# CONSEQUENCES OF THE KAHN–PRIDDY THEOREM IN HOMOTOPY AND GEOMETRY

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§1. *Introduction and statement of results.* Much recent attention has been given to geometric representation of elements of the stable homotopy groups of spheres, $\pi_*^S$. A particular example concerns non-singular bilinear maps $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{m+n+1-p}$; on restriction and normalisation these become biskew maps $S^m \times S^n \to S^{m+n-p}$. Now the Hopf construction $\mathscr{H}$ applied to any map $f : S^m \times S^n \to S^{m+n-p}$ yields

$$\mathscr{H}f : S^{m+n+1} \to S^{m+n+1-p}, \qquad \mathscr{H}f(x\cos\theta, y\sin\theta) = (\cos 2\theta, f(x,y)\sin 2\theta).$$

If $m+n > 2p$, so that $\alpha = [\mathscr{H}f] \in \pi_p^S$, then $\alpha$ is said to be *represented* by $f$ or by the original bilinear map from which $f$ arose.

The key results on this kind of representation, to be found in [**1**, Theorem 1], [**11**, Theorem 1.1] and [**18**, Theorem 5.1] are

THEOREM 1. (a) *If $\alpha \in \pi_p^S$, then $2\alpha$ is bilinearly representable.*

(b) *There exist $\alpha \in \pi_*^S$ which are not so representable.*

Here we explore representability by means of immersed spheres in Euclidean space. Although the geometry is of course quite different, the homotopy theory involved has some overlap with the bilinear case. In particular we use the Kahn–Priddy theorem, in the form of the following strengthening of Mukai's corollary [**15**, Theorem 2].

THEOREM 2. *If $m \geqslant \mu(p)$ and $k \equiv 0 \,(\mathrm{mod}\, a_{\mu(p)+1})$, $k \geqslant p+2$, then the generalized J-homomorphism*

$$J_{k+m,m} : \pi_{k+p}(V_{k+m,m}) \to \pi_{k+p+m}(S^{k+m})$$

*is an epimorphism.*

In this assertion $\mu(p)$ is the minimum value of $m$ such that

$$\pi_{h+p}(V_{h+m,m}) \twoheadrightarrow \pi_{h+p}(V_{2h,h})$$

whenever the Hurwicz–Radon number $a_{p+2}$ divides $h$. It is a consequence of James periodicity [**10**, (2.7)] that surjectivity need be checked for only one such $h$. So $\mu(p) \leqslant p+1$ by connectivity. Moreover, for $p \neq 1, 3, 7$, one has $\mu(p) \leqslant p$ after Adams' vector fields on spheres data [7, (4.4)]; further information is deducible from, *e.g.*, [**9**]. The proof of this theorem is deferred to §3 below. In §2 we prove

THEOREM 3. (i) *Every element of $\pi_p^S$ is representable by an immersion of $S^n$ in $\mathbb{R}^{2n-p}$, whenever $n \equiv p \,(\mathrm{mod}\, a_{\mu(p)+1})$.*

(ii) *For all* $p, n$, *if* $\alpha \in \pi_p^S$ *is representable by an immersion of* $S^n$ *in* $\mathbb{R}^{2n-p}$ *which is regular homotopic to an embedding, then* $\alpha = 0$.

We consider too the case where the immersion of $S^n$ factors through real projective space $P^n$, and prove

THEOREM 4. (i) (a) *If* $\alpha \in \pi_p^S$, *then, for infinitely many* $n$, $2\alpha$ *is representable by an immersion of* $P^n$ *in* $\mathbb{R}^{2n-p}$.

(b) *There exist* $\alpha \in \pi_*^S$ *which are not so representable.*

(ii) *For all even* $p$, *if* $\alpha \in \pi_p^S$ *is representable by an embedding of projective space, then* $2\alpha = 0$.

