ALGEBRAIC K-THEORY — WORKSHOP NOTES

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These notes are essentially a fully referenced version of those handed out to the thirty or so participants in the Workshop on Algebraic K-Theory held at the National University of Singapore in June 1985 in the week preceding the Singapore Topology Conference. References [3] and [9] were suggested for more detailed background material. Notes from the companion workshop, on Homology of Groups (P.J. Hilton), appear as [7].

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0. Introduction

Our starting-point is the following pair of theorems from a typical undergraduate course in the linear algebra of finite-dimensional vector spaces.

Theorem 0. For each vector space over a given field there is a positive integer (called dimension), corresponding to the size of a minimal generating set (called a basis), which determines the vector space uniquely up to isomorphism.

Theorem 1. Every invertible matrix over a field may be reduced to the identity matrix by a sequence of elementary row operations.

For a more general ring R (associative, with 1), "classical" K-theory (the study of groups $K_0(R)$, $K_1(R)$ [2]) measures how well these theorems hold also for R.

Later, the group $K_2(R)$ was invented [12]. It is a measurement of the uniqueness of the sequence of row operations required to reduce a given matrix over R.

These groups were given the letter "K" for two reasons:

(i) they have properties formally similar to those of the groups $K^i(X)$ already existing for any nice topological space X and $i \in \mathbb{Z}$;

(ii) in the case where $i = 0, 1, 2$ and R = ring of continuous functions from X to $\mathbb{R}$ or $\mathbb{C}$, $K_i(R)$ is closely related to (and gives much information about) $K^{-i}(X)$.

This suggested that for a ring R there should be groups $K_i(R)$ for **all** integers i, enjoying properties suggested by (i) above. When these were defined, it turned out that they weren't very interesting, e.g. were often zero. However the question of defining "higher" K-groups $K_i(R)$ $i > 2$ remained open and very interesting. It was solved by Quillen (1970) by means of the following sequence of constructions.

R	$\to$	GLR	general linear group
	$\to$	BGLR	classifying space
	$\to$	$BGLR^+$	plus-construction
	$\to$	$\pi_i(BGLR^+)$	homotopy group
	$=$	$K_i(R)$	

Three of these constructions were already well-known; on the other hand the plus-construction was new and of independent interest.

1. K_1

1.1 Elementary row and column operations

Recall from undergraduate linear algebra the reduction of an arbitrary invertible matrix over a field F to the identity matrix by means of "elementary row operations".

These are of three types:—

(i) interchanging two rows;

(ii) multiplying a row by a non-zero scalar;

(iii) adding a multiple of one row to another.

Each of these operations may evidently be achieved by pre-multiplication of the given matrix (or postmultiplication if one chooses to perform the like operations on columns) by one of the three following corresponding types. (Since the operation is the identity on all but at most two rows, the relevant matrix may conveniently be given a 2×2 format.)

(i) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$, $d\neq 0$

(iii) $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$

So Theorem 1 can also be written as follows.

1.1.1. Proposition. Any invertible matrix over a field is equal to a product of matrices of the above three types. ◇

This is usually about as far as the analysis proceeds at undergraduate level. However it is quite easy to make the following observations which are **valid over any ring R and not just a field**. First, note the redundancy of type (i) in view of

(a) $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is a product of a type (ii) matrix with type (iii) matrices because, for any invertible d,

(b) $$\begin{pmatrix} 0 & d \\ -d^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

In fact, among type (ii) matrices it is only necessary to look at those which differ from the identity matrix in respect of the first diagonal entry, since for a unit d

(c) $$\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix},$$

while

(d) $$\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & d^{-1} \\ -d & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is by (b) above expressible as a product of type (iii) matrices. Finally, it is not difficult to see that at most one type (ii) matrix (in our restricted form) is required in the product. For, such matrices evidently multiply to form a group isomorphic to the group of units of R. Moreover they commute with those type (iii) matrices whose corresponding row operations do not involve the first row.

The remaining case to consider is that of

(e) $$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d^{-1}a \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix}1&0\\a&1\end{pmatrix}\begin{pmatrix}d&0\\0&1\end{pmatrix}=\begin{pmatrix}d&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\ad&1\end{pmatrix}$$

It follows that by means of type (iii) operations alone any product of matrices of the three given types may be reduced to a diagonal form in which all entries after the first are 1. When the ring is **commutative** so that the determinant homomorphism is defined, the first entry here is simply the determinant of the original matrix since type (iii) operations clearly do not change determinants. As our discussion has focussed upon matrices of type (iii), we shall in future refer to these—and these alone—as *elementary matrices*. So we have the following.

1.1.2. Proposition. Over a field any matrix of determinant 1 is equal to a product of elementary matrices. ◇

This naturally suggests that for an arbitrary ring R one look at the extent to which the *nth general linear group* GL_nR of invertible $n \times n$ matrices over R may be approximated by its subgroup E_nR of products of elementary matrices ($n \geq 2$). This forms the subject matter of K_1-theory.

1.2 The Whitehead Lemma

Now we show that the problem of an absence of group structure on the set of cosets GL_nR/E_nR may be got around by allowing n to increase to infinity.

To begin with, consider, for $n \geq 1$, the ring M_nR of $n \times n$-matrices with entries in R. Here the ring structure is given by the usual matrix addition and multiplication. The inclusion $M_nR \hookrightarrow M_{n+k}R$ corresponds to matrix direct sum with the $k \times k$-identity matrix I_k; thus

$$\alpha \longmapsto \alpha \oplus I_k = \begin{pmatrix}\alpha & 0\\ 0 & I_k\end{pmatrix}.$$

The group of units $(M_nR)^*$ of M_nR is just GL_nR. From our point of view its most interesting elements ($n \geq 2$) are the *elementary matrices*, those with (at most) one non-zero off-diagonal entry and 1s right down the diagonal. Thus the elementary matrix e_{ij}^a associated to the row operation which adds a times the jth row to the ith has $i \neq j$ and $a \in R$ and $e_{ij}^a - I_n = ae_{ij}$ is the zero matrix apart from a in the (i,j)-slot. E_nR is the subgroup of GL_nR they generate:

$$E_nR = gp\{e_{ij}^a \mid a \in R, i \neq j \in \{1, \ldots, n\}\}.$$

On occasion it is also convenient to adopt the convention that E_1R is the trivial group. Certain relations in E_nR follow easily from the fact (written with Kronecker delta) that

$$e_{hi}e_{jk} = \delta_{ij}e_{hk},$$

namely

(1.2.1) $$e_{ij}^a e_{ij}^b = e_{ij}^{a+b},$$

and, if $h \neq k$ (so that $n \geq 3$)

(1.2.2) $$[e_{hi}^a, e_{jk}^b] = \begin{cases} 1 & i \neq j \\ e_{hk}^{ab} & i = j. \end{cases}$$

Although this formula does not depend on which way round we define our commutators, for future reference we shall be using the notation.

$$[x, y] = xyx^{-1}y^{-1}.$$

A clear effect of the last of these relations is that when $n \geq 3$ E_nR is generated by commutators and so equal to its commutator subgroup $[E_nR, E_nR]$ (in other words, *perfect*).

1.2.3. Lemma. For $n \geq 3$, E_nR is a perfect group, that is

$$E_nR = [E_nR, E_nR]. \quad \diamond$$

To study the relationship between E_nR and GL_nR we note that if our ring is already a ring of matrices M_kR, then we may equate $M_n(M_kR)$ with $M_{nk}R$. Here we do not claim compatibility with inclusions $M_nR \hookrightarrow M_{n+1}R$, etc.

1.2.4. Lemma. $E_2(M_nR) \leq E_{2n}R$.

Proof. Exercise. $\diamond$

The inclusion $GL_nR \hookrightarrow GL_{n+k}R$, $\alpha \mapsto \alpha \oplus I_k$, is implicit in the statements of the next lemmas.

1.2.5. Lemma. If $\alpha \in GL_nR$, then in $GL_{2n}R$

(a) α and $I_n \oplus \alpha = \begin{pmatrix} I_n & 0 \\ 0 & \alpha \end{pmatrix}$ represent the same coset modulo $E_{2n}R$, and

(b) $\alpha \oplus \alpha^{-1} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}R$.

Proof. After (1.2.4), we may restrict our attention to $GL_2(M_nR)$. Then the results follow from equations (c) and (d) of (1.1). $\diamond$

1.2.6. Lemma. If $\beta_n \in GL_nR$, then in $GL_{2(m+n)}R$ β_n and $I_m \oplus \beta_n$ represent the same coset modulo $E_{2(m+n)}R$.

