

Ex-Homotopy Comparison Theorems

A. J. Berrick

Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511

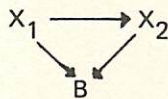
ABSTRACT

By means of abstract homotopy theory, it is shown that an ex-map which is a homotopy equivalence is in fact an ex-homotopy equivalence (given suitable conditions). The power of this categorical approach is that it has much broader applicability than directly topological arguments.

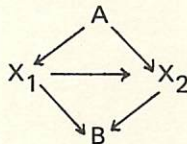
INTRODUCTION AND NOTATION

This note argues that abstract homotopy theory [17], [19], [20] provides a convenient framework for the study of ex-homotopy theory [12], [15], [16]. The set $[X, E]_B$ of B-homotopy classes of B-maps (defined in the next paragraph) in a category C with homotopy system is considered in §1, generalising to C_B the treatment of Top_B appearing in [5]. The advantage of this approach is seen in §2 when we choose C embedded in Top^A . Then abstract homotopy theorems about fibre-homotopy equivalence become theorems on ex-homotopy equivalence. Independent (but contemporaneous) treatments of an abstract homotopy-theoretic approach to ex-homotopy theory are presented in [4] and [18].

Our terminology is standard, namely, for an object B in a category C , the category C_B "over B " has as objects morphisms $X \rightarrow B$ in C and as morphisms commuting triangles over B in C :



Then, for any object A in C , the category C^A (of objects "under A ") is most readily defined as the opposite category of C_A . Finally, by performing both constructions simultaneously, one obtains C_B^A with objects $A \rightarrow X \rightarrow B$ and as morphisms commuting diagrams in C :



Thus, when C is the category Top of topological spaces and continuous maps, then Top_B^A provides the setting for ex-homotopy theory. (When A or B is the empty space, then Top_B^A specialises to Top_B , Top^A respectively.)

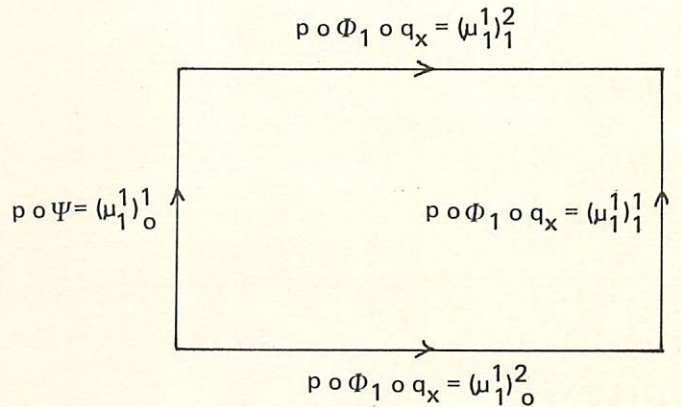
1. THE SET $[X, E]_B$

We begin with a technical lemma. Notation is as in [17]; however, to aid the uninitiated we make the following brief remarks. A homotopy system $\underline{J} = (I, j_0, j_1, q)$ on a category C comprises a covariant functor $I : C \rightarrow C$ together with three natural transformations

$j_0, j_1 : id_C \rightarrow I, q : I \rightarrow id_C$ satisfying $q \circ j_0 = q \circ j_1 = id$. The principal example is the homotopy system \underline{J} of Top , where I corresponds to the functor $X \rightarrow X \times [0, 1]$ and natural transformations j_0, j_1, q to the respective maps $x \mapsto (x, 0), x \mapsto (x, 1), (x, t) \mapsto x$ ($x \in X, t \in [0, 1]$). Kan conditions on \underline{J} are assumptions of the existence of morphisms ("solutions") which allow certain diagrams to be filled in to form commuting polygons; in the case of \underline{J} on Top (and Top_B^A) the conditions DNE(n), E(n) are satisfied by virtue of the n-dimensional "bucket lemma": $[0, 1]^{n-1}$ is a retract of $[0, 1]^n$. Finally, the term h-fibration is defined prior to Lemma 1.6 below, since an h-fibration in C is just what is there called an h-fibration with respect to C' in the special case $C' = C$.