Remarkably, Theorem 1(a) is deducible from Theorem 4(i)(a). Theorems 3 and 4 were announced in [**5**, (1.6)] where further connections between bilinear maps and immersions of projective spaces were discussed.

§2. *Representations of elements by immersions.* We obtain elements of $\pi_p^S$ from immersions of $S^n$ in $\mathbb{R}^{2n-p}$. The construction may be described in any of the following three ways.

(1) Compose the tangent bundle monomorphism with projection to the fibre, to yield a proper map $\tau S^n \to \mathbb{R}^{2n-p}$. Take its one-point compactification, yielding a map $S^n \times S^n \to S^{2n-p}$ to which the Hopf construction is applied.

(2) The geometric Euler class (called the *dual* in [**16**]) of the normal bundle is a $p$-manifold embedded in $S^n$ and canonically framed in $S^n \times \mathbb{R}^{p+1} \to \mathbb{R}^{n+p+1}$ (indeed, in $S^n$ itself [**16**]). Apply the Pontrjagin–Thom construction. Proposition 2 of [**6**] relates this description to that discussed below. In this context Theorem 3(i) asserts that, up to framed bordism, any framed $p$-manifold in $\mathbb{R}^{2(n-1)}$ arises in this way. This version was first suggested to me by Dr. B. Sanderson.

(3) Again compose the tangent bundle monomorphism with projection to the fibre, but this time only after having added a trivial line bundle and restricted to sphere bundles. Apply the Hopf construction to this (right-skew) map $f: S^n \times S^n \to S^{2n-p}$.

The third description is the most readily interpreted in homotopy theory. For, because $f$ comes from a vector-bundle morphism, it has adjoint $f': S^n \to V_{2n+1-p,n+1}$. Now recall that the generalized $J$-homomorphism

$$J_{2n+1-p,n+1}: \pi_n(V_{2n+1-p,n+1}) \to \pi_{2n+1}(S^{2n+1-p})$$

is defined by the Hopf construction on the evaluation map

$$\rho: S^n \times V_{2n+1-p,n+1} \to S^{2n-p},$$

as detailed in the proof of Lemma 2.1 below. Thus $J_{2n+1-p,n+1}(f')$ coincides with the element of $\pi_{2n+1}(S^{2n+1-p}) \cong \pi_p^S$ obtained by applying the Hopf construction directly to $f$.

However, in [3] we related $[f'] \in \pi_n(V_{2n+1-p,n+1})$ to the Smale invariant of (the regular homotopy class of) the original immersion. In consequence, the above construction factors as

$$\mathrm{Imm}(S^n, \mathbb{R}^{2n-p}) \xrightarrow{\text{Smale}} \pi_n(V_{2n-p,n}) \longrightarrow \pi_n(V_{2n+1-p,n+1}) \xrightarrow{J_{2n+1-p,n+1}} \pi_{2n+1}(S^{2n+1-p}).$$

When $n \equiv p \pmod{a_{\mu(p)+1}}$, then each of these three maps is surjective, by [17, Theorem A], stability, and Theorem 2 above, respectively, proving Theorem 3(i). For Theorem 3(ii), note that an embedding has a normal fibre bundle that is homotopy trivial [12]. It therefore has a normal vector field so that its Smale invariant factors through $\pi_n(V_{2n-p-1,n}) \cong \pi_n(V_{2n+1-p,n+2})$. Moreover, there is an exact sequence

$$\pi_n(V_{2n+1-p,n+2}) \longrightarrow \pi_n(V_{2n+1-p,n+1}) \longrightarrow \pi_{n-1}(S^{n-p-1}),$$

with $J_{2n+1-p,n+1}$ factoring through the latter map, as in (3.1) below.