Proof. An immediate application of (1.2.5) (a) gives $(\beta_n \oplus I_m) \oplus (I_m \oplus I_n)$ congruent to $I_m \oplus I_n \oplus \beta_n \oplus I_m$ modulo $E_{2(m+n)}R$. A second application of (1.2.5) gives $I_n \oplus \beta_n = (\beta_n \oplus I_n)\,\alpha_{2n}$ where $\alpha_{2n} = \Pi e_{i_\lambda j_\lambda}^{a_\lambda} \in E_{2n}R$. Therefore $I_m \oplus I_n \oplus \beta_n = (I_m \oplus \beta_n \oplus I_n)(I_m \oplus \alpha_{2n})$ with $I_m \oplus \alpha_{2n}^{\lambda} = \prod_\lambda e_{m+i_\lambda,\, m+j_\lambda}^{a_\lambda} \in E_{m+2n}R$. $\diamond$

Now suppose $\alpha_1, \alpha_2 \in GL_nR$. Then

$$\begin{aligned}[\alpha_1 \oplus I_n, \alpha_2 \oplus I_n] &= [\alpha_1, \alpha_2] \oplus I_n \\ &= (\alpha_1 \oplus \alpha_1^{-1})(\alpha_2 \oplus \alpha_2^{-1})(\alpha_1^{-1}\alpha_2^{-1} \oplus \alpha_2\alpha_1),\end{aligned}$$

an element of $E_{2n}R$ by (1.2.5). Thus $[GL_nR, GL_nR] \leq E_{2n}R$ and so, using (1.2.3), we have

$$[E_nR, E_nR] \leq [GL_nR, GL_nR] \leq E_{2n}R = [E_{2n}R, E_{2n}R].$$

We now pass to the limit, by the following definitions.

$$MR = \text{dirlim } M_nR = \bigcup_{n \geq 1} M_nR$$

$$GLR = \bigcup_{n \geq 1} GL_nR = (MR)^*$$

$$ER = \bigcup_{n \geq 2} E_nR \leq GLR.$$

1.2.7. Lemma (Whitehead).

$$[ER, ER] = [GLR, GLR] = ER. \quad \diamond$$

This has, as an immediate consequence, the result we have been seeking.

1.2.8. Corollary. ER is a normal subgroup of GLR and GLR/ER = $(GLR)_{ab}$. ◇

1.2.9. Corollary. Two matrices in GL_nR are congruent modulo ER if and only if one may be obtained from the other by elementary row and column operations of type (iii). ◇

Note that this assertion does not claim congruence modulo E_nR, a point taken up in (1.4) below.

One can provide a more fruitful description of ER. To do this, recalling that a *perfect* group is one equal to its commutator subgroup, observe (as in (3.3) below) that each group possesses a unique maximal perfect subgroup which, being equal to its own commutator subgroup, must be contained in the commutator subgroup of the full group.

1.2.10. Corollary. ER is the maximal perfect subgroup of GLR. ◇

1.3 K_1: *definition and examples*

We now define $K_1(R)$ to be the abelian group

$$K_1(R) = GLR/ER = (GLR)_{ab}.$$

Before turning to some of the easier examples, we comment on the group structure in $K_1(R)$. While the multiplication inherited by any quotient of GLR is of course given by matrix multiplication (of coset representatives), in this instance (1.2.6) offers another description. For, given $\alpha_m \in GL_mR$, $\beta_n \in GL_nR$, the multiplication in $K_1(R)$

$$(\alpha_m, \beta_n) \longmapsto (\alpha_m \oplus I_n). (\beta_n \oplus I_m)$$

coincides with

$$(\alpha_m, \beta_n) \longmapsto (\alpha_m \oplus I_n). (L_m \oplus \beta_n) = \alpha_m \oplus \beta_n.$$

Exercise. Any quotient of GLR where $\alpha\beta = \alpha \oplus \beta$ is a quotient of K_1R.

We note two simple results of value in calculations; the former follows immediately from invertibility.

1.3.1. Lemma. For a matrix (a_{ij}) in GL_nR, the entries in any column generate R, that is,

$$Ra_{1j} + \ldots + Ra_{nj} = R. \quad \diamond$$

1.3.2. Lemma. If a proper subset of the entries of any column of a matrix in GL_nR generates R, then that matrix is congruent modulo ER to a matrix in $GL_{n-1}R$.

Proof. Exercise. ◇

The easiest examples for the purposes of calculation occur when R is commutative. For the determinant homomorphism $GL_nR \to R^*$ is invariant under the inclusion $GL_nR \hookrightarrow GL_{n+1}R$. It may therefore be regarded as being defined on GLR, where its kernel, the *special linear group* SLR, certainly includes all elementary matrices and thus the subgroup ER that they generate. So it induces on the quotient group $K_1(R)$ a homomorphism *det*: $K_1(R) \to R^*$. Now *det* has a right inverse given by $R^* = GL_1R \hookrightarrow GLR \to K_1(R)$. (In other words, if $a \in R^*$, then the matrix $\begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}$ has determinant a.) If we define $SK_1(R) = SLR/ER$ to be the kernel of *det* we then have a split exact sequence of abelian groups

$$SK_1(R) \rightarrowtail K_1(R) \overset{det}{\twoheadrightarrow} R^*.$$

1.3.3. Proposition. For R commutative, the determinant homomorphism induces

$$K_1(R) = SK_1(R) \oplus R^*. \quad \diamond$$

We may now restate (1.1.2) in K-theoretic terms.

1.3.4. Corollary. If R is a field, then $SK_1(R) = 0$ and *det*: $K_1(R) \to R^*$ is an isomorphism. ◇

The next examples are exercises using (1.3.2).

1.3.5. Proposition. If R is a Euclidean ring, then $SK_1(R) = 0$ and $K_1(R) \cong R^*$.

1.3.6. Proposition. If R is a principal ideal domain, then the homomorphism

$$SL_2R \hookrightarrow SLR \to SK_1(R)$$

is a surjection.

Note that a ring homomorphism $f: R \to S$ induces group homomorphisms $GLf: GLR \to GLS$ and $Ef: ER \to ES$. So the following is easy.

1.3.7. Proposition. $f: R \to S$ induces $K_1(f): K_1(R) \to K_1(S)$ such that

(i) $K_1(id) = id$, and

(ii) $K_1(f \circ g) = K_1(f) \circ K_1(g)$.

A fancier way of saying this is: K_1 is a functor from the category of rings and ring homomorphisms to the category of abelian groups and group homomorphisms.

1.4 Stability

We briefly discuss the "finite" approximation GL_nR/E_nR to $K_1(R)$. Here one is primarily interested in whether a complete approximation exists, in the sense that for some n $GL_nR/E_nR \cong K_1(R)$; more precisely, one seeks n satisfying the following condition.

1.4.1. Condition. The maps

$GL_nR/E_nR \to GL_{n+1}R/E_{n+1}R \to GL_{n+2}R/E_{n+2}R \to \dots \to K_1(R)$, induced from inclusions $GL_nR \to GL_{n+1}R$ and $E_nR \to E_{n+1}R$, are isomorphisms.

In this event the sequence

$$GL_1R \to GL_2R/E_2R \to GL_3R/E_3R \to \dots$$

is said to *stabilize* after n terms. Of course the condition presupposes a group structure on each of these sets of cosets, requiring that for all $m \geq n$ the following holds.

1.4.2. Condition. E_mR is a normal subgroup of GL_mR.

This condition arises quite naturally from 1.2.9. For, from the equality of left and right cosets of E_mR, it implies that if $\alpha \in GL_mR$ is obtainable from β by elementary row-operations then it may also be obtained from β by column-operations, and vice versa. See also 1.3.2, where knowledge that $E_nR \trianglelefteq GL_nR$ would lead to the strengthened conclusion of congruence modulo E_nR.

One motivation for the study of stability comes from the computationally important question of whether the groups involved are finitely generated. For example, let us check

1.4.3. Proposition. Suppose that 1.4.1 holds for some $n \geq 3$ and for R which is finitely generated as a ring. Then GL_nR is a finitely generated group if and only if $K_1(R)$ is too.

Proof. Since a homomorphic image of a finitely generated group is also finitely generated, necessity is clear. For sufficiency, one recalls that a group is finitely generated if it has finitely generated quotient by a finitely generated normal subgroup (a fact which follows readily from the definitions). The remaining point, that for $n \geq 3$ E_nR is finitely generated, is left as an exercise. ◇

Although in general further conditions on R are required in order to guarantee (1.4.2), here is a partial result valid for any ring.

1.4.4. Lemma. For $n \geq 3$, E_nR is normalized by $GL_{n-1}R$ in GL_nR.

Proof. Exercise. ◇

There are two common ways to have E_nR normalized by all of GL_nR.

1.4.5. Proposition [5]. Suppose that R is a commutative ring and that $n \geq 3$. Then E_nR is a normal subgroup of GL_nR.

