GROUP EPIMORPHISMS PRESERVING PERFECT RADICALS,

AND THE PLUS-CONSTRUCTION

A. J. Berrick

If one uses the plus-construction approach to algebraic K-theory, then various key problems (for example, obstruction to excision) have natural topological formulations (such as a fibre sequence being plus-constructive). Here we swing the pendulum back, to obtain an algebraic setting for discussing these matters. Central to this viewpoint is a certain class of group homomorphisms.

1. Definition

To define the class, recall that a group P is perfect if equal to its commutator subgroup [P,P]. Because the homomorphic image of a commutator is a commutator, so

1.1 *the homomorphic image of a perfect group is again perfect.*

The class of perfect subgroups of a given group G is therefore closed under automorphisms of G. It is also evidently closed under group union, because if each subgroup H_α's generated by its commutators, then so is the subgroup the H_α's all generate. Thus the class admits a maximum element, the *perfect radical* PG of G, which must be a fully invariant subgroup. The construction is functorial because, from 1.1,

1.2 *if* $\phi : G \to H$ *is a homomorphism, then* $\phi(PG) \leq P(\phi G) \leq PH$.

An alternative approach here is to view PG as the intersection of the transfinite derived series of G.

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We enquire under what conditions equality holds in (1.2). Since it is the image ϕG to which we wish to restrict consideration, suppose ϕ to be an epimorphism. We seek hypotheses to ensure that $\phi(PG) = P(\phi G)$, in other words, that ϕ is EP^2R - an Epimorphism (or, in context, Extension) Preserving Perfect Radicals.

For an extreme example of an epimorphism which is not EP^2R, let the extension $R \rightarrowtail F \overset{\phi}{\twoheadrightarrow} P$ correspond to a free presentation of a perfect group P. Thus F, being free, has only free non-trivial subgroups, and no free group can be perfect. So although $PF = 1$, making $\phi PF = 1$, we have $PP = P$. On the other hand, there are no examples from finite group theory : we shall see that any surjection with finite kernel is EP^2R.

Note that solubility of G forces triviality of PG (that is, G is *hypoabelian*) since among n-th derived groups we must have $(PG)^{(n)} \leq G^{(n)}$. In fact,

1.3 *The following three conditions are equivalent :-*

(i) G *is soluble;*

(ii) $PG = 1$ *and for some* i $G^{(i)}$ *is finite;*

(iii) $PG = 1$ *and for some* j $G^{(j)}/Z(G^{(j)})$ *is finite.*

The proof is an easy exercise, save for a lemma of Schur to the effect that $G^{(i)}/Z(G^{(i)})$ finite implies $G^{(i+1)}$ finite.

2. Algebraic results

Our main purpose here is to establish (in (2.3) below) conditions on a group G and normal subgroup $N = \text{Ker } \phi$ sufficient to ensure that an epimorphism $\phi : G \twoheadrightarrow Q$ is EP^2R. Several such conditions are collated as Proposition 2.3. We find it useful (for Lemma 3.7) to relate one of these to the following question. Since evidently the composition of EP^2R maps is EP^2R, what can be said about the converse? Again, it is easy to see that if $\psi : G \to H$, $\phi : H \to Q$ have composite $\phi \circ \psi : G \to Q$ EP^2R then ϕ must also be EP^2R; so, under what conditions will ψ be EP^2R as well?

LEMMA 2.1 : *Let* $\psi : G \to H, \phi : H \to Q$ *be epimorphisms such that, for some finite* n,

$$(\text{Ker } \phi)^{(n)} \leq \psi PG.$$

Then

(a) ϕ *is* EP^2R; *and*

(b) ψ *is* EP^2R *if and only if* $\phi\circ\psi$ *is* EP^2R.