We now study the case where $S^n$ is immersed as the composition of the double covering $S^n \to P^n$ with an immersion of $P^n$ in $\mathbb{R}^{2n-p}$, whence $f: S^n \times S^n \to S^{2n-p}$ is biskew. In order to prove Theorem 4(i)(a) we require as before that $n \equiv p \pmod{a_{\mu(p)+1}}$; we further need the existence of an immersion $g: P^n \to \mathbb{R}^{2n-p-1}$. So let $n = p + t(2^p - 1)a_{p+2}$. This gives

$$\begin{aligned} \alpha(n) &\geqslant \alpha(p) + \alpha(t(2^p - 1)) \\ &= \alpha(p) + \nu_2\left(\binom{t2^p}{t}\right) \\ &\geqslant \alpha(p) + p. \end{aligned}$$

Use [13] to guarantee the existence of $g$. From [4, Theorem 1.1(c), Lemma 6.1], $g$ determines an element $\gamma \in \pi_n(V_{2n-p,n+1})$ (there called its Smale invariant) such that, for any $\beta \in \pi_n(V_{2n+1-p,n+1})$,

$$i_*(\gamma) + \beta + (-1)^{n+1}\sigma_*(\beta) \in \pi_n(V_{2n+1-p,n+1})$$

is similarly induced from an immersion of $P^n$ in $\mathbb{R}^{2n-p}$. Here, composition of $\gamma$ by $i: V_{2n-p,n+1} \hookrightarrow V_{2n+1-p,n+1}$ corresponds to composition of $g$ by $\mathbb{R}^{2n-p-1} \hookrightarrow \mathbb{R}^{2n-p}$, and $\sigma$ is the involution on $V_{2n+1-p,n+1}$ which changes the sign of every element of the $(n+1)$-frame. Now suppose $\alpha \in \pi_{2n+1}(S^{2n+1-p})$. From Theorem 2, $\alpha = J_{2n+1-p,n+1}(\beta)$ for some $\beta$ as above, while (3.1) implies that

$$J_{2n+1-p,n+1} \circ i_* = \pm E^{n+2} \circ {}_{n+1}\delta_0 \circ i_* = 0.$$

So Theorem 4(i)(a) follows from the next lemma.

Lemma 2.1.

$$J_{s,t} \circ \sigma_* = (-1)^t J_{s,t}: \pi_r(V_{s,t}) \longrightarrow \pi_{r+t}(S^s).$$

*Moreover, if* $r+t \leqslant 2s-1$, *then*

$$(-1)^s J_{s,t} = (-1)^t J_{s,t}.$$

*Proof.* Let $A_t : S^{t-1} \to S^{t-1}$ be the antipodal involution, of degree $(-1)^t$. Define $\rho : S^{t-1} \times V_{s,t} \to S^{s-1}$ by

$$\rho((x_1, \ldots, x_t), (\underline{v}_1, \ldots, \underline{v}_t)) = \Sigma x_i \underline{v}_i .$$

For $f : S^r \to V_{s,t}$, a representative of $J_{s,t} \circ \sigma_*[f]$ is given by the Hopf construction on the composition

$$S^{t-1} \times S^r \xrightarrow{1 \times f} S^{t-1} \times V_{s,t} \xrightarrow{1 \times \sigma} S^{t-1} \times V_{s,t} \xrightarrow{\rho} S^{s-1} .$$

The required results follow from the commutativity of the diagram.

$$\begin{array}{ccccc}
 & & S^{t-1} \times V_{s,t} & & \\
 & {}^{A_t \times 1}\nearrow & & \searrow^{\rho} & \\
S^{t-1} \times V_{s,t} & \xrightarrow{1 \times \sigma} & S^{t-1} \times V_{s,t} & \xrightarrow{\rho} & S^{s-1} \\
 & {}_{\rho}\searrow & & \nearrow_{A_s} & \\
 & & S^{s-1} & &
\end{array}$$

An immediate consequence of (2.1) is that elements of $\pi_{2n+1}(S^{2n+1-p})$ derived from our construction must have order 2, if $n+p$ is odd. Similarly, if we begin with an embedded $P^n$ in $\mathbb{R}^{2n-p}$, then the map $S^n \times S^n \to S^{2n-p}$ is invariant under the interchange involution on $S^n \times S^n$, [2, Theorem 1(b)], and therefore yields an element of $\pi_{2n+1}(S^{2n+1-p})$ of order 2, if $n$ is even. Together these two observations imply Theorem 4(ii).