Alternatively, write $\mathrm{sr}R \leq n$, terming srR the *stable rank* of R, if for any *unimodular* $(a_0, a_1, \dots, a_n)$ of R^{n+1} (that is, $Ra_0 + \dots + Ra_n = R$) there exists $(b_0, b_1, \dots, b_{n-1})$ in R^n such that $(a_0 + b_0a_n, \dots, a_{n-1} + b_{n-1}a_n)$ in R^n is also unimodular. So (1.3.1) states that any invertible matrix has unimodular columns. Refinement of the proof of (1.3.2) yields

1.4.6. Proposition. If $\mathrm{sr}R \leq n-1$, then $GL_nR = E_nR.GL_{n-1}R$. ◇

There is now an immediate corollary from 1.4.4.

1.4.7. Corollary. If $\mathrm{sr}R \leq n-1$ where $n \geq 3$, then E_nR is a normal subgroup of GL_nR. ◇

Thus the concept of stable rank is well-suited to discussion of the normality condition (1.4.2). We next apply it to (1.4.1). The fact (1.4.6) that for $\mathrm{sr}R \leq n$ $GL_{n+1}R$ can be replaced by $E_{n+1}R.GL_nR$ implies the surjectivity of $GL_nR \to GL_{n+1}R/E_{n+1}R$ (right cosets) and hence of $GL_nR/E_nR \to GL_{n+1}R/E_{n+1}R$

through which it factors. This establishes the easier part of the key stability theorem:

1.4.8. Theorem [2], [20]. The canonical function

$$GL_nR/E_nR \to GL_{n+1}R/E_{n+1}R$$

is

(a) surjective when $n \geq srR$, and
(b) bijective when $n > srR$.

Of course from (1.4.7) this function is a homomorphism if also $n \geq 3$. Note too that if $n \geq \max(srR, 2)$ then

$$\begin{aligned} GL_{n+1}R/E_{n+1}R &= E_{n+1}R.GL_nR/E_{n+1}R \\ &\cong GL_nR/(E_{n+1}R \cap GL_nR), \end{aligned}$$

making bijectivity equivalent to the condition

$$E_nR = E_{n+1}R \cap GL_nR.$$

We do not prove bijectivity here. ◇

1.4.9. Theorem [18]. If R is a (left) Noetherian ring, then

$$srR \leq \text{Krull dim } R + 1.$$

Thus the concept of stable rank is widely applicable.

1.5. Relation with topological K^{-1}

For k the field ℝ or ℂ of real or complex numbers, the set k(X) of continuous k-valued functions on a topological space X acquires a commutative ring structure from that of k, namely addition and multiplication of values. Provided X is reasonably well-behaved, it is possible to recapture the **topology** of X from the **ring** k(X). We therefore assume X to be a compact CW-complex, that is, obtained from a finite set of points by attaching in turn a finite number of cells. Specifically, the *maximal spectrum* Max(A) of a commutative ring A comprises the set of maximal ideals of A endowed with the Zariski topology: closed sets are the collections of all maximal ideals which contain a given ideal of A. These constructions are classically related as follows.

1.5.1 Proposition. Mak (k(X)) is homeomorphic to X.

In order to appreciate the topological counterpart to Theorem 1.4.8, we need to have a description of K_k^{-1} of X [1]. Although the standard definition is as the Grothendieck group (discussed in §2) of k-vector bundles over the suspension space SX, more helpful here is the isomorphism

$$K_k^{-1}(X) \cong \varinjlim\, [X, GL_nk]$$

Here the set $[X, GL_nk]$ of homotopy classes of maps from X to GL_nk takes on the group structure of GL_nk (i.e. matrix multiplication).

1.5.2. Theorem [1]. If X has dimension d, then

$$[X, GL_nk] \to [X, GL_{n+1}k]$$

is surjective whenever $n \geq d+1$ and bijective whenever $n \geq d+2$.

(Here, the dimension of X refers to the highest dimension of any of the (finitely many) cells which make up X.) Now the determinant homomorphism GLk → k* gives rise to a homomorphism

$$[X, GL_n k] \to [X, k^*]$$

whose kernel $[X, SL_n k]$ becomes in the limit a group $SK_k^{-1}(X)$.

1.5.3. Theorem [2]. $SK_1(k(X)) \cong SK_k^{-1}(X)$.

1.5.4. Corollary. If X has dimension d, then

$$SL_n k(X)/E_n k(X) \to SK_1(k(X))$$

is surjective whenever $n \geq d+1$ and bijective whenever $n \geq d+2$. ◇

Note that in general srk(X) is undefined, so (1.5.4) is not a corollary of (1.4.8) but has to be proved topologically!

2. K_0 and K_2

Although the study of K_0 has great historical and practical significance, it is less important in this treatment. So we deal with it very briefly.

2.1 Vector spaces and modules

We study the effect on Theorem 0 when "field" is replaced by "ring". Instead of vector spaces over a field we look at left (unital, that is, with 1.m = m always) *modules* M over a ring R. The generalisation of a finite-dimensional vector space is a *finitely generated* (f.g.) module M, meaning that there exist $m_1, \ldots, m_n \in M$ so that arbitrary $m \in M$ can be written as

$$m = r_1 m_1 + \ldots + r_n m_n$$

for some $r_1, \ldots, r_n \in R$. (For a general ring we do not claim that this choice of $r_1, \ldots, r_n$ is unique.) An example is R itself, written R^1. The direct sum construction ⊕ generalises to modules; it satisfies $(M_1 \oplus M_2) \oplus M_3 \cong M_1 \oplus (M_2 \oplus M_3)$ and $M_1 \oplus M_2 \cong M_2 \oplus M_1$. We define the *free* module R^k by the formula $R^k = R^1 \oplus R^{k-1}$ $(k \geq 2)$.

Note that over a field any f.g. module is free (called a vector space) and any submodule of a free module is also free. In general this is not what happens, so one has to study separately those modules which are submodules of free modules — the f.g. *projective* modules. (For example, over $R = \mathbb{Z}/6$ the modules $\mathbb{Z}/2$, $\mathbb{Z}/3$ are not free, but are projective since $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$.)

2.1.1. Proposition (e.g. [7]*).* A f.g. module P is projective if and only if for some module Q and free module R^k

$$P \oplus Q \cong R^k.$$

Note that in general $P \oplus R^n \cong R^{n+k}$ does **not** imply that $P \cong R^k$ (e.g. R the ring $End_F(V)$ of endomorphisms of an infinite-dimensional vector space V over a field F).

2.2 Grothendieck groups

It is already clear that to generalise Theorem 0 we must first reformulate it. Given a field F, we consider the abelian group *Groth* generated by isomorphism classes [V] of vector spaces V with $[V_1] + [V_2] = [V_1 \oplus V_2]$ by definition. So all elements of *Groth* have the form [V] — [W] for some vector spaces V, W. Then Theorem 0 says the function

$$Groth \to \mathbb{Z}, \quad [F^n] - [F^m] \longmapsto n - m$$

is actually a group isomorphism.

This is an example of a *Grothendieck group*. More generally, given a collection (actually, a category) of objects with some operation ⊕ defined (satisfying obvious properties like associativity, commutativity), then its Grothendieck group is again the abelian group generated by isomorphism classes [A] subject to $[A \oplus B] = [A] + [B]$. (A still more general version works for any "category with exact sequences".)

2.3 K_0 : definition and examples

Now define $K_0(R)$ (sometimes known as the *projective class group*) to be the Grothendieck group of the category of f.g. projective R-modules.

So Theorem 0 reads

2.3.1. Proposition. $K_0(\text{field}) \cong \mathbb{Z}$.

In fact over any principal ideal domain R any f.g. projective module is free, so that $K_0(R) \cong \mathbb{Z}$ generated by $[R^1]$. However, because as rings $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$, so as groups

$$K_0(\mathbb{Z}/6) \cong K_0(\mathbb{Z}/2) \oplus K_0(\mathbb{Z}/3) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Also,

$$K_0(\text{End}_F(V)) = 0$$

where V is an infinite-dimensional vector space over a field F and $\text{End}_F(V)$ its ring of endomorphisms. The example which gives K_0 its notation is the ring k(X) of k-valued continuous functions ($k = \mathbb{R}, \mathbb{C}$) on a compact CW-complex X. For, after Swan [19],

$$K_0(k(X)) \cong K_k^0(X),$$

where $K_k^0(X)$ (defined by topologists around 1960) is the Grothendieck group for the category of k-vector bundles over X (a k-vector bundle being, loosely, a k-vector space varying nicely over the points of X)[1].