LEMMA 1.1. Let I be a homotopy system on a category C which satisfies the Kan-conditions DNE(2), E(3). Let $p : E \rightarrow B$ be an h-fibration, and suppose $\Phi, \Phi' : IX \rightarrow E$ have $po\Phi = po\Phi' : IX \rightarrow B$ and $\Phi_0 \cong \Phi'_0$. Then $\Phi_1 \cong \Phi'_1$.

Proof: Since \underline{J} satisfies DNE(2), so does \underline{J}_B by (3.7) of [17]. Let $\Psi : IX \rightarrow E$ be obtained as λ_1^1 of a solution λ to the (2,1,1)-equation γ with $\gamma_0^1 : \Phi_0 \cong \Phi'_0, \gamma_0^2 = \Phi, \gamma_1^2 = \Phi'$. Now consider the (3,1,1)-equation γ with $\gamma_0^1 = p \circ \Phi_0 \circ q_X \circ q_{IX}, \gamma_0^2 = \gamma_1^2 = \gamma_1^3 = p \circ \Phi \circ q_{IX}, \gamma_0^3 = p \circ \lambda$. Its solution α has $\alpha_1^1 = \lambda_1^1 \alpha : IIX \rightarrow B$ representable as



Since $p : E \rightarrow B$ is an h-fibration, there exists $\Theta : IIX \rightarrow E$ with $p \circ \Theta = \mu_1^1$ and $\Theta \circ j_0 \cong \Psi$. So

$$\Phi_1 = \Psi_0 \tilde{B} \Theta \circ j_{0IX} \circ j_{0XB} \tilde{B} \Theta \circ j_{1IX} \circ j_{0XB} \tilde{B}$$

$$\tilde{B} \Theta \circ j_{1IX} \circ j_{1XB} \tilde{B} \Theta \circ j_{0IX} \circ j_{1XB} \tilde{B} \Psi_1 = \Phi'_1,$$

whence $\Phi_1 \tilde{B} \Phi'_1$ (\tilde{B} being an equivalence relation, by (3.8) of [17]).

Henceforth assume that \underline{J} satisfies DNE(2), E(3). Define, for $p: E \rightarrow B$ an h-fibration,

$$[X(h), E(p)]_B$$

– with h and/or p omitted in the absence of ambiguity – as the set of ‘homotopy over B ’ equivalence classes of morphisms $f: X \rightarrow E$ such that $p \circ f = h: X \rightarrow B$. We define certain mappings among these sets, to show that they may be considered as homotopy invariants of $h: X \rightarrow B$ and $p: E \rightarrow B$.

Given a morphism $\varphi: IX \rightarrow B$, define

$$\varphi_* : [X(\varphi_0), E]_B \rightarrow [X(\varphi_1), E]_B$$

by setting, for $f: X \rightarrow E$ with $p \circ f = \varphi_0: X \rightarrow B$,

$$\varphi_*(f) = \Phi_1 \in [X(\varphi_1), E]_B$$

where $\Phi: IX \rightarrow E$ has $p \circ \Phi = \varphi$, $\Phi_0 \tilde{B} f$. Lemma 1.1 shows φ_* to be well-defined. (Note that there tends to be the usual confusion in notation of equivalence classes with their representatives.)

Given morphisms $\varphi: IX \rightarrow B$, $\psi: IX \rightarrow B$ with $\psi_0 = \varphi_1$, a morphism $\psi\varphi: IX \rightarrow B$ may be defined as λ^2 of a solution λ to the (2,2,0)-equation γ with $\gamma_0^1 = \varphi$, $\gamma_1^2 = \psi$, $\gamma_1^1 = \psi_1 \circ q_X$. If λ is lifted to $\Lambda: IIX \rightarrow E$ with $p \circ \Lambda = \lambda$, $\Lambda_0^1 \circ j_{0IX} \tilde{B} \Phi$ where $p \circ \Phi = \varphi$, $\Phi_0 \tilde{B} f: X \rightarrow E$, then

$$\psi_* \circ \varphi_*(f) = (\Lambda_1^2)_1 = (\Lambda_1^1)_1 \tilde{B} (\Lambda_1^1)_0 =$$

$$(\Lambda_0^2)_1 = (\psi\varphi)_*(f).$$

So $\psi_* \circ \varphi_* = (\psi\varphi)_*$.

