a quotient of $\mathrm{GL}_2(R)$. Then Proposition 6.5 shows that $F^* \neq (F^*)^2$. □

On combining Theorem 6.7 and Proposition 6.8 with Proposition 6.5, we get the final result.

Theorem 6.9. *Let R be a commutative semilocal ring which does not have the field of two elements as a quotient ring, and let $n > 1$. Then $\mathrm{GL}_n(R)$ is imperfect if and only if $F^* \neq (F^*)^n$ for every field F which is a quotient ring of R.* □

For example, let π be a cofinite set of primes containing 2, and take R to be the localization of $\mathbb{Z}$ at the primes in π'. Then R satisfies the hypotheses of Theorem 6.9. Hence $GL_n(R)$ is imperfect if and only if $\gcd(p-1, n) > 1$ for all $p \notin \pi$.

References

[1] H. Bass, Algebraic K-Theory (Benjamin, New York, 1968).
[2] H. Bass, J. Milnor and J.-P. Serre, Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and $Sp_{2n}(n \geq 2)$, Publ. I.H.E.S. 33 (1967).
[3] A.J. Berrick, Homology of torsion generated groups, in preparation.
[4] A.J. Berrick and P. Menal, The abelianization of $SL_2(\mathbb{R})$, in preparation.
[5] H.S.M. Coxeter and W.O.J. Moser, Generators and Relations for Discrete Groups (Springer, Berlin, 4th ed., 1980).
[6] D. Holt and W. Pleskins, Perfect Groups (Oxford, 1989).
[7] B. Hupper, Endliche Gruppen, I (Springer, Berlin, 1967).
[8] A. Karass and D. Solitar, The subgroups of a free product of two groups with an amalgamated subgroup, Trans. Amer. Math. Soc. 150 (1970) 227–255.
[9] R.C. Lyndon and P.E. Schupp, Combinatorial Group Theory (Springer, Berlin, 1977).
[10] J.C. McConnell and J.C. Robson, Non-commutative Noetherian Rings (Wiley/Interscience, New York, 1987).
[11] D.H. McLain, A characteristically-simple group, Proc. Cambridge Philos. Soc. 50 (1954) 641–642.
[12] J. Mennicke, Finite factor groups of the unimodular group, Ann. of Math. (2) 81 (1965) 31–37.
[13] M. Nakaoka, Homology of the infinite symmetric group, Ann. of Math. (2) 73 (1961) 229–257.
[14] D.J.S. Robinson, Groups in which normality is a transitive relation, Proc. Cambridge Philos. Soc. 60 (1964) 21–38.
[15] D.J.S. Robinson, On the theory of subnormal subgroups, Math. Z. 89 (1965) 30–51.
[16] D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups (Springer, Berlin, 1972).
[17] J.E. Roseblade, On certain subnormal coalition classes, J. Algebra 1 (1964) 132–138.
[18] U. Stammbach, Homology in Group Theory, Lecture Notes in Mathematics, Vol. 359 (Springer, Berlin, 1973).
[19] M. Suzuki, Group Theory (Springer, Berlin, 1982).
[20] L.N. Vaserstein, On the stabilization of the general linear group over a ring, Mat. USSR-Sb. 8 (1969) 383–400.
[21] L.N. Vaserstein, On normal subgroups of GL_2 over rings with many units, Compositio Math. 74 (1990) 157–164.
[22] J.S. Wilson, The normal and subnormal structure of general linear groups, Proc. Cambridge Philos. Soc. 71 (1972) 163–177.
[23] J. Zimmerman, Complete groups with nonabelian composition factors, Trans. Amer. Math. Soc. 302 (1987) 151–159.