"Functoriality" works too as in (1.3.7). Here a ring homomorphism f: R → S makes S a right R-module. If P is a f.g. projective (or free) left module over R, then so too is $S \otimes_R P$ over S. (Because $S \otimes_R (P \oplus Q) \cong (S \otimes_R P) \oplus (S \otimes_R Q)$) this gives

2.3.2. Proposition. f: R → S induces $K_0(f): K_0(R) \to K_0(S)$ such that

(i) $K_0(\text{id}) = \text{id}$, and
(ii) $K_0(f \circ g) = K_0(f) \circ K_0(g)$.

2.4. Definition of K_2

Recall that the relations (1.2.1), (1.2.2) always hold in the groups $E_n(R)$ ($n \geq 3$). For given R, do any others also hold?

To study this, we define the n-th *Steinberg group* St_nR which is designed to map onto E_nR. Thus it has generators x_{ij}^a, one corresponding to each e_{ij}^a, subject to the relations corresponding to (1.2.1), (1.2.2). Since the relations between x_{ij}'s are also valid among e_{ij}'s, the function $x_{ij}^a \longmapsto e_{ij}^a$ defines a homomorphism ϕ: $St_nR \to GL_nR$ whose image is E_nR, and which is compatible with $GL_nR \hookrightarrow GL_{n+p}R$ and the natural homomorphism ι: $St_nR \to St_{n+p}R$ obtained by extending the domain of subscripts. In other words, if we pass to the limit of each directed set of groups and write $StR = \varinjlim_n St_nR$, then ϕ is a homomorphism from StR to GLR with image ER. We define $K_2(R)$ to be its kernel, which we now study. Here Z(G) denotes the centre of a group G.

2.4.1. Lemma. The homomorphism ϕ: $St_nR \to GL_nR$ has as kernel

$$Y_n = St_nR \cap \iota^{-1}Z(St_mR) \ (2 \leq n < m).$$

Proof. A matrix (a_{ij}) commutes with e_{rs}^1 if and only if $a_{sr} = 0$ and $a_{rr} = a_{ss}$. Because the embedding $E_nR \hookrightarrow E_mR$ adjoins 1's down the extension of the diagonal, it follows that

$$E_nR \cap Z(E_mR) = 1,$$

whence

$$St_nR \cap \iota^{-1}Z(St_mR) \leq Y_n.$$

To complete the proof it suffices to apply the next lemma to show that $\iota\, Y_n \leq Z(St_mR)$. (Here we take the opportunity of rectifying the corresponding argument in [3].)

2.4.2. Lemma. Suppose a group G is the group union of subgroups G_1 and G_2. If a normal subgroup Y of G has $Y \cap G_i = 1$ ($i = 1, 2$), then $\overline{Y} = Y \cap N_G(G_1) \cap N_G(G_2)$ is central in G.

Proof. For $i = 1, 2$, certainly $[\overline{Y}, G_i] \leq [Y, G_i]$ which must be contained in the normal subgroup ; on the other hand, because $\overline{Y}$ normalises G_i we also have $[\overline{Y}, G_i] \leq G_i$. So $[\overline{Y}, G_i]$ is contained in both Y and G_i, hence trivial. Thus elements of $\overline{Y}$ commute with all elements G_1, G_2 and thereby with all products of such elements, which is to say, with all elements of G. ◇

We shall choose for our G_1, G_2 the "last row" and "last column" subgroups of St_mR, defined respectively as the subgroups

$$R_m = gp\{x_{mj}^a \mid a \in R, 1 \leq j < m\}, \quad C_m = gp\{x_{im}^a \mid a \in R, 1 \leq i < m\},$$

and for our example of Y we take Y_m, normal in $G = St_m(R)$. Evidently, from the final class of relations, $St_m(R) = gp(R_m, C_m)$. Now any two elements x_{mj}^a, x_{mk}^b commute, so that R_m is abelian, enabling a typical element to be shuffled into the standard form $x_{m1}^{a_1}\, x_{m2}^{a_2} \ldots x_{m,m-1}^{a_{m-1}}$, which ϕ maps to

$$e_{m1}^{a_1} \cdots e_{m,m-1}^{a_{m-1}} = \left(\begin{array}{c|c} I_{m-1} & 0 \\ \hline a_1 \cdots a_{m-1} & 1 \end{array}\right).$$

Such a matrix can only be the identity matrix if $a_1 = \ldots = a_{m-1} = 0$. Thus Y_m does indeed avoid R_m (and similarly C_m). So (2.4.2) reveals that the intersection of Y_m with any subgroup of $St_m(R)$ normalizing both R_m and C_m is central. In particular, $\iota Y_m = Y_m \cap \iota\, St_n(R)$ can be seen to be central once it is shown that $\iota St_n(R)$ is just such a subgroup. Here is the verification.

$$\begin{aligned} & x_{ij}^{-a}\, x_{mk}^{b}\, x_{ij}^{a} \\ &= x_{mk}^{b}\, [x_{mk}^{-b}, x_{ij}^{-a}] \\ &= \begin{cases} x_{mk}^{b} & k \neq i, \\ x_{mk}^{b}\, x_{mj}^{ba} & k = i, \end{cases} \end{aligned}$$

which in either event is in R_m. Similarly for C_m. ◇

In (2.4.1) we let first m, then n, tend to infinity.

2.4.3. Theorem. $K_2(R) = \mathrm{Ker}\,[\phi\colon StR \to ER] = Z(StR)$. ◇

In terms of elementary row operations on matrices over R, $K_2(R)$ gives information about the question: "Which sequences of row operation always yield the same result as each other?" For, two sequences which always have the same effect give two products of generators in StR whose images in ER are equal. An example when $R = \mathbb{Z}$ is that

$$\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in E_2(\mathbb{Z}),$$

giving a product $(x_{21}^{-1}\, x_{12}^{2})^4 \in St_2(\mathbb{Z})$ with trivial image in $E(\mathbb{Z})$. In fact this element becomes the only non-trivial element of $K_2(\mathbb{Z})$, the cyclic group of order 2. Similarly over $R = \mathbb{Z}/n$ ($n \equiv$ (mod 4)), while $K_2(\mathbb{Z}/n) = 0$ if $n \not\equiv 0$ (mod 4).

Functoriality here follows, for any ring homomorphism $f\colon R \to S$, from the commutative diagram

$$\begin{array}{ccc} St(R) & \xrightarrow{St(f)} & St(S) \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ E(R) & \xrightarrow{E(f)} & E(S) \end{array}$$

(that is, $E(f) \circ \phi = \phi \circ St(f)$).

2.4.4. Proposition. $f\colon R \to S$ induces $K_2(f)\colon K_2(R) \to K_2(S)$ such that

(i) $K_2(id) = id$, and
(ii) $K_2(f \circ g) = K_2(f) \circ K_2(g)$.

An amusing exercise here is to consider $K_2(f)$ when f is the surjection $\mathbb{Z} \to \mathbb{Z}/2$.

2.5 Properties

The previously-studied topological K-groups [1] have been a great source of inspiration for their algebraic namesakes. For example, there is a simple device for defining negative(ly indexed) K-groups in topology. Recall that the *suspension* SX of a space X with basepoint x_0 is the *smash product* space

$$S^1 \wedge X = ([0, 1] \times X)/\{(1, x) \sim (t, x_0) \sim (0, x), \text{ any } t \in [0, 1], x \in X\}.$$

Then one takes $K_k^{-n}(X)$ to be $\tilde{K}_k^0(S^nX)$. There is indeed an analogous algebraic process: embed the pseudo-ring (= ring without 1) mR of all finite matrices in the ring CR of all *locally finite* matrices over R (that is, those with only finitely many non-zero entries in each row or column). In this ring mR forms a two-sided ideal, and the *suspension* SR is then defined to be the quotient ring (just as topological suspension is formed from the cone

$$CX = ([0, 1] \times X)/\{(1, x) \sim (t, x_0), \text{ any } t \in [0, 1], x \in X\}$$

by identifying an embedded copy of the original space $\{0\} \times X$ with the apex $(1, x_0)$ of the cone). Again we set, for $n > 0$, $K_{-n}(R) = K_0(S^nR)$ [2]. (For some years these negative algebraic K-groups were thought uninteresting; recently, topological applications have been discovered.)

Note that in §1.5 we compared $K_1(R)$ with $K_k^{-1}(X)$ but now are looking instead for $K_k^1(X)$. At least in the complex case the difficulty is easily cleared up. When $k = \mathbb{C}$ the *periodicity theorem* reveals that $\tilde{K}_{\mathbb{C}}^0(S^2X) \cong K_{\mathbb{C}}^0(X)$. This allows $K_{\mathbb{C}}^n(X)$ to be defined for all integers n, with $K_{\mathbb{C}}^n(X) \cong K_{\mathbb{C}}^{n+2}(X)$ and in particular $K_{\mathbb{C}}^{-1}(X) \cong K_{\mathbb{C}}^1(X)$. Note too that $K_{\mathbb{C}}^0(X) \cong K_{\mathbb{C}}^1(SX)$. So the next result is very welcome.