Now ψ can always be chosen ‘inverse’ to φ , that is as λ_1^2 of a solution to the (2,2,1)-equation with $\gamma_0^1 = \varphi$, $\gamma_0^2 = \gamma_1^1 = \varphi_0 \circ q_X$. This immediately yields the following generalisation of [5] (2.5).

PROPOSITION 1.2. Let $\varphi: IX \rightarrow B$ be a homotopy in C where I satisfies DNE(2), E(3). Then

$$\varphi_* : [X(\varphi_0), E]_B \rightarrow [X(\varphi_1), E]_B$$

is a bijection.

Given an h-fibration $p: E \rightarrow B$ in C , we now define a contravariant functor H from C_B to Ens , the category of sets and set functions. If $h: X \rightarrow B$ is an object in C_B , set

$$H(h: X \rightarrow B) = [X(h), E]_B,$$

If $g: X_1 \rightarrow X_2$ with $h_2 \circ g = h_1: X_1 \rightarrow B$, $h_2: X_2 \rightarrow B$, let $H(g)$, written g^* , be the function

$$g^* : [X_2(h_2), E]_B \rightarrow [X_1(h_1), E]_B$$

sending $k: X_2 \rightarrow E$ over B to $k \circ g: X_1 \rightarrow E$. It is trivial to verify that H is indeed a contravariant functor.

Suppose given $g_0, g_1: X_1 \rightarrow X_2$, $h: X_2 \rightarrow B$ and a homotopy $\varphi: IX_1 \rightarrow X_2$ from g_0 to g_1 . Then $h \circ \varphi: IX \rightarrow B$ is a homotopy from $h \circ g_0$ to $h \circ g_1$; by Proposition 1.2, it induces a bijection

$$(h \circ \varphi)_* : [X_1(h \circ g_0), E]_B \rightarrow [X_1(h \circ g_1), E]_B.$$

Thus H is in a sense a homotopy functor. This sense may be made more precise when $g: X_1 \rightarrow X_2$ is an h-equivalence with inverse $f: X_2 \rightarrow X_1$ say, and $\varphi: \text{id}_{X_2} \simeq g \circ f$. It follows from Lemma 1.1 and Proposition 1.2 (since both $f^* g^*(k) = k \circ g \circ f$ and $(h \circ \varphi)_*(k)$ are homotopic to $k: X_2 \rightarrow E$ through homotopies ‘covering’ $h \circ \varphi$) that

$$f^* g^* = (h \circ \varphi)_* : [X_2(h), E]_B \rightarrow [X_2(h \circ g \circ f), E]_B,$$

is a bijection. Likewise $g^* f^*$ is a bijection, whence g^*, f^* are too. This establishes the ‘only if’ part of the following characterisation of h-fibrations; the ‘if’ part follows a fortiori on letting $g = j_{0X}: X \rightarrow IX$ (with h-inverse $q_X: IX \rightarrow X$).

THEOREM 1.3. $p: E \rightarrow B$ is an h-fibration if and only if for all h-equivalences $g: X_1 \rightarrow X_2$ the function

$$g^* : [X_2, E]_B \rightarrow [X_1, E]_B$$

is a bijection.

One may alternatively concentrate on E , considering the map $\pi_* : [X, E(p_1)]_B \rightarrow [X, E(p_0)]_B$ induced from a homotopy $\pi: IE \rightarrow B$ from p_0 to p_1 (both h-fibrations). Analogously to Proposition 1.2 we obtain

PROPOSITION 1.4. Let $\pi: IE \rightarrow B$ be a homotopy from p_0 to p_1 , $p_v: E \rightarrow B$ an h-fibration, $v = 0, 1$. Then

$$\pi_* : [X, E(p_1)]_B \rightarrow [X, E(p_0)]_B$$

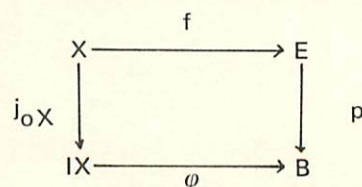
is a bijection.

