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AXIAL MAPS WITH FURTHER STRUCTURE

A. J. Berrick

ABSTRACT. For $F = R, C$ or $H$ an $F$-axial map is defined to be an axial map $RP^m \times RP^m \to RP^{m+k}$ equivariant with respect to diagonal and trivial $F^*$-actions. Analogously to the real case, it is shown that $C$-axial maps correspond to immersions of $CP^n$ in $R^{2n+k}$ while (for $F = R$ and for $F = C$, $k$ odd) embeddings induce $F$-symmaxial maps. Examples are thereby given of symmaxial maps not induced by embeddings of $RP^n$, and of $R$-axial maps which are not $C$-axial. Furthermore, the relationships which hold when $F = R, C$ are no longer valid for $F = H$.

Let $F$ be one of the fields $R$, $C$ or $H$ of dimension $d$ (= 1, 2, 4 respectively) over $R$, whose units $F^*$ act on the right on $S(F^{n+1})$ to induce the projective space $FP^n$. Since the action of $R^*$ extends to the action of $F^*$, we may regard $F^*$ as acting also on $RP^n$ and hence diagonally on $RP^m \times RP^n$, $n \equiv -1 (d)$. By way of generalisation of the usual definitions ($F = R$—see [2], [4], [12]), we say $f: RP^m \times RP^n \to RP^{m+k}$ is $F$-axial of type $(n, k)$ if $f$ restricts to homotopy essential maps on the axes of the product and is equivariant with respect to the above $F^*$-action on its domain and trivial $F^*$-action on its range. If further $f$ is homotopy equivariant—through an $F^*$-equivariant homotopy—with respect to interchanging the factors of the domain and trivial $Z_2$-action on the range, $f$ is $F$-symmaxial. (When $F = R$ it is sometimes omitted from the notation.) This note explores the relationship between $F$-axial (resp. $F$-symmaxial) maps and the existence of an immersion (resp. embedding) of $FP^n$ in $R^m$, denoted $FP^n \subseteq (m)$ (resp. $FP^n \subseteq (m)$).

1. THEOREM. Let $F = R$ or $C$, with $N = n$ or $(2n + 1)$ respectively.
(a) If $FP^n \subseteq (dn + k)$, then there exists an $F$-axial map of type $(N, k)$.
(b) If $FP^n \subseteq (dn + k)$, then there exists an $F$-symmaxial map of type $(N, k)$, provided $k$ is odd if $F = C$.
(c) If $FP^n \subseteq (dn + k)$, then the $F$-axial maps given by the constructions of (a) and (b) are homotopic through an $F^*$-equivariant homotopy.
(d) If there exists an $F$-axial map of type $(N, k)$ with $2k \geq dn + 1$, then $FP^n \subseteq (dn + k)$.

PROOF. (a), (d). Let $\gamma$ be the realisation of the Hopf line bundle, $\epsilon$ the trivial real line bundle, and $\tau$ the real tangent bundle over $FP^n$. In the following sequence of implications, † indicates the use of the condition $2k \geq dn + 1$.

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$FP^n \subseteq (dn + k) \iff r$ is a subbundle of $(dn + k)e \{6\}$
\[ \iff \tau \oplus de = (n + 1)\gamma^* \text{ is a subbundle of} \]
\[ (dn + k + d)e \{7, p. 100\} \]
\[ + \iff \text{there exists a skew map} \]
\[ (n + 1)\gamma^* \to (d(n + 1) + k)e \{5, (1.2)\} \]
\[ \iff \text{there exists a map} S^N \times S^N \to S^{N+k} \text{ which induces} \]
\[ \text{an} F\text{-axial map of type} (N, k). \]

(b) Let $f: F^P \to \mathbb{R}^{dn+k}$ be an embedding. (To use conventional matrix notation, we shall assume here that $F^*$ acts on $\mathbb{R}^d$ on the left.) Write $\mathbb{R}_0^m = \mathbb{R}^m \setminus \{0\}$; $\nu: \mathbb{R}_0^m \to S^{m-1}$, $x \mapsto x/\|x\|$; $\pi: S^N \to FP^N$, and set $\bar{\Delta} = \{(x,wx) \in \mathbb{R}_0^{N+1}; \ w \in F^*\}$, $\Delta' = \bar{\Delta} \cap (S^N \times S^N)$, $e = (1,0, \ldots, 0) \in \mathbb{R}^{N+1+k}$, and $j: \mathbb{R}^{dn+k} \to \mathbb{R}^d \oplus \mathbb{R}^{dn+k}$ for the inclusion of the orthogonal complement of $Fe$ in $\mathbb{R}^{N+1+k}$. For $u,v \in S^N$, write $a = \langle v,u \rangle_F$; and define
\[ G: (S^N \times S^N, S^N \times \Delta') \times I \to (\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}, \mathbb{R}_0^{N+1} \times \mathbb{R}_0^{N+1} \setminus \bar{\Delta}), \]
\[ g(u,v,t) = \begin{cases} \left[\begin{array}{c} 1 - |a|^2 t^2 \\ at \\ 1 \end{array}\right] \left[\begin{array}{c} u \\ -\bar{a}t \\ v - au \end{array}\right] \\ 0, \end{cases} \]
\[ g(x,y) = \begin{cases} 1/\|x\| \cdot \|y\| \cdot \|f\nu(x) - f\nu(y)\| \cdot [f\nu(x + y) - f\nu(x - y)] \\ (x,y) \in \mathbb{R}_0^{N+1} \times \mathbb{R}_0^{N+1} \setminus \bar{\Delta}, \\ 0, \quad (x,y) \in (\mathbb{R}^{N+1} \cup \mathbb{R}^{N+1}) \cup \bar{\Delta}. \end{cases} \]

Hence, define
\[ F: S^N \times S^N \times I \to S^{N+k}, \quad F(u,v,t) = v(\alpha e + jgG(u,v,t)). \]

The reader may verify that these maps behave as required, so that $F_0: S^N \times S^N \to S^{N+k}$ induces an $F$-symaxial map of type $(N,k)$. (When $F = C$, the involution on $\mathbb{R}P^{2n+1+k}$ given by $\pm (\alpha e + j(z)) \mapsto \pm (\bar{a}e + j(z))$ is homotopic to the identity provided $k$ is odd.)