2.5.1. Theorem [2], [21]. For $n \leq 2$

$$K_{n-1}(R) \cong K_n(SR).$$

One formulation of the periodicity theorem is $(n \geq 0)$

$$K_{\mathbb{C}}^{-n}(X \times S^1) \cong K_{\mathbb{C}}^{-n}X \oplus K_{\mathbb{C}}^{-n-1}X.$$

Thus $K_{\mathbb{C}}^{-n-1}(X)$ is the "non-$K_{\mathbb{C}}^{-n}$ part" of $K_{\mathbb{C}}^{-n}(X \times S^1)$. Now consider the "Laurent polynomial ring" $\mathbb{C}(X)[t, t^{-1}]$: we have

$$\mathbb{C}(X) \subseteq \mathbb{C}(X)[t, t^{-1}] \subseteq \mathbb{C}(X \times S^1)$$

where the projection function

$$t: X \times S^1 \twoheadrightarrow S^1 \hookrightarrow \mathbb{C}$$

is non-zero and thereby invertible. (Moreover, under a suitable norm $\mathbb{C}(X \times S^1)$ can be regarded as the completion of $\mathbb{C}(X)[t, t^{-1}]$, so that the more algebraically presented $\mathbb{C}(X)[t, t^{-1}]$ serves as a fair approximation.) On the other hand, there is also an embedding (with Kronecker delta notation)

$$R[t, t^{-1}] \hookrightarrow SR,$$
$$t^r \longmapsto (\delta_{i,\, j+r}).$$

(While the locally finite matrix $(\delta_{i,\, j-1})$ is only a left inverse to $(\delta_{i,\, j+1})$ in CR, it becomes a two-sided inverse modulo finite matrices.) So

$$\Sigma\, a_r t^r \longmapsto \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Based on the above, our hope would be that $K_{-n}(R) = K_{-n+1}(SR)$ is a direct summand of $K_{-n+1}(R[t, t^{-1}])$, the other summand comprising K_{-n+1} terms. This fact is often described as "the fundamental theorem".

2.5.2. Theorem ([2], for $n \leq 1$). For $n \leq 2$, the inclusions

$$R \hookrightarrow R[t^{\pm 1}] \hookrightarrow R[t, t^{-1}] \hookrightarrow SR$$

induce an exact sequence

$$0 \to K_n(R) \to K_n(R[t]) \oplus K_n(R[t^{-1}]) \to K_n(R[t, t^{-1}]) \to K_{n-1}(R) \to 0$$

whose monomorphism and epimorphism are (naturally) split.

3. *The plus-construction*

3.1. *Review of topology*

We very rapidly recall some standard machinery from algebraic topology needed in the sequel. Details may be found among the numerous graduate-level texts, although to my knowledge no single book includes all the facts below.

Maps $f_0, f_1: X \to Y$ are *homotopic*, $f_0 \simeq f_1$, if there is a homotopy $F: X \times I \to Y$ with $F(\ ,t) = f_t$ $(t = 0, 1)$ where $I = [0, 1] \subseteq \mathbb{R}$. If $f: X \to Y$ and $g: Y \to X$ satisfy $f \circ g \simeq id$, $g \circ f \simeq id$, they are inverse *homotopy equivalences*, and X, Y have the same *homotopy type*. For spaces with *basepoint* (as below) all maps, homotopies are taken to preserve basepoint. The set of such homotopy classes of maps from X to Y is denoted [X, Y]. We consider here spaces of the homotopy type of connected *CW-complexes* with basepoint; such spaces are built up by a sequence of attachments of cells.

Every map $f: A \to B$ can be approximated (in that $f = p \circ h$ where h is a homotopy equivalence) by a *fibration* $p: E \to B$ whose defining property is that any $G: W \times I \to B$ lifts to E (so there exists $\bar{G}: W \times I \to E$ with $p \circ \bar{G} = G$) provided $G(\ , 0)$ does. For $b_0 \in B$ the basepoint, $p^{-1}(b_0)$ is called the *fibre* of p, or the *homotopy fibre* F_f of f. Dual to a fibration (reverse the direction of all arrows in the definition) is a *cofibration*.

Algebraic invariants (actually, functors) of spaces that we use are the *homotopy* π_i and *homology* H_i groups ($i \geq 0$ – but X connected has $\pi_0(X)$ and $H_0(X)$ trivial). Their definition is not important here, only a few formal properies. Unless stated otherwise, H_* has *trivial integer coefficients*, that is $H_i(X) = H_i(X; \mathbb{Z})$, but in general coefficients may lie in any abelian group or even any *local coefficient system of abelian groups* {L}. Such an {L} is a family of isomorphic abelian groups, one at each point of a given space. So if {L} is defined with respect to Y, then $f: X \to Y$ defines another system $f^*\{L\}$ on X.

Although $\pi_i (i \geq 2)$ and $H_i (i \geq 0)$ are abelian groups, π_1 need not be. In fact $H_1(X) = \pi_1(X)_{ab}$. This relation extends: if X is n-*connected* ($\pi_i(X)$ is trivial for $i \leq n$), then G. Whitehead's theorem asserts that $\pi_{n+1}(X) \cong H_{n+1}(X)$. The usual applications of homotopy and homology groups are to dual situations: π_* applies best to fibrations, and H_* to cofibrations. If $f: X \to Y$ has homotopy fibre F_f, there results the *homotopy exact sequence*

$$\ldots \to \pi_n(F_f) \to \pi_n(X) \xrightarrow{\pi_n(f)} \pi_n(Y) \to \pi_{n-1}(F_f) \to \ldots \to \pi_0(Y)$$

(that is, the image of each homomorphism is the kernel of its successor).

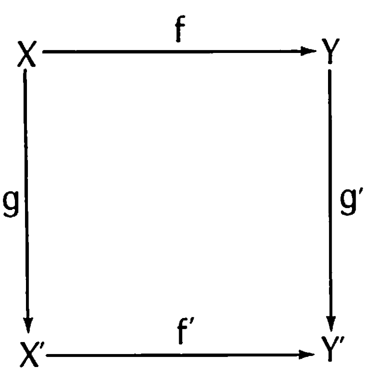
For example, a theorem of J.H.C. Whitehead asserts that a space (of our not-too-general kind) which has all its homotopy groups trivial must have the homotopy type of a one-point space (that is, is *contractible*). Now if f: X → Y has $\pi_n(f)$ always an isomorphism, then the homotopy exact sequence makes $\pi_n(F_f)$ always trivial. However, F_f contractible results in f being a homotopy equivalence.

Without further restriction on X, Y (as occurs in (3.3.9) (v) below) one cannot in general conclude that $H_n(f)$ always an isomorphism implies that f is a homotopy equivalence. This distinction between *homology equivalences* and homotopy equivalences is crucial to the whole of this chapter. If f: X → Y is a cofibration then there is an exact sequence

$$\to H_n(X) \xrightarrow{H_n(f)} H_n(Y) \to H_n(Y/f(X)) \to H_{n-1}(X) \to \ldots \to H_0(Y/f(X))$$

($H_n(Y/f(X))$ often being written as $H_n(Y, f(X))$), where the coefficient system can be taken arbitrarily over Y. This sequence can be applied to the homology of two of the three spaces X, Y, Y/f(X) in order to compute the homology of the third.

Homology groups fit in less neatly with fibrations, but machinery does exist. The *Serre spectral sequence* exhibits $H_*(X)$ as a limit of exact sequences, whose starting-points are the groups $H_p(Y; \{H_q(F_f)\})$. Here { } refers to the fact that f: X → Y induces an *action* of $\pi_1(Y)$ on each $H_q(F_f)$ and so a local coefficient system on Y. When the action is trivial (that is, only induces identity maps on $H_*(F_f)$) then f is called *orientable*. For orientable fibrations there is a handy *spectral sequence comparison theorem*. Given a commuting diagram



($f' \circ g = g' \circ f$), inducing Fg: $F_f \to F_{f'}$, one can compare homotopy groups of F_f and $F_{f'}$ via the map of homotopy exact sequences determined by $\pi_*(g)$, $\pi_*(g')$. The comparison theorem allows a similar comparison of homology groups. Supposing $\pi_1(g')$ to be an isomorphism, the theorem states that if two of the three families $H_*(g')$, $H_*(g)$, $H_*(Fg)$ comprise isomorphisms, then so does the third.