In this case the result derived analogously to the ‘only if’ part of Theorem 1.3 takes the following relatively familiar form.

THEOREM 1.5. Let $g: E_1 \rightarrow E_2$ be an h-equivalence with $p: E_2 \rightarrow B$, $p \circ g: E_1 \rightarrow B$ h-fibrations. Then composition with g induces a bijection

$$g_* : [X, E_1]_B \rightarrow [X, E_2]_B.$$

Note that (6.1) of [17] is a corollary of this theorem. Actually, our methods prove two stronger results than the above statement of the theorem. Firstly, we can say that g_* has a left (resp. right) inverse as g has a left (resp. right) h-inverse. Secondly, let C' be a full subcategory of C closed under the homotopy system \underline{J} in C . We say that a morphism $p: E \rightarrow B$ in C is an h-fibration with respect to C' if for any object X in C' and commuting diagram



there exists a "diagonal" $\Phi: IX \rightarrow E$ with $p \circ \Phi = \varphi: IX \rightarrow B$ and $\Phi \circ \tilde{f}: X \rightarrow E$. For such a C' we then have

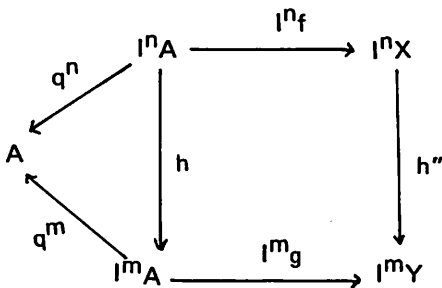
LEMMA 1.6. The conclusion of Theorem 1.5 holds if $p, p \circ g$ are taken merely to be h-fibrations with respect to C' , where X is an object in C' .

At this point it is worth observing that the derivations of g^*, g_* , dualise to provide a characterisation of h-cofibrations; for example, the dual of Theorem 1.3 generalises (2.2) of [3].

2. EX-HOMOTOPY RESULTS

We turn now to consideration of ex-homotopy theory, that is, Top_B^A . As in [17] a homotopy system \downarrow on a category C induces a homotopy system on C_B . However, if $A \subset C, C^A$ does not in general inherit a homotopy system from \downarrow yielding the relation of homotopy under A . We remedy this deficiency.

Define $C^A \downarrow$ as the category whose objects are morphisms of C of the form $I^n f: I^n A \rightarrow I^n X, n \geq 0$, and whose morphisms (h', h'') are given by commuting diagrams



Then \downarrow induces a homotopy system $\downarrow^A = (I^A, j_0^A, j_1^A, q^A)$ on $C^A \downarrow$ with $I^A(I^n f: I^n A \rightarrow I^n X) = (I^{n+1} f: I^{n+1} A \rightarrow I^{n+1} X)$, $I(h', h'') = (Ih', Ih'')$, $j_V^A(I^n f: I^n A \rightarrow I^n X) = (j_V, j_V), V = 0, 1, q^A(I^n f: I^n A \rightarrow I^n X) = (q, q)$. Then $h'' \overset{\sim}{\circ} h': X \rightarrow Y$ in $C^A \downarrow$ if and only if $(id_A, h'') \simeq (id_A, h')$ with respect to \downarrow^A in $C^A \downarrow$.

Lemma 1.6 now yields the following theorem (cf. (6.1) of [15]). (Recall that \mathbb{I}, \mathbb{I}^A satisfy DNE(2), E(3), where \mathbb{I} is the standard homotopy system on Top .)

THEOREM 2.1 If $s: A \rightarrow X$ is a closed cofibration and $p_V: E_V \rightarrow B$ is a fibration ($v = 1, 2$) with $g: E_1 \rightarrow E_2$ in Top_B^A a homotopy equivalence, then the function

$$g_*: [X, E_1]_B^A \rightarrow [X, E_2]_B^A$$

induced by composition with g , is a bijection.

Proof: Set $C = \text{Top}^{A, \mathbb{I}}$. Let the objects of C' be maps $j \times id: A \times [0, 1]^n \rightarrow Y \times [0, 1]^n$ where $j: A \rightarrow Y$ is a closed cofibration, $n \geq 0$. By virtue of [30] Theorem 1, [30] Theorem 4 (cf. [13]) states that p_1, p_2 are fibrations with respect to C' . Lemma 1.6 now applies. (Recall from [7] Satz (2.18) that g is already an h-equivalence in Top^A .)