Finally, Theorem 4(i)(b) may be proved by an argument involving biskew maps analogous to that used in [1] for bilinear maps. Indeed that was the original proof, as this work was originally done in ignorance of [11], owing to the present author having been in Nigeria at the time of its publication. More simply, however, one can instead use the striking result [11, Theorem 3.1], in the form that any element of $\pi_*^S$ obtained from the Hopf construction on a biskew map is bilinearly representable. Then, in view of our analysis above, Theorem 4(i)(b) is an immediate consequence of Theorem 1(b); likewise Theorem 1(a) follows at once from Theorem 4(i)(a).

§3. *Homotopy details.* We indicate how Theorem 2 is proved, and mention a couple of its homotopy consequences. The key diagram is the following, which commutes (up to sign) for *all* $k, p, m$ after naturality, [15, Proposition 3], [8, Theorem 2] and [6, (7.14)].

$$\begin{array}{ccccccc}
\pi_{k+p}(P_k^{k+m-1}) & \xrightarrow{i_*} & \pi_{k+p}(V_{k+m,m}) & \xrightarrow{\delta} & \pi_{k+p-1}(O_k) & \xrightarrow{J} & \pi_{2k+p-1}(S^k) \\
\downarrow q_* & & \downarrow J_{k+m,m} & \searrow {}_{m}\delta_0 & \downarrow P_* & & \downarrow H \\
\pi_{k+p}(S^k) & \underset{\sim}{\xrightarrow{E^m}} & \pi_{k+p+m}(S^{k+m}) & \xleftarrow{E^{m+1}} & \pi_{k+p-1}(S^{k-1}) & \xrightarrow{E^k} & \pi_{2k+p-1}(S^{2k-1})
\end{array} \qquad (3.1)$$

Note that $i_*$, $E^{m+1}$, $E^k$ are epimorphisms for $k = p+2$, isomorphisms for $k > p+2$.

Fixing $p > 0$, suppose $a_{p+2}|h$. In [15], Mukai deduced from the Kahn–Priddy theorem that

$$_{p+2}\delta_0 : \pi_{h+p}(V_{2h,h}) \twoheadrightarrow \pi_{h+p-1}(S^{h-1})$$

so long as $h > p+2$. However this proviso may be dropped after consideration of the two cases $p+2 = h$, namely

$$p = 2: \quad \pi_6(V_{8,4}) \xrightarrow{{}_4\delta_0} \pi_5(S^3) \longrightarrow \pi_5(V_{8,5}),$$

$$p = 6: \quad \pi_{14}(V_{16,8}) \xrightarrow{{}_8\delta_0} \pi_{13}(S^7) \longrightarrow \pi_{13}(V_{16,9}).$$

In each case the latter map of the exact sequence is trivial, after [9]. From the definition of $\mu(p)$,

$$_{\mu(p)}\delta_0 : \pi_{h+p}(V_{h+\mu(p),\mu(p)}) \longrightarrow \pi_{h+p}(V_{2h,h}) \longrightarrow \pi_{h+p-1}(S^{h-1})$$

is therefore onto.