We can embed group theory within topology by considering the *classifying space* BG of a group G. This has $\pi_n(BG) = G$ (n = 1) and 0 otherwise.

3.2. Acyclic maps

There are two obvious kinds of equivalences among maps, the *homotopy equivalences*, which induce isomorphisms on all homotopy groups, and the *homology equivalences*, inducing isomorphisms on $H_*(\ ;\mathbb{Z})$. Our key notion is intermediate between these two. We say a map is *acyclic* if it satisfies the conditions of (3.2.1).

3.2.1. Theorem [13], [3]. The following conditions on $f: X \to Y$ are equivalent:

(i) f is an homology equivalence and the homotopy fibration $F_f \to X \to Y$ is orientable;

(ii) the homotopy fibre F_f is acyclic (that is, $H_i(F_f) = 0$ for $i \geq 1$);

(iii) for any local coefficient system $\{L\}$ of abelian groups on Y,

$$H_*(f): H_*(X; f^*\{L\}) \to H_*(Y; \{L\})$$

is an isomorphism.

(Here $f^*\{L\}$ is the coefficient system on X induced from $\{L\}$ by f.) Equivalence of (i) and (ii) comes by applying the spectral sequence comparison theorem to the map of fibrations

$$\begin{array}{ccc} F_f & \xrightarrow{Ff} & * \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id} \\ Y & \xrightarrow{id} & Y \end{array}$$

Equivalence of (ii) and (iii) uses a (generalised) spectral sequence argument. From (iii) there is an easy corollary.

3.2.2. Corollary. Suppose $f: X \to Y$ is acyclic. Then $g: Y \to Z$ is acyclic if and only if $g \circ f$ is. ◇

3.2.3. Corollary. Suppose

$$\begin{array}{ccc} X & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{g_1} & Y \cup_X Z \end{array}$$

is a push-out, with f a cofibration. If f is acylic, then so is f_1.

Proof. Let $\{L\}$ be a local coefficient system on $Y \cup_X Z$. Because f is a cofibration the right-hand vertical homomorphism in

$$\begin{array}{ccccccc} \ldots \to H_q(X; f^* g_1^* \{L\}) & \to & H_q(Y; g_1^* \{L\}) & \to & H_q(Y, f(X); g_1^* \{L\}) & \to & \ldots \\ \downarrow & & \downarrow & & \downarrow & & \\ \ldots \to H_q(Z; f^* f_1^* \{L\}) & \to & H_q(Y \cup_X Z; \{L\}) & \to & H_q(Y \cup_X Z, g_1(Y); \{L\}) & \to & \ldots \end{array}$$

is an excision isomorphism for all q. By exactness, the left-middle arrow in each row is an isomorphism if and only if the right-hand group is zero (for all q). ◇

The dual situation to (3.2.3) uses (3.2.1) (ii) instead of (iii). Because pull-backs preserve homotopy fibres (up to homotopy type)

3.2.4. Corollary. Suppose

$$\begin{array}{ccc} Y' \times_Y X & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is a pull-back. Then f is acyclic if and only if f′ is too. ◇

Note that if F_f is acyclic, then $0 = H_1(F_f) = (\pi_1(F_f))_{ab}$, making $\pi_1(F_f)$ a *perfect* group. (The key example of a perfect group is of course ER for any ring R — see (1.2.7).) So if f: X → Y is acyclic, then (ii) implies that in the exact homotopy sequence

$$\ldots \to \pi_2(Y) \to \pi_1(F_f) \xrightarrow{\pi_1(i)} \pi_1(X) \xrightarrow{\pi_1(f)} \pi_1(Y) \to \pi_0(F_f)$$

$\pi_0(F_f)$ is trivial, so $\pi_1(f)$ is surjective and with kernel the image of the perfect group $\pi_1(F_f)$—hence perfect.

3.2.5. Corollary. If f: X → Y is acyclic, then $\pi_1(Y) \cong \pi_1(X)/P$ where P is some perfect normal subgroup of $\pi_1(X)$. ◇

If P is trivial, then $\pi_2(Y) \twoheadrightarrow \pi_1(F_f)$ maps an abelian group onto a perfect one, hence is trivial, making $\pi_1(F_f)$ trivial too. Therefore F_f is both acyclic and 1-connected, hence contractible.

3.2.6. Corollary. If f: X → Y is acyclic and $\pi_1(f)$ an isomorphism, then f is a homotopy equivalence. ◇

3.2.7. Example. The Poincare 3-sphere is a manifold M with $H_*(M) \cong H_*(S^3)$, that is, an *homology* S^3 (the isomorphism being induced by a map collapsing the complement of a small ball in M). It is derived from the faithful smooth representation of the perfect alternating group A_5 in the rotation group SO(3). $M = SO(3)/A_5 = \mathbb{R}P^3/PSL(2, 5) = S^3/SL(2, 5)$. Because $\pi_1(S^3) = 1$, the map $M \to S^3$ is orientable, and so acyclic by (3. 2. 1) (i). It is not a homotopy equivalence (e.g. $\mathrm{Ker}\pi_1$ is non-trivial).

3.2.8. Example. Given a ring R, let UT be the ring of upper-triangular 2×2-matrices over R. Define π: UT → R ⊕ R by

$$\begin{pmatrix} a & b \\ o & c \end{pmatrix} \longmapsto (a, c).$$

It turns out that $GL\pi$: GLUT → GL(R ⊕ R) induces an isomorphism on $H_*(\ ;\mathbb{Z})$, (that is, $BGL\pi$ is an homology equivalence) [17], [3]. However, its kernel is isomorphic to the abelian group MR of all finite matrices (under addition). So $GL\pi = \pi_1(BGL\pi)$ has its kernel MR non-perfect. Hence $BGL\pi$: BGLUT → BGL(R ⊕ R) is an homology equivalence which is (by (3.2.5.)) not acyclic.

Returning to (3.2.3), note that the van Kampen theorem computes $\pi_1(Y \cup_X Z)$ as $\pi_1(Y) *_{\pi_1(X)} \pi_1(Z)$. Applying (3.2.5) when f is acyclic gives $\pi_1(Y) \cong \pi_1(X)/\mathrm{Ker}\pi_1(f)$.

3.2.9. Corollary. In (3.2.3), Ker $\pi_1(f_1)$ is the normal closure of the perfect subgroup $\pi_1(g)\,\mathrm{Ker}\pi_1(f)$ of $\pi_1(Z)$. ◇

3.3 Definition of the plus-construction

Here is the key classification theorem for a cyclic maps from a space X.

3.3.1. Theorem. Let P be a perfect normal subgroup of $\pi_1(X)$. Then there exists an acyclic cofibration f: X → Y with $\mathrm{Ker}\pi_1(f) = P$, unique in the sense that if g: X → Z is another then there exists a homotopy equivalence h: Z → Y such that $h \circ g = f$.

Proof. For brevity we only sketch the proof of existence. By attaching, for each (normal) generator of P, a 2-cell to the covering X′ of X with $\pi_1(X') = P$ one obtains a space whose homology differs from $H^*(X')$ only in dimension 2. This can be corrected by attaching 3-cells to form a space Y′, with $\pi_1(Y') = 1$ by van Kampen's theorem. Then by (3.2.1) $X' \to Y'$ is acyclic, so that the push-out space $Y = X \cup_{X'} Y'$ has $X \to Y$ acyclic by (3.2.3) with $\pi_1(X)/P$ by (3.2.9). Uniqueness follows from

3.3.2. Lemma. Let $f\colon X \to Y$ and $g\colon X \to Z$ have f an acyclic cofibration and $\mathrm{Ker}\,\pi_1(f) \leq \mathrm{Ker}\,\pi_1(g)$. Then there exists $h\colon Y \to Z$ such that $h \circ f = g$; moreover, any two such are homotopic.

Proof. Apply (3.2.3), (3.2.9) to the push-out

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{\;g_1\;} & Y \cup_X Z \end{array}$$

to deduce that f_1 is acyclic with $\mathrm{Ker}\,\pi_1(f_1)$ the normal closure of the trivial (!) subgroup $\pi_1(g)\,(\mathrm{Ker}\,\pi_1(f))$. So by (3.2.6) f_1 is a homotopy equivalence; composing its unique inverse homotopy class with g_1 gives the required homotopy class for h. ◇

Canonical choice of P in (3.3.1) is provided by a little group theory. Because homomorphisms send commutators to commutators they also map perfect subgroups to perfect subgroups. So in particular any conjugate of a perfect subgroup is perfect. Since a product of perfect subgroups must also be generated by its commutators, it follows that the product of **all** perfect subgroups of a given group G, the unique *maximal perfect subgroup* PG, is normal in G (and thus the *perfect radical* of G).