An application of this result may be found in [3].

Using (7.6.22), (7.8.9) of [29] instead of [30] leads to the same conclusion when p_1, p_2 are weak fibrations (Serre fibre spaces), provided that (X, A) is a relative CW-complex. Similarly one may argue on g^* to obtain the following theorem.

THEOREM 2.2. If $s_V: A \rightarrow X_V$ is a closed cofibration [resp. (X_V, A) is a relative CW-complex] ($v = 1, 2$) with $g: X_1 \rightarrow X_2$ in Top_B^A a homotopy [resp. weak] equivalence, and if $p: E \rightarrow B$ is a [resp. weak] fibration, then composition with g induces a bijection

$$g^*: [X_2, E]_B^A \rightarrow [X_1, E]_B^A$$

Two-fold application of this theorem yields a sufficient condition for a homotopy equivalence to be an ex-homotopy equivalence. (The obvious statement for weak fibrations is omitted.) (cf. (6.1) of [8].)

COROLLARY 2.3. Let $g: E_1 \rightarrow E_2$ be a map in Top_B^A which is a homotopy equivalence. If $s_V: A \rightarrow E_V$ is a closed cofibration and $p_V: E_V \rightarrow B$ is a fibration ($v = 1, 2$), then g is an ex-homotopy equivalence.

Other forms of this comparison theorem may be obtained by combining the above techniques with a diligent search through the literature. Applications include metric ANRs [1] (2.4), simplicial sets and Kan fibrations [23] (7.17), [28], shape and s-fibrations [14], [24], and equivariant fibre homotopy equivalence [2] § 10, [25], [26], [27]. (Also cf. [22] § 8, discussing $A \subset E_1$.) Local-global versions of the result follow from [6], [10], [11].

Suppose $A = B$ in Corollary 2.3: it is natural to ask "When is the section of a fibration a closed cofibration?". Theorem 2.4 below provides a sufficient condition.

As in [29] p. 256, define a bundle pair, base B , to comprise a total pair (E, \dot{E}) , fibre pair (F, \dot{F}) and a projection map $p: E \rightarrow B$ such that there exists an open covering $\{V\}$ of B and for each $V \in \{V\}$ a homeomorphism $\Phi_V: V \times (F, \dot{F}) \rightarrow (p^{-1}(V), p^{-1}(V) \cap \dot{E})$ over V . In the case $\dot{F} = f_0$ a point in F , the maps $\Phi_V: V \times f_0 \rightarrow E$ combine to induce a canonical section $s: B \rightarrow E$ of p . We say the bundle pair is numerable if some such covering $\{V\}$ is, i.e. $\{V\}$ admits a refinement by a locally finite partition of unity.

THEOREM 2.4. The canonical section of a numerable bundle pair, whose fibre pair consists of a space and its non-degenerate base-point, is a cofibration; the projection map is a fibration. If further, either

- (i) the base is compact and the fibre is a T_1 -space, or
- (ii) both base and fibre are Hausdorff spaces, then the section is a closed cofibration.

Proof: The projection map is a fibration by [8] Theorem (4.8). Since f_0 is a non-degenerate base-point of the fibre F , the inclusion map is a cofibration. Then, by [31] Theorem 6, so is the inclusion $V \times f_0 \hookrightarrow V \times F$. Thus, by [7] (1.8), (1.10)

$s|_V: V \xrightarrow{id} V \times F_0 \hookrightarrow V \times F \xrightarrow{\Phi_V} p^{-1}(V)$ is a cofibration. In other words, the pair $(p^{-1}(V), s(B) \cap p^{-1}(V))$ has the homotopy extension property, for all sets V in the

numerable covering $\{V\}$ of B . By [9], so does the pair $(E, s(B))$,— i.e. s is a cofibration.

Finally, (i) trivially ensures that $s(B)$ is closed in E , while (ii) is equivalent to E 's being Hausdorff. In which case [7] (1.17) implies that $s(B)$ is closed in E .

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