Proof. Let $J \trianglelefteq H$ denote the inverse image $\phi^{-1}(PQ)$. From the lemma (2.2) below, $J^{(n)} = J^{(n+1)}(\text{Ker }\phi)^{(n)}$. However it is known that $(\text{Ker }\phi)^{(n)} \le \psi PG \le PH = (PH)^{(n+1)} \le J^{(n+1)}$ (after (1.2)). So $J^{(n)} \le J^{(n+1)}$, whence $J^{(n)}$, being perfect, lies in - and is therefore equal to - PH. Thus PH has the same image as $J^{(n)}$, namely $PQ^{(n)} = PQ$. This establishes (a), and thence one implication of (b), when $\phi\circ\psi$ is the composite of EP^2R maps.

Conversely, if $\phi\circ\psi$ is EP^2R, then PH and ψPG both have image PQ, from which one deduces that $PH = (\psi PG).\text{Ker }\phi$. Then

$$PH = PH^{(n)} = ((\psi PG).\text{Ker }\phi)^{(n)} = \psi PG.(\text{Ker }\phi)^{(n)}$$

because $\psi PG \trianglelefteq H$. By assumption this last subgroup lies in ψPG, leaving $\psi PG = PH$ as required.

I am grateful to Prof. B. Hartley for suggesting an argument which led me to the following lemma.

LEMMA 2.2 : *Let* $K \hookrightarrow J \twoheadrightarrow P$ *be an extension where* P *is a perfect group. Then for all finite* $m \ge n \ge 0$,

$$J^{(n)} = J^{(m)}K^{(n)}.$$

Proof. For each fixed m, argue by induction on n. The case $n = 0$ results from the fact that $P = P^{(m)}$, with the latter the image of $J^{(m)}$. On the other hand, whenever $J^{(n-1)} = J^{(m)}K^{(n-1)}$ it follows that

$$J^{(n)} = [J^{(m)}K^{(n-1)}, J^{(m)}K^{(n-1)}] = J^{(m)}K^{(n)}.$$

An immediate application of Lemma 2.1(a) (or (b) with ψ as the identity map) is (iii) of the following result.

PROPOSITION 2.3 : *An extension* $N \hookrightarrow G \xrightarrow{\phi} Q$ *is* EP^2R *provided either*

(i) *it is split;*

(ii) $G^{(m)} \le N.PG$ *for some finite* m;

(iii) $N^{(n)} \le PG$ *for some finite* n; *or*

(iv) *the homomorphism* $G \to \mathrm{Aut}(N/PN)$, *induced by conjugation, has hypoabelian image in* $\mathrm{Out}(N/PN) = \mathrm{Aut}(N/PN)/\mathrm{Inn}(N/PN)$.

REMARKS. Of the above hypotheses, (iii) is perhaps the most useful, in its application to central and perfect extensions, or, more generally, to extensions whose kernel has finite derived length. The finite ordinals occurring in (ii), (iii) may not be replaced by infinite ones, as the example of a free presentation of a perfect group reveals. Finally, the study of the outer automorphisms induced by an extension is pursued further in [4].

Proof. (i) is just a two-fold application of (1.2), since if $\psi : Q \to G$ is right-inverse to ϕ, then $\phi PG \geq \phi\psi PQ = PQ$. For (ii) we also borrow from (1.2) the fact that $\phi PG \leq PQ$. Now $PQ \leq Q^{(m)}$ for all finite m, so that $PQ \leq \phi(G^{(m)}.N) \leq \phi(PG.N) = \phi PG$.

Thus only (iv) demands further discussion, in the course of which we shall make use of the following immediate consequence of (iii).

COROLLARY 2.4 : *If* $P \trianglelefteq G$ *is perfect, then* $P(G/P) = PG/P$.