3.3.3. Lemma. If $\psi\colon G \twoheadrightarrow H$ is a group epimorphism with perfect kernel, then $PH = \psi PG$.

Proof. Write $K = \mathrm{Ker}\,\psi$, $L = \psi^{-1}PH$; so $LK/K \cong PH$ is perfect. Then

$$L = [L, L]\,K \leq [L.PG, L.PG],$$

making $L.PG$ perfect. So $L.PG \leq PG$, which leaves $L \leq PG$. ◇

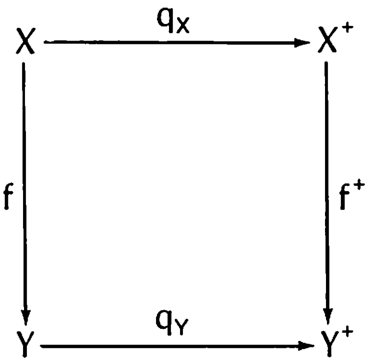
3.3.4. Corollary. For a perfect normal subgroup K of G, $P(G/K) = 1$ if and only if $K = PG$. ◇

To define the *plus-construction* on X we take $P = P\pi_1(X)$ in (3.3.1), denoting the corresponding acyclic cofibration by $q_X\colon X \to X^+$. So by (3.2.5) $\pi_1(X^+) \cong \pi_1(X)/P\pi_1(X)$.

3.3.5. Corollary. Acyclic f: X → Y is equivalent to q_X if and only if $P\pi_1(Y) = 1$. ◇

Similarly, (3.3.2) and the group-theoretic remarks above yield

3.3.6. Proposition. Given f: X → Y, there is a unique homotopy class of maps $f^+ : X^+ \to Y^+$ making the following square commute.



◇

Since $P(G \times H) = PG \times PH$, we have

3.3.7. Lemma. $(X \times Z)^+ = X^+ \times Z^+$ with $q_{X \times Z} = (q_X, q_Z)$. ◇

Applying this when Z = I, along with (3.3.6), gives

3.3.8. Proposition. If $f_0 \simeq f_1 : X \to Y$, then $f_0^+ \simeq f_1^+ : X^+ \to Y^+$. ◇

3.3.9. Examples.

(i) If $P\pi_1(X) = 1$, then we may take q_X as id: X → X.

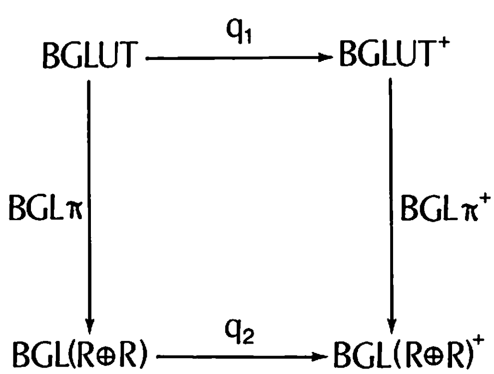
(ii) If X is acyclic, then X^+ is contractible.

(iii) For M as in (3.2.7) the acyclic map $M \to S^3$ corresponds to q_M because $P\pi_1(S^3) = 1$.

(iv) Of key interest here is $q_{BGLR} : BGLR \to BGLR^+$. It turns out that the space $BGLR^+$ has very nice properties (see (4.3.1)). In particular it is a *simple* space (that is, π_1 acts trivially on each π_n, n ≥ 1). From (1.2.10)

3.3.10. Proposition. $\pi_1(BGLR^+) \cong K_1(R)$.

(v) After (3.2.8) there is a commuting square of homology equivalences



Now one property of simple spaces like $BGLR^+$ is that $BGL\pi^+$ being an homology equivalence forces it to be a homotopy equivalence. So $q_2 \circ BGL\pi$ and q_2 are acyclic, although by (3.2.8) $BGL\pi$ is not (cf. (3.2.2)).

3.4. *The plus-construction and fibrations*

Since the plus-construction leaves homology unchanged, the main interest is in its effect on homotopy groups. Here the chief tool is the homotopy exact sequence of a fibration. So we have the key question: "Which fibrations are *plus-constructive*?"; that, is when does F the fibre of $p: E \to B$ also have F^+ as the fibre of $p^+: E^+ \to B^+$?

3.4.1. Example. In (3.3.9) (v) the fibration $BMR \to BGLUT \xrightarrow{BGL\pi} BGL(R \oplus R)$ is **not** plus-constructive because $BMR^+ = BMR$ (MR is abelian) while $BGL\pi^+$, being a homotopy equivalence, has contractible fibre.

3.4.2. Theorem [3]. $F \to E \to B$ is plus-constructive if $P\pi_1(B) = 1$.

This is quite an easy argument similar to (3.2.4). Much deeper is

3.4.3. Theorem [4]. Suppose that F^+ is a simple space. Then $F \to E \to B$ is plus-constructive if and only if $P\pi_1(B)$ acts trivially on $H_*(F)$.

3.4.4. Corollary. If R is commutative, then $BSLR^+$ is the fibre of $Bdet^+$: $BGLR^+ \to BR^*$. ◇

3.4.5. Corollary. BER^+ is the universal covering space of $BGLR^+$.

Proof. After (3.4.2) there is a fibration $BER^+ \to BGLR^+ \to BK_1(R)$. So apply (3.3.10). ◇

3.4.6. Corollary. Let $N \rightarrowtail G \twoheadrightarrow Q$ be a central extension of groups. Then BN is the fibre of $BG^+ \to BQ^+$.

Proof. Recall that the action of π_1 on itself is by conjugation. So $\pi_1(BN)$ acts trivially on $\pi_n(BN)$ ($=0$, $n > 1$; abelian, $n = 1$). Since Q acts trivially on N it also acts trivially on $H_*(N) = H_*(BN)$. So (3.4.3) applies. ◇

3.4.7. Corollary. $BK_2(R)$ is the fibre of $BStR^+ \to BER^+$. ◇

4. *Higher K-groups*

4.1 *Definition*

We can now define [13], for any ring R and $n \geq 3$,

$$K_n(R) = \pi_n(BGLR^+).$$

Since K_n is a composition of functors, it too is a functor. To justify this definition we check the other cases $n = 1, 2$. Now (3.3.10) handles $n = 1$, while in §4.2 we show

4.1.1. Theorem. $\pi_2(BGLR^+) \cong K_2(R)$.

4.1.2. Example. Because

$$GL(R_1 \oplus R_2) \cong GLR_1 \times GLR_2,$$

so (after (3.3.7))

$$BGL(R_1 \oplus R_2)^+ = BGLR_1^+ \times BGLR_2^+,$$

yielding, for all $n \geq 1$,

$$K_n(R_1 \oplus R_2) \cong K_n(R_1) \oplus K_n(R_2).$$

4.1.3. Example. Since $BGL\pi^+ : BGLUT^+ \to BGL(R \oplus R)^+$ is a homotopy equivalence (3.3.9) (v), we have, for $n \geq 1$,

$$K_n(UT) \cong K_n(R) \oplus K_n(R).$$

4.1.4. Example [21]. The cone CR on a ring R has BGLCR acyclic. So by (3.3.9) (ii), for $n \geq 1$,

$$K_n(CR) = 0.$$

4.2. K_2, K_3 *as* H_2, H_3

The homotopy exact sequence from (3.4.7) reads in part

(4.2.1) $\pi_2(BStR^+) \to \pi_2(BER^+) \to \pi_1(BK_2(R)) \to \pi_1(BStR^+)$

and, further along,

(4.2.2) $\pi_3(BK_2(R)) \to \pi_3(BStR^+) \to \pi_3(BER^+) \to \pi_2(BK_2(R))$.

Now StR, ER are perfect groups, so that $BStR^+$, BER^+ have π_1 trivial. Then by G. Whitehead's theorem they have $\pi_2 \cong H_2$. Assume for the moment

4.2.3. Theorem. $H_2(StR) = 0$.

An immediate consequence (Whitehead again) is that $\pi_3(BStR^+) = H_3(StR)$. Of course $\pi_n(BG) = 0$ for $n \geq 2$ (any discrete group G). Then from the homotopy exact sequence of $BER^+ \to BGLR^+ \to BK_1(R)$ (3.4.5) we also have $\pi_n(BER^+) \cong \pi_n(BGLR^+)$, $n \geq 2$. Thus (4.2.1) reduces to (4.1.1) with immediate consequence

4.2.4. Corollary. $H_2(ER) \cong \pi_2(BER^+) \cong \pi_2(BGLR^+) \cong K_2(R)$. ◇

Likewise (4.2.2) gives

4.2.5. Theorem. $H_3(StR) \cong \pi_3(BStR^+) \cong \pi_3(BER^+) \cong K_3(R)$. ◇

The proof of (4.2.3) uses the standard fact that central extensions with quotient Q and kernel K are classified by $H^2(Q; K)$. When $H_1(Q) = 0$ the universal coefficient theorem has the cohomology group isomorphic to $Hom(H_2(Q), K)$. So, with $Q = StR$ and $K = H_2(StR)$, (4.2.3) follows from

4.2.6. Theorem. Any central extension $K \rightarrowtail G \overset{\psi}{\twoheadrightarrow} StR$ is trivial (that is, admits a splitting homomorphism $\sigma: StR \rightarrowtail G$ with $\psi \circ \sigma = id$).