(c) Clearly it suffices to establish that the tangent bundle monomorphism
\[ \tau(f): \tau FP^n \to \tau \mathbb{R}^{dn+k} = \mathbb{R}^{dn+k} \times \mathbb{R}^{dn+k} \]
is fibre-homotopic to
\[ g'(x,y), F^* = (f\nu(x), g(x,y)) \]
(f, g as in (b)), since the $F$-axial maps of both (a) and (b) come from composition with $G_0: S^N \times S^N \to \pi^* \tau FP^n$ specified in (b).

But this is evident from the following homotopy (cf. [5, Lemma 2.2]):
\[ H: \tau FP^n \times I \to \mathbb{R}^{dn+k} \times \mathbb{R}^{dn+k}, \]
\[ H([x,y], F^*, t) = (f\nu(x), \{f\nu(x + (1 - t)y) - f\nu(x - (1 - t)y)\} / (1 - t^2)). \]
(Note that, as $t \to 1$, $1 - t^2 = 2(1 - t) + O(1 - t^2)$.)

By [2], the numerical condition of 1(d) is satisfied when $n > 7$ if $F = C$ and may be omitted if $F = R$. Thus 1(a),(d) yield that $CP^n \subseteq (2n + k)$ implies
$\mathbb{R}P^{2n+1} \subseteq (2n + k + 1)$—cf. [12, (5.2)]. When $F = \mathbb{R}$, 1(b),(c) answer affirmatively a question raised in [2] (for which, I understand, Professors Feder and Gitler also have a proof); we now show the converse is not true.

2. Example. Let $n$ be a power of 2. Then by [8], $\mathbb{C}P^n \subset (4n - 1)$; 1(b) now implies the existence of a C-symmaxial (and so R-symmaxial) map of type $(2n + 1, 2n - 1)$. But [9], [10] $\mathbb{R}P^{2n+1} \not\subset (4n)$, so that the existence of a symmaxial map of type $(n,k)$ does not imply $\mathbb{R}P^n \subset (n + k)$.

The next result is perhaps more predictable. Nevertheless, it illustrates the falsity of the converse to [12, (5.2)].

3. Example. Let $n + 1 = 2^r$, where $r \equiv 2, 3 (4)$. Then by [4] $\mathbb{R}P^{2n+1} \subseteq (4n - 2r)$; so by [11] there exists an R-axial map of type $(2n + 1, 2n - 2r - 1)$. However, by [13], $\mathbb{C}P^n \not\subseteq (4n - 2r - 1)$, whence, from 1(c), the existence of an R-axial map of type $(2n + 1, k)$ does not imply the existence of a C-axial map of type $(2n + 1, k)$.

Since 1 shows that the situation for $\mathbb{R}P^n$ largely carries over to $\mathbb{C}P^n$, one might naively hope that a comparable result holds for $\mathbb{H}P^n$. However, [3, §4] casts doubt upon, and 5 below puts paid to, such hopes.

4. Lemma. If there exists an H-axial (resp. H-symmaxial) map $f$ of type $(4n + 3, k)$, then there exists a C-axial (resp. C-symmaxial) map $g$ of type $(4n + 3, k)$.

Proof. Write $\mathbb{R}^{4n+4} = \mathbb{C}^{2n+2} \oplus \mathbb{C}^{2n+2}$ which we identify with $\mathbb{H}^{n+1}$ as $\mathbb{C}^{2n+2} \oplus \mathbb{C}^{2n+2}$. For $x_i, y_i \in \mathbb{C}^{2n+2}, i = 1, 2$, $f$ induces $g$ by setting

$$g(\pm (x_1, x_2), \pm (y_1, y_2)) = f(\pm (x_1 + x_2j), \pm (y_1 + y_2j)),$$

since $(x_1a + (x_2a))j = (x_1 + x_2j)a$ for $a \in \mathbb{C}^*$. If $f$ is symmaxial then clearly $g$ is too.

5. Example. Let $n$ be a power of 2. From [8], $\mathbb{H}P^n \subset (8n - 3)$. But if there were an H-symmaxial—or even H-axial—map of type $(4n + 3, 4n - 3)$, then by 4 above there would exist a C-axial map of type $(4n + 3, 4n - 3)$. So by 1(c) $\mathbb{C}P^{2n+1} \subseteq (8n - 1)$, which is contradicted by [1], [13]. Hence, $\mathbb{H}P^n \subset (4n + k)$ does not imply the existence of an H-axial map of type $(4n + 3, k)$.

As for positive results in the quaternionic case, we must content ourselves with the following observation.

6. Note. If there exists an H-axial map of type $(4n + 3, k)$ with $2k \geq 4n + 1$, then $\mathbb{H}P^n \subset (4n + 3 + k)$. The proof is as for 1(c) above, save that one uses the characterisation of the tangent bundle given in [3, §4].

References


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