By a little extra work, we shall in fact establish the following strengthening of (iv), whose converse will be proved by topological means in §3 below. That its automorphism homomorphism condition is weaker than (iv) may be seen from the commuting diagram

$$\begin{array}{ccccc} G & \to & \mathrm{Aut}(N) & \to & \mathrm{Aut}(N/PN) \\ \downarrow & & \downarrow & & \downarrow \\ Q & \to & \mathrm{Out}(N) & \to & \mathrm{Out}(N/PN) \end{array} .$$

PROPOSITION 2.5 : *An extension* $N \hookrightarrow G \overset{\phi}{\twoheadrightarrow} Q$ *induces the trivial homomorphism* $PQ \to \mathrm{Out}(N/PN)$ *if and only if both*

a) ϕ *is* EP^2R; *and*

b) $[PG,N] = PN$.

In fact, it is only necessary to prove this result when $PN = 1$, since by (2.4) ϕ is EP^2R if and only if $N/PN \hookrightarrow G/PN \twoheadrightarrow Q$ is. (2.4) also implies the equivalence of $[PG,N] = PN$ and $[P(G/PN), N/PN] = 1$. We shall therefore demonstrate this notationally simpler, special case...

LEMMA 2.6 : *An extension* $N \hookrightarrow G \overset{\phi}{\twoheadrightarrow} Q$ *with* $PN = 1$ *induces the trivial map* $PQ \to \mathrm{Out}(N)$ *if and only if both*

a) ϕ *is* EP^2R; *and*

b) $[PG,N] = 1$.

To obtain (a) first, we restrict to the induced extension $N \hookrightarrow H \twoheadrightarrow PQ$ over PQ. Proceeding as in [4], we observe that the homomorphism $H \to \mathrm{Out}(N)$ has kernel $N.C_H(N)$ and so embeds $H/N.C_H(N)$ in $\mathrm{Out}(N)$. This makes $H/N.C_H(N)$ precisely the image of $PQ \cong H/N$ in $\mathrm{Out}(N)$ and therefore trivial. Consequently $H = N.C_H(N)$ and there is an extension $N \cap C_H(N) = Z(N) \hookrightarrow C_H(N) \twoheadrightarrow PQ$. Since the centre of N has derived length at most 1, condition (2.3)iii) is satisfied, whence $C_H(N) \twoheadrightarrow PQ$ is EP^2R. However, $C_H(N) \le G$, so that $\phi : G \twoheadrightarrow Q$ is EP^2R too.

From the fact that $\phi(PC_H(N)) = PQ$ we have that $PG \le N.PC_H(N)$, giving $PG = P(N.PC_H(N))$. So (2.6)(b) is an immediate application of the next lemma.

LEMMA 2.7 : *Suppose* $A = BC$ *where* B, C *are commuting normal subgroups of* A. *Then*

a) $[PA,B] = PB$, $[PA,C] = PC$; *and*

b) $PA = PB.PC$.

REMARK 2.8 : *Conditions* (2.7)(a) *and* (b) *are equivalent.*

Note that by applying (2.7)(b) instead of (2.7)(a) we obtain the following. (From (2.6)(a) we may identify H above as $N.PG$.)

COROLLARY 2.9 : *If the extension* $N \hookrightarrow G \twoheadrightarrow Q$ *induces the trivial map* $PQ \to \mathrm{Out}(N)$, *then* $PG = PN.PC_{N.PG}(N)$.

To prove the lemma we verify the assertions P_α, Q_α below, for each ordinal α.

$$P_\alpha : PA \le B^{(\alpha)}.C \ .$$

$$Q_\alpha : [PA,B] \le B^{(\alpha)} \ .$$

Evidently P_0 is true. If α is a limit ordinal, then the truth of Q_β for all $\beta < \alpha$ clearly forces that of Q_α; otherwise, the truth of $P_{\alpha-1}$ gives that of P_α, since

$$PA = [PA,PA] \le [B^{(\alpha-1)}.C,\ B^{(\alpha-1)}.C] \le B^{(\alpha)}.C \ .$$

On the other hand, because when Q_α holds

$$PA = [PA,B.C] \le [PA,B].C \le B^{(\alpha)}.C \ ,$$

we have Q_α implying P_α; the converse is even easier. So transfinite induction clinches all P_α, Q_α. From the latter,

$$[PA,B] \le \bigcap_\alpha B^{(\alpha)} = PB = [PB,B] \le [PA,B] \ .$$

Since the hypotheses of (2.7) require $B \cap C$ to be central in A, that lemma has an interesting generalization whose proof uses Proposition 2.3 (and thereby Lemma 2.7).

PROPOSITION 2.10 : *Suppose* $A = B.C$ *for normal subgroups* B,C *of* A. *If* $B \cap C$ *has either its derived series of finite length or* $\mathrm{Out}(B \cap C)$ *or* $\mathrm{Out}(B \cap C/P(B \cap C))$ *hypoabelian, then*

$$PA.(B \cap C) = PB.PC.(B \cap C) \ .$$

Let us first check that (2.10) does indeed imply (2.7). For,

$$\begin{aligned} PA &= [PA.(B \cap C),\ PA.(B \cap C)] \\ &= [PB.PC.(B \cap C),\ PB.PC.(B \cap C)] \\ &= [PB.PC,\ PB.PC] = PB.PC. \end{aligned}$$

Now the the proof of (2.10). From the decomposition

$$A/B \cap C \cong B/B \cap C \times C/B \cap C \ ,$$

we have (after, e.g., (2.3)i))

$$P(A/B \cap C) \cong P(B/B \cap C) \times P(C/B \cap C) \ .$$

We identify these perfect radicals by appealing to (2.3)iii), iv). Thus any extension $B \cap C \hookrightarrow D \twoheadrightarrow D/B \cap C$ is EP^2R. In particular, we deduce that

$$PA.(B \cap C)/B \cap C \cong PB.(B \cap C)/B \cap C \times PC.(B \cap C)/B \cap C \ ;$$

whence the result.

3. Connections with the plus-construction

In algebraic K-theory, the starting-point for the definition of K_1A is the Whitehead lemma on the general linear group over the ring A. This identifies $\mathcal{P}$GLA as EA, the subgroup generated by elementary matrices. Moreover, EA = [GLA,GLA], making GLA a member of the class of groups of finite derived length, which we have seen is well-suited to a discussion of epimorphisms preserving perfect radicals. It was of course the classical fact that $K_1A = GLA/\mathcal{P}GLA$ which led Quillen to the plus-construction and thence the definition $K_iA = \pi_i(BGLA^+)$. Since then there has been much further use of the plus-construction, in topology and geometry. The main motivation for the study of EP^2R maps (at the present time) lies in their close relation to natural questions concerning the plus-construction. Here are two examples (respectively [1 (5.11)] and [1 (6.8)]).

(3.1) *Suppose* $f : X \to Y$ *has connected fibre. Then the commuting diagram*

$$\begin{array}{ccc} X & \xrightarrow{q_X} & X^+ \\ f\downarrow & & \downarrow f^+ \\ Y & \xrightarrow{q_Y} & Y^+ \end{array}$$

is co-Cartesian if and only if $\pi_1(f)$ *is* EP^2R.

(3.2) *If the fibre sequence* $F \to E \overset{p}{\to} B$ *(with* F,E,B *connected) is plus-constructive (that is,* F^+ *is also the homotopy fibre of* p^+*), then* $\pi_1(p)$ *is* EP^2R.

The proof of (3.2) in [1] involves a diagram-chasing argument whose evident irreversibility casts considerable doubt on the converse. This suspicion can be validated in two ways.

First, it is possible to state precisely what further condition is needed in order to characterise those fibre sequences (of connected spaces) which are plus-constructive. Thus [3] reveals the following.

(3.3) *A fibre sequence* $F \to E \overset{p}{\to} B$ *is plus-constructive if and only if* $\pi_1(p)$ *is* EP^2R *and the induced action of* $\mathcal{P}\pi_1(E)$ *on* $\pi_*(F^+)$ *is trivial.*

In the special case where F,E,B are classifying spaces of groups with $\pi_1(F)$ hypoabelian, this simplifies greatly.

(3.4) *Let* $N \hookrightarrow G \xrightarrow{\phi} Q$ *be a group extension, with* $PN = 1$. *Then the fibre sequence* $BN \to BG \to BQ$ *is plus-constructive if and only if both* ϕ *is* EP^2R *and* $[PG,N] = 1$.

This comes very close to implying the as-yet unproved part of (2.5) above. The gap is filled by the following corollary to the main theorem of [3].

PROPOSITION 3.5 : *An extension* $N \hookrightarrow G \twoheadrightarrow Q$ *with* $PN = 1$ *induces the trivial homomorphism* $PQ \to \mathrm{Out}(N)$ *if and only if the fibre sequence* $BN \to BG \to BQ$ *is plus-constructive.*

The proof of (3.5) (given [3]) simply consists in identifying the group $AUT(BN^+)$ of (free) homotopy classes of self-homotopy equivalences of BN^+. Since $PN = 1$, this is just $AUT(BN) = \pi_0(\mathrm{Aut}(BN))$ which is indeed $\mathrm{Out}(N)$ [5 p.42].

This argument does not require that the total space and base space be classifying spaces. There is therefore the following corollary, in some sense dual to the result [2] that every fibre sequence whose base has hypoabelian fundamental group is plus-constructive.

COROLLARY 3.6 : *Suppose both* N *and* Out(N) *are hypoabelian. Then every fibre sequence with fibre* BN *is plus-constructive.*

Algebraic K-theory provides important examples concerning (2.5) and (3.4); it demonstrates the irredundancy of the condition on the subgroup $[PG,N]$. We take for G the general linear group GLUT (after [1 p.27]) on the ring $\begin{pmatrix} A & A \\ O & A \end{pmatrix}$ of upper triangular 2 × 2 - matrices over a given ring A. The split ring epimorphism

$$\begin{pmatrix} A & A \\ O & A \end{pmatrix} \longrightarrow \begin{pmatrix} A & O \\ O & A \end{pmatrix}$$

induces a split epimorphism $\phi : GLUT \to GL(A \oplus A)$ of general linear groups. By (2.3)i) above, ϕ is certainly EP^2R. Now $N = \mathrm{Ker}\,\phi$ intersects with GL_nUT as matrices $\begin{pmatrix} I_n & M \\ O & I_n \end{pmatrix}$ where M is an arbitrary n × n - matrix over A. So multiplication in N corresponds to addition of finite matrices, making N an abelian subgroup. However, N fails to commute with $PGLUT = EUT$. For example, in GL_2UT the matrices

$$\left[\begin{array}{c|cc} I_2 & \begin{matrix}0 & 0\\ 1 & 0\end{matrix} \\ \hline 0 & I_2 \end{array}\right] \in N_{\cap} GL_2UT$$

and $e^1_{12} \oplus I_2 \in E_2UT$ have non-trivial commutator. Hence (by (2.5)) the image of $E(A \oplus A)$ in $Out(N)$ is non-trivial, and from (3.4) the fibre sequence $BN \to BGLUT \xrightarrow{B\phi} BGL(A \oplus A)$ is not plus-constructive. This fact allows the existence of the homotopy equivalence $B\phi^+ : BGLUT^+ \to BGL(A \oplus A)^+$ (since $BN^+ = BN$ is patently not contractible), whose significance embraces the non-extension of lower K-theory Mayer-Vietoris sequences [1 ch.3] and the additivity of the natural map from the A-representation ring of a group G to the algebraic K-group $KA^0(BG)$ [1 (12.3)], [6]. The latter enables one to define further structure on algebraic K-theory [1 chs 12,13].

More generally, for any ring epimorphism $A \twoheadrightarrow A_1$ the induced map $\psi : EA \to EA_1$ is evidently EP^2R (EA is after all perfect). However, since EA has trivial centre, $B\psi$ is only plus-constructive if ψ is an isomorphism. Again, this is the fact which makes so difficult the description of the fibre of $BEA^+ \to BEA_1^+$, and thereby the relative terms of the long exact sequence linking K_*A to K_*A_1.

The second discussion concerning a converse to (3.2) above consists in a characterisation of fibre sequences $F \to E \xrightarrow{p} B$ in terms of the relation between the (homotopy) fibres $F = F_p$ and F_{p^+}. Recall that, to have $F_{p^+} = F^+$, one requires of the map $F \to F_{p^+}$ that it induce an epimorphism on fundamental groups (with kernel $P\pi_1(F)$) and an isomorphism on homology (with arbitrary coefficients). It is thus to be expected that these two data should arise as distinct aspects of the original fibre sequence. This expectation is fulfilled in (3.8), (3.9) below. First, a lemma based on (2.1).

LEMMA 3.7 : *Suppose* $A \xrightarrow{f} B \xrightarrow{g} C$ *is a fibre sequence with* $P\pi_1(C) = 1$, *and that a map* $p : E \to B$ *(with connected homotopy fibre) induces* $p' : A\times_B E \to A$. *Then* $\pi_1(p)$ *is* EP^2R *if and only if* $\pi_1(p')$ *is.*

To prove (3.7), we consider the following commuting diagram, which is exact only in respect of the first pairs of horizontal maps.

$$\begin{array}{ccccccccc}
\pi_2(C) & \to & \pi_1(A\times_B E) & \xrightarrow{\pi_1(f')} & \operatorname{Im}\pi_1(f') & \hookrightarrow & \pi_1(E) & \xrightarrow{\pi_1(g\circ p)} & \pi_1(C) \\
\| & & \downarrow{\scriptstyle \pi_1(p')} & & \downarrow{\scriptstyle \pi_1(p)|} & & \downarrow{\scriptstyle \pi_1(p)} & & \| \\
\pi_2(C) & \to & \pi_1(A) & \xrightarrow{\pi_1(f)} & \operatorname{Ker}\pi_1(g) & \hookrightarrow & \pi_1(B) & \xrightarrow{\pi_1(g)} & \pi_1(C)
\end{array}$$

Since $\pi_2(C)$ is abelian, Lemma 2.1 yields (a) that $\pi_1(f')$ is EP^2R, and (b) that $\pi_1(p')$ is EP^2R precisely when $\pi_1(p\circ f') = \pi_1(f\circ p')$ is, which by (a) occurs precisely when the restriction $\pi_1(p)|$ is EP^2R. However, the hypothesis that $P\pi_1(C) = 1$ forces $P\pi_1(B)$ to lie inside Ker $\pi_1(g)$, making $\pi_1(p)|$ EP^2R if and only if $\pi_1(p)$ is.

PROPOSITION 3.8 : *For any fibre sequence* $F \to E \xrightarrow{p} B$ *with* F *connected, the induced fundamental group homomorphism* $\pi_1(F) \to \pi_1(F_{p^+})$ *is surjective if and only if* $\pi_1(p)$ *is* EP^2R.

<u>Proof</u>. By [2 Lemma 2.1] and the lemma above, neither assertion is affected by pulling-back over the fibre sequence $AB \to B \to B^+$ (since of course $P\pi_1(B^+) = 1$). Therefore assume B is already an acyclic space. Then B^+ is contractible, so that the fibre sequence $F_{p^+} \to E^+ \to B^+$ pulls back over $B \to B^+$ to induce a map of fibre sequences over B:

$$\begin{array}{ccc}
F & \xrightarrow{Fg} & F_{p^+} \simeq E^+ \\
\searrow & & \swarrow \\
 & E \xrightarrow{g} E^+\times B & \\
 & p \searrow \quad \swarrow & \\
 & B &
\end{array}$$

As the maps of total spaces and of fibres share a common homotopy fibre [1 p.35], it follows that $\pi_1(Fg)$ maps $\pi_1(F)$ onto $\pi_1(F_{p^+})$ if and only if

$$\pi_1(g) : \pi_1(E) \to \pi_1(E^+\times B) = \pi_1(B) \times \pi_1(E)/P\pi_1(E)$$

is surjective. Now write π for $\pi_1(E)$, P for $P\pi_1(E)$ and K for Ker $\pi_1(p)$, so that $\pi_1(B) \cong \pi/K$. There is a map of group extensions

$$\begin{array}{ccccc}
KP & \rightarrowtail & \pi & \twoheadrightarrow & \pi/KP \\
\downarrow & & \downarrow{\scriptstyle \pi_1(g)} & & \downarrow{\scriptstyle \Delta} \\
KP/K \times KP/P & \rightarrowtail & \pi/K \times \pi/P & \twoheadrightarrow & \pi/KP \times \pi/KP
\end{array} ,$$

where the left-hand surjection is the canonical projection

$$KP \twoheadrightarrow KP/K_\cap P \cong P/K_\cap P \times K/K_\cap P \cong KP/K \times KP/P .$$

Since the diagonal inclusion Δ is an isomorphism precisely when its domain is trivial, the diagram shows that $\pi_1(g)$ is onto if and only if $\pi = KP$, which is to say, $\pi_1(E) = \mathrm{Ker}\ \pi_1(p).P\pi_1(E)$. Now the image $\pi_1(B)$ of $\pi_1(p)$ is perfect; thus the last equality is equivalent to $\pi_1(p)$ being EP^2R, as required.

An alternative, and potentially faster, proof (kindly suggested by the referee) uses a generalisation of the Snake Lemma to the case of non-abelian groups. This is applied to the map of the exact sequences of fundamental groups corresponding to p and p^+. The obvious difficulty, of defining the relevant cokernel (to $\pi_1(Fg)$), is circumvented, because the assumption that B is acyclic ensures that $\pi_1(Fg)$ has its image normal in $\pi_1(F_{p^+})$.

Of course (3.2) is an immediate corollary to this proposition. By means of one of the techniques introduced in the proof above, we can also characterise those fibre sequences for which the induced map $F \to F_{p^+}$ is an integral homology equivalence.

PROPOSITION 3.9 : *For any fibre sequence* $F \to E \xrightarrow{p} B$ *with* F *connected, the induced homology homomorphism* $H_*(F) \to H_*(F_{p^+})$ *is an isomorphism if and only if* $P\pi_1(B)$ *acts trivially on* $H_*(F)$.

The sufficiency of the $P\pi_1(B)$-action condition is just [2 Lemma 4.2]; here we demonstrate its necessity. We do this by again taking the induced fibre sequences over AB

$$\begin{array}{ccccc}
F & \to & F_{p^+} & \simeq & \bar{E}^+ \\
& \searrow & & \swarrow & \\
& \bar{E} & \to & \bar{E}^+ \times AB & \\
& & \searrow \quad \swarrow & & \\
& & AB & &
\end{array}$$

(This is the same diagram as previously. The notation differs here because we shall have to refer to the original fibre sequence.) Since the right-hand fibre sequence is a product fibration, its action of $\pi_1(AB)$ on $H_*(F_{p^+})$ is certainly trivial. The given homology isomorphism forces the action on $H_*(F)$ to be trivial too. However this action is induced from the original action of $\pi_1(B)$ on $H_*(F)$ via the pull-back over $AB \to B$ of the original fibre sequence. Hence the result follows from the fact that $P\pi_1(B)$ is the image of the homomorphism $\pi_1(AB) \to \pi_1(B)$.

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Department of Mathematics
National University of Singapore
Kent Ridge, Singapore 0511
Republic of Singapore