Proof. It suffices to define σ on generators of StR. So put

$$\sigma(x_{ij}^a) = [\psi^{-1}(x_{i,\,i+j}^1),\ \psi^{-1}(x_{i+j,\,j}^a)],$$

a uniquely defined element of G because $K \leq Z(G)$. The defining relations for StR simply imply that $\psi \circ \sigma = id$. The check that σ respects all these relations, and so is a homomorphism, is an exercise involving some delicate manipulation of commutators. ◇

4.3. Delooping of $BGLR^+$

One of the most topologically desirable classes of spaces is that of *infinite loop-spaces*, spaces X_n ($n \geq 0$) such that $X_n \simeq \Omega X_{n+1}$.

4.3.1. Theorem [21]. $BGLR^+$ is an infinite loop-space.

For (3.3.9) (v) we note that any infinite loop-space is simple. Also, for any space X, $[X, BGLR^+]$ will be an abelian group which may be regarded as the zeroth group of some extraordinary cohomology theory groups of X. Moreover, the group addition here corresponds to matrix direct sum in GLR (cf. (1.3)). The proof of (4.3.1) contains two main steps. For the first, the fact (2.5) that the suspension SR is the quotient ring of the cone CR by the extended ideal of $R \hookrightarrow CR$ leads to an extension

$$GLR \rightarrowtail GLCR \twoheadrightarrow ESR$$

(using GLCR = ECR (4.1.4)), and thus fibration

$$BGLR \to BGLCR \to BESR.$$

4.3.2. Proposition. This fibration is plus-constructive.

The proof of (4.3.2) applies (3.4.3). Then one compares the fibration

$$BGLR^+ \to BGLCR^+ \to BESR^+$$

with the usual path-fibration

$$\Omega(BESR^+) \to P(BESR^+) \to BESR^+,$$

where $P(BESR^+)$ is the contractible space of paths in $BESR^+$ which start at its basepoint. Since $BGLCR^+$ is also contractible (4.1.4), comparison of the homology exact sequences of these last two fibrations shows that $BGLR^+ \simeq \Omega BESR^+$. As $BESR^+$ is the universal cover of $BGLSR^+$ (3.4.5), the process may be repeated, yielding (4.3.1) and, more particularly, for $n \geq 2$,

$$K_{n-1}(R) = \pi_{n-1}(\Omega BESR^+) = \pi_n(BESR^+) = \pi_n(BGLSR^+).$$

4.3.3. Theorem. For $n \geq 2$,

$$K_{n-1}(R) \cong K_n(SR). \quad \diamond$$

(Compare (2.5.1).)

4.4. Properties

To justify their definition as higher K-groups, these $K_n(R)$ ($n \geq 3$) of Quillen should satisfy the same formal properties as the lower K-groups. Hence the significance of (4.3.3).

4.4.1. Theorem [6]. Theorem 2.5.2. holds for all integers n.

There is a nice method for converting information about projective R-modules into information about $K_n(R)$, $n \geq 1$. For a given group G (usually GL_mR) let $Rep_R(G)$ be the exact-sequence Grothendieck group (see (2.2)) formed from exact sequences of f.g. projective R-modules admitting a linear G-action (which is preserved by all homomorphisms).

4.4.2 Theorem [17]. There is a natural homomorphism $Rep_R(G) \to [BG, BGLR^+]$.

The function in (4.4.2) is constructed by considering a free R-module containing a given projective one. The G-action, being linear, corresponds to a homomorphism $G \to GLR$ and so a map $BG \to BGLR^+$. The check that addition is preserved involves comparing

$$G \to \begin{pmatrix} GL_nR & M_nR \\ 0 & GL_nR \end{pmatrix} \subseteq GL_{2n}R$$

(associated to addition in $Rep_R(G)$) with addition in $[BG, BGLR^+]$, given by

$$G \to \begin{pmatrix} GL_nR & 0 \\ 0 & GL_nR \end{pmatrix} \subseteq GL_{2n}R$$

This means comparing the composition

$$BG \to BGLUT^+ \to BGLR^+$$

with

$$BG \to BGL(R \oplus R)^+ \to BGLR^+.$$

However, by (3.3.9) (v) $BGL\pi^+ : BGLUT^+ \to BGL(R \oplus R)^+$ is a homotopy equivalence, so that addition is after all preserved.

As an application of (4.4.2), let $f: R \to R_1$ be a ring homomorphism making R_1 f.g. projective over R. This determines a transfer homomorphism $Rep_{R_1}(G) \to Rep_R(G)$ for any G. Setting $G = GL_mR$ and letting $m \to \infty$ determines a homotopy class in $[BGLR_1^+, BGLR^+]$ and so a *transfer homomorphism* $K_n(R_1) \to K_n(R)$. This most commonly arises in "localisation sequences", such as (4.4.3) below (where $R_1 = R/m$).

Let R be the ring of integers (roots of monic polynomials with integer coefficients) in some number field (finite $\mathbb{Q}$-extension), and let F be its field of fractions. Write M for the set of maximal ideals m of R.

4.4.3. Theorem [15]. There is an exact sequence

$$\ldots \to K_n(R) \to K_n(F) \to \sum_{m \in M} K_{n-1}(R/m) \to K_{n-1}(R) \to \ldots .$$

For such an R there is also

4.4.4. Theorem [16]. The group $K_n(R)$ is finitely generated for all n.

A few computations of specific K-groups have also been achieved.

4.4.5. Theorem [14]. The field $\mathbb{F}_q$ of q elements has, for $m \geq 1$, $K_{2m}(\mathbb{F}_q) = 0$ and $K_{2m-1}(\mathbb{F}_q)$ is cyclic of order q^{m-1}.

4.4.6. Theorem [10]. $K_3(\mathbb{Z})$ is cyclic of order 48.

Only partial information is known for $K_n(\mathbb{Z})$, $n > 3$.

4.5. Graded ring structure

Recall from (4.3) that $K_n(R)$ has an abelian group structure coming from the direct sum of matrices. There is also defined matrix tensor product, distributive over direct sum. This leads to families of group homomorphisms from $GL_pR \times GL_qR'$ to $GL(R \otimes_{\mathbb{Z}} R')$. A check of compatibility of families allows one to deduce that there is a well-defined homotopy class of maps

$$BGLR^+ \wedge BGLR'^+ \to BGL(R \otimes_{\mathbb{Z}} R')^+.$$

Standard machinery converts this to a pairing on deloopings of $BGLR^+$, leading to

4.5.1. Theorem [11]. There is a bilinear, associative multiplication

$$K_m(R) \times K_n(R') \to K_{m+n}(R \otimes_{\mathbb{Z}} R');$$

for commutative R *this induces*

$$K_m(R) \times K_n(R) \to K_{m+n}(R).$$

For m, n ≤ 0 the pairings are defined by their effect on suspensions (using (2.5)). So for commutative R, $K_*(R)$ has the structure of a *graded ring* (as $K^*(X)$ was known to have). We finally mention how, like its topological counterpart, algebraic K-theory admits λ-*operations* [8]. These are functions $\lambda^k: K_n(R) \to K_n(R)$ ($k \geq 0$, $n \in \mathbb{Z}$) with similar formal properties to the topological ones. In topology, exterior powers of vector spaces generalise to exterior powers of vector bundles. Here we generalise to exterior powers of R-modules carrying a G-action, to define λ^k: $Rep_R(G) \to Rep_R(G)$. Again, the interesting case is $G = GL_mR$, where (4.4.2) leads to a well-defined class Λ^k in $\varprojlim [BGL_mR, BGLR^+]$ and thence to λ^k: $K_n(R) \to K_n(R)$.

It is common to study a variant of λ^k, the group homomorphism $\psi^k: K_n(R) \to K_n(R)$ (*Adams' operation*). When R is *perfect* of characteristic $p > 0$, so that the *Frobenius* p-th power homomorphism is an automorphism, then it can be shown that the automorphism induced on $K_n(R)$ ($n \geq 1$) is just ψ^p. Then the formula

$$\lambda^p \circ p. = (-1)^{p+1} \psi^p$$

makes p. an automorphism too. Hence $K_n(R)$ is uniquely p-divisible.

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