Now suppose given $k \geqslant p+2, k \equiv 0 \ (\mathrm{mod}\, a_{\mu(p)+1})$ as in the statement of the theorem. Retain $h \equiv 0 \,(\mathrm{mod}\, a_{p+2})$ but specialize to $h \geqslant k$ (*e.g.* $h = ka_{p+2}$ would do). Because $k \equiv 0 \,(\mathrm{mod}\, a_{\mu(p)+1})$ there exists a cross-section class $\theta \in \pi_{h-k-1}(V_{h-k,\mu(p)+1})$ and so, by [10, (2.7)], a commuting square

$$\begin{array}{ccc} \pi_{k+p}(V_{k+\mu(p),\mu(p)}) & \xrightarrow{{}_{\mu(p)}\delta_0} & \pi_{k+p-1}(S^{k-1}) \\ \big\downarrow \theta''_{\#} & & \big\downarrow E^{h-k} \\ \pi_{h+p}(V_{h+\mu(p),\mu(p)}) & \xrightarrow{{}_{\mu(p)}\delta_0} & \pi_{h+p-1}(S^{h-1}) \end{array}$$

in which $E^{h-k}$ is an isomorphism provided $p < k-2$ (or $p = k-2 = 2, 6$ from the classical exactness of (3.2) below), and both $\theta''_{\#}$ (for $p \leqslant k-1$) and the lower $_{\mu(p)}\delta_0$ are epimorphisms. Moreover, $_{\mu(p)}\delta_0$ factors as

$$_{\mu(p)}\delta_0 : \pi_{k+p}(V_{k+\mu(p),\mu(p)}) \longrightarrow \pi_{k+p}(V_{k+m,m}) \longrightarrow \pi_{k+p-1}(S^{k-1})$$

whenever $m \geqslant \mu(p)$. Hence the theorem and its variants follow from (3.1).

One immediate consequence concerns the sequence

$$0 \longrightarrow \pi_q(S^n) \xrightarrow{E} \pi_{q+1}(S^{n+1}) \xrightarrow{H} \pi_{q+1}(S^{2n+1}) \longrightarrow 0 \qquad (3.2)$$

which is classically known to be (split) exact for $n = 1, 3, 7$, and otherwise is exact for $q < 2n-1$ but not for $q = 2n-1, 2n$. Application of Theorem 2 to (3.1) reveals when $H$ is surjective in the $E$–$H$–$P$ sequence.

COROLLARY 3.3. *Suppose $r < 3n$ is such that $a_{\mu(p)+1}|(n+1)$ whenever $p \leqslant r-2n$. Then* (3.2) *is exact in the range $2n < q < r$.*

Geometrically, this says that two compact manifolds embedded in $\mathbb{R}^q$ with codimension $n$ are framed bordant, if, and only if, they are framed bordant in $\mathbb{R}^{q+1}$.

We conclude with an application to the set $\pi_{n+p}^{\text{Proj}}(S^n)$ of homotopy classes of maps $S^{n+p} \to S^n$ which factor through $P^{n+p}$.

COROLLARY 3.4. *If $a_{\mu(p)+1}|(n+p+1) < 2n$, then*

$$\pi_{n+p}^{\text{Proj}}(S^n) = \pi_{n+p}(S^n).$$

For the proof, choose $m \geqslant \mu(p)+1$ and $M \equiv 0 \pmod{a_{p+2}}$ with $M \geqslant 2n+p+2$. Let $j: S^{M-n-p-2} \to V_{M+m-n-p-1,m+1}$ be the inclusion map. From Theorem 2, (3.1) and [14, §2 Corollary 2]

$$\begin{aligned}\pi_{M-n-2}(S^{M-n-p-2}) &= \operatorname{Im}_m \delta_0 \\ &= \operatorname{Ker} j_* \\ &= \operatorname{Ker} i_*^{-1} \circ j_* \\ &= E^{M-2n-p-2}\pi_{n+p}^{\text{Proj}}(S^n),\end{aligned}$$

so that the result follows on desuspension.

As an example, (3.4) shows immediately that for $p = 9$, say, $\pi_{31}^{\text{Proj}}(S^{22}) = \pi_{31}(S^{22})$, whereas previously an Adams spectral sequence argument was required for this. Further, when $p > 20$, so that the calculations of [14] no longer apply, (3.4) provides new information.

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