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GROUP EXTENSIONS AND THEIR TRIVIALISATION

by A. J. Berrick [1])

Abstract

This is a discussion of the set of all (equivalence classes of) extensions with a given kernel K and quotient Q, with emphasis on the topological approach. A phenomenon studied is of groups Q for which all such extensions are trivial whenever K belongs to a broad class of groups. It is shown that this applies to many McLain groups which are also given a significant generalisation here.

Introduction

The following remarks were prompted by the paper [12] of de la Harpe and McDuff (whose encouragement in writing up this material is gratefully acknowledged). An interesting phenomenon was noted there: certain groups, so large as to have all homology groups and countable homomorphs trivial, admit no non-trivial extensions with finitely generated abelian kernel. Here we place this observation in a broader setting and record two further classes of example, one of which involves a strong generalisation of groups such as Steinberg and McLain groups that are closely related to matrix groups. Although some of the general results will be more or less known to group-theorists, the topological methods introduced should be of interest. We shall work only with discrete groups, even when there is an obvious non-discrete topology available. Helpful comments of B. Hartley, T. B. Ng and D. Robinson have been of assistance to this work.

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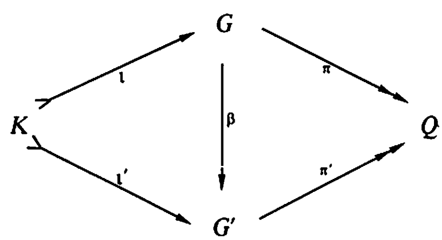
1. An exact sequence

Fixing two groups K and Q, we consider extensions G with kernel K and quotient Q. (The phraseology is intended to evade the "of K by Q" versus "of Q by K" controversy.) Strictly speaking, K is only isomorphic to the kernel, for we take an extension to be a short exact sequence of groups

$$K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q\,,$$

often referring to this simply as "G".

Two extensions G, G' then are *equivalent* (also known as *congruent*) if there exists a (necessarily bijective) homomorphism $\beta: G \to G'$ making



commute.

The set of equivalence classes, $\mathscr{E}\mathrm{xt}(Q, K)$, is a pointed set in that it admits a distinguished element (basepoint), namely the class of the *trivial extension*

$$K \overset{in_1}{\rightarrowtail} K \times Q \overset{pr_2}{\twoheadrightarrow} Q\,.$$

It is usual either to consider more tractable subsets of this set or to specialise to the case of abelian K, so as to obtain richer algebraic structure. However here we look at $\mathscr{E}\mathrm{xt}$ in full generality. We determine it to the extent of placing this set in an exact sequence of pointed sets. (Recall that a sequence of pointed set functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if $f(A) = g^{-1}(c_0)$, where c_0 is the basepoint of C.) For discussion of naturality of the sequence, we observe that the pointed set functor $\mathscr{E}\mathrm{xt}(\ ,\ )$ is contravariant in the quotient group via the existence of induced (pulled-back) extensions. On the other hand, in the absence of a commutativity condition it fails to be a (covariant) functor in the kernel. (More on this, later.)

PROPOSITION 1.1. *There is an exact sequence of pointed sets*

$$H^2(Q; Z(K)) \overset{A}{\rightarrowtail} \mathscr{E}\mathrm{xt}(Q, K) \overset{B}{\to} \mathrm{Hom}(Q, Out(K)) \overset{\Gamma}{\to} \coprod_{\alpha} H^3(Q; \{Z(K)\}_\alpha)$$

where the functions A, B, Γ *are defined below.*

Proof. First, an explanation of notation. $H^2(Q; Z(K))$ refers to cohomology with trivial coefficients $Z(K)$, the centre of K. On the other hand, $\{\ \}_\alpha$ indicates that the coefficients in $H^3(Q; \{Z(K)\}_\alpha)$ may be non-trivial, corresponding to a non-trivial homomorphism α from Q to the group $\mathrm{Aut}(Z(K))$ of all automorphisms of $Z(K)$. Cohomology groups, being abelian, have 0 as natural basepoint; $\coprod$ refers to the coproduct in the category of pointed sets, that is, the one-point union obtained by identifying every 0 in the disjoint union. In this case the union is taken over all possible choices of local systems of coefficients; in other words, is indexed by

$$\mathrm{Hom}(Q, \mathrm{Aut}(Z(K))\,.$$

Finally, $\mathrm{Out}(K)$ denotes the outher automorphism group of K, the quotient of $\mathrm{Aut}(K)$ by its group $\mathrm{Inn}(K)$ of inner automorphisms.

Although this result may be deduced from [9] (see also [15] ch. IV, [11]), I have chosen to outline a more geometric, less ad hoc treatment here. (Equivalence of the corresponding functions occurring in the different approaches has been verified in [13].)

It is of course a standard fact (recaptured below) that $H^2(Q; Z(K))$ corresponds to the subset of $\mathscr{E}\mathrm{xt}(Q, Z(K))$ comprising central extensions. (A further topological proof, in the spirit of some of the discussion below, is presented in [2 ch. 8]. That treatment also permits a topological proof of the fact [9] that our function A generalises, to provide a bijection of each inverse image under B with the corresponding $H^2(Q; \{Z(K)\})$.)

The function A is usefully considered in somewhat fuller generality. Therefore let $\tau: Z \to L$ be a group homomorphism with domain abelian and image central in L. We define $A: H^2(Q; Z) \to \mathscr{E}\mathrm{xt}(Q, L)$ as follows. Given a central extension $Z \overset{\iota'}{\rightarrowtail} E \overset{\phi'}{\twoheadrightarrow} Q$ representing an equivalence class $[\phi'] \in H^2(Q; Z)$, let its image under A be the class of the extension

$$L \overset{\iota''}{\rightarrowtail} L \times E/\tilde{Z} \overset{\phi''}{\twoheadrightarrow} Q$$

Here the subgroup $\tilde{Z}$ of $L \times E$ consists of all pairs $(\tau(z), \iota'(z^{-1}))$, $z \in Z$, and is normal precisely because $\tau(Z)$ and $\iota'(Z)$ are both central. The homomorphisms ι'' and ϕ are the predictable ones: $\iota''(x) = (x, 1)$ and $\phi''(x, e) = \phi'(e)$. The various checks, for example that ϕ'', then A, is well-defined, are straightforward and assigned to the reader. Our proof that A is injective follows the definition of B given below. Note (for (1.2) below) that when L is abelian the resulting extension is central, so that A may be regarded as a map

$$H^2(Q; Z) \to H^2(Q; L) \hookrightarrow \mathrm{Ext}(Q, L).$$

In this form, it reduces to the Baer construction, which coincides with the obvious cohomological homomorphism

$$\tau_*: H^2(Q; Z) \to H^2(Q; L).$$

The function B is often referred to as the *coupling* [11 p. 65]. For a given extension $K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ it comes from conjugation in K by inverse images in G of elements in Q. Such inverse images being determined only up to multiplication by elements of $\iota(K)$, the G-conjugation automorphism of K is defined only modulo $\mathrm{Inn}(K)$. Again, it is simple to check that B is an invariant of equivalence and thus well-defined.

Now observe that conjugation by $K \times E/\tilde{Z}$ on $\iota''(K)$ has the same effect as K-conjugation. Therefore $\mathrm{B} \circ \mathrm{A}$ is trivial. If, on the other hand, $K \overset{\iota}{\rightarrowtail} G \overset{\pi}{\twoheadrightarrow} Q$ induces trivial $Q \to \mathrm{Out}(K)$, then G coincides with the kernel $\iota K \,.\, C_G(\iota K)$ of the trivial composition of homomorphisms in the commuting square

$$\begin{array}{ccc} G & \to & \mathrm{Aut}(K) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ Q & \underset{\mathrm{B}\pi}{\to} & \mathrm{Out}(K). \end{array}$$

So

$$Q \cong \iota K \,.\, C_G(\iota K)/\iota K \cong C_G(\iota K)/\iota Z(K);$$

in other words, there is a central extension

$$Z(K) \overset{\iota|}{\rightarrowtail} C_G(K) \overset{\pi|}{\twoheadrightarrow} Q.$$

From the isomorphism

$$\begin{aligned} K \times C_G(\iota K)/\tilde{Z} &\to G \\ (k, g) &\mapsto kg \end{aligned}$$

we infer that $A[\pi|] = [\pi]$, as required for exactness at $\mathscr{E}xt(Q, K)$. Again, if we begin with a central extension $Z(K) \overset{\iota'}{\rightarrowtail} G \overset{\phi'}{\twoheadrightarrow} Q$, then the extension $K \overset{\iota''}{\rightarrowtail} K \times E/\tilde{Z} \overset{\phi''}{\twoheadrightarrow} Q$ representing $A[\phi']$ has $Z(K) \overset{\iota''|}{\rightarrowtail} C_{K \times E/\tilde{Z}}(\iota'' K) \overset{\phi''|}{\twoheadrightarrow} Q$ equivalent to ϕ'. Thus A is a bijection onto Ker B, with inverse given by restriction.

We turn now to the definition of the function Γ. At this stage classifying spaces (of topological monoids in the case of the set of self-homotopy equivalences $\mathscr{E}(X)$ and its basepoint-preserving counterpart $\mathscr{E}(X; x_0)$, otherwise of discrete groups) enter the picture. From Corollary A.5 there is a fibration

$$\mathscr{K}(Z(K), 2) \to B\mathscr{E}(X) \to BOut(K)$$

where $X = BK = \mathscr{K}(K, 1)$. A homomorphism $\psi: Q \to Out(K)$ induces $B\psi: BQ \to BOut(K)$.

$$\begin{array}{ccc} & & \mathscr{K}(Z(K), 2) \\ & & \downarrow \\ & \nearrow & B\mathscr{E}(X) \\ & & \downarrow \\ BQ & \to & BOut(K) \end{array}$$

The question as to when $B\psi$ lifts to a map $BQ \to B\mathscr{E}(X)$ (making the above triangle commute) is solved by standard obstruction theory (e.g. [23 VI (6.14)]), which asserts that there is an element of $H^3(BQ; \{Z(K)\}) = H^3(Q; \{Z(K)\})$, uniquely determined by ψ and therefore safely labelled as $\Gamma\psi$, whose vanishing is equivalent to the existence of the desired lifting. (Note that the local coefficient system $\{Z(K)\}$ is also determined by ψ via its composition with the restriction homomorphism $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$.) Now our present claim is that $\Gamma\psi$ vanishes precisely when ψ is derived, via B, from a group extension. The link between these assertions is provided by the universality of the fibration

$$BK \to B\mathscr{E}(X; x_0) \to B\mathscr{E}(X)$$

(e.g. [7]). That is, every fibration with fibre BK is induced from this one by a map of its base space into $B\mathscr{E}(X)$. So liftings $BQ \to B\mathscr{E}(X)$ of $B\psi$ correspond one-to-one with fibrations $BK \to BG \to BQ$. (The homotopy exact sequence here shows that the total space must be a $\mathscr{K}(G, 1)$.) Finally, since the fundamental group functor is left inverse to the classifying space functor, fibrations of this form correspond one-to-one to extensions $K \rightarrowtail G \twoheadrightarrow Q$. This clinches exactness at $\mathrm{Hom}(Q, \mathrm{Out}(K))$.

In fact, the argument shows more, for it reveals that the first three terms of (1.1) are none other than those of the exact sequence

$$1 \to [BQ, \mathscr{K}(Z(K), 2)] \to [BQ, B\mathscr{E}(X)] \to [BQ, \mathrm{BOut}(K)]$$

arising from the fibration (A.5) (where the first term, $[BQ, \Omega\mathrm{BOut}(K)]$, is trivial because $\Omega\mathrm{BOut}(K) = \mathrm{Out}(K)$ is discrete). Although this does not yield that A is injective (merely that its kernel is trivial), it does provide a topological proof that $H^2(Q; Z(K)) = [BQ, \mathscr{K}(Z(K), 2)]$ maps with trivial kernel onto Ker B, which we have seen corresponds to the set of equivalence classes of central extensions with quotient Q and kernel $Z(K)$.

We now take up the matter of naturality of this sequence in the kernel K (naturality in the quotient being regarded as obvious). This has significant ramifications for us later on.

PROPOSITION 1.2. *Let H be a characteristic subgroup of K. Then the quotient homomorphism $\kappa: K \to K/H$ induces a map of exact sequences*

$$\begin{array}{ccccc} H^2(Q; Z(K)) & \overset{\mathrm{A}}{\rightarrowtail} & \mathscr{E}\mathrm{xt}(Q, K) & \overset{\mathrm{B}}{\to} & \mathrm{Hom}(Q, \mathrm{Out}(K)) \\ \downarrow \kappa_* & & \downarrow \kappa_* & & \downarrow \kappa_* \\ H^2(Q; Z(K/H)) & \overset{\mathrm{A}}{\rightarrowtail} & \mathscr{E}\mathrm{xt}(Q, K/H) & \overset{\mathrm{B}}{\to} & \mathrm{Hom}(Q, \mathrm{Out}(K/H)) . \end{array}$$

Moreover, if $Z(K) \leqslant H$ and $H/Z(K)$ is normal when regarded as a subgroup of $\mathrm{Aut}(K)$, *then there is a partial splitting*

$$\Delta: \mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathscr{E}\mathrm{xt}(Q, K/H)$$

such that

$$\Delta \circ \mathrm{B} = \kappa_* \quad \textit{and} \quad \mathrm{B} \circ \Delta = \kappa_* .$$

Note that the condition on $H/Z(K)$ is clearly satisfied whenever $H/Z(K)$ is characteristic in $K/Z(K) = \mathrm{Inn}(K)$, as, for example, happens when H is a member of the upper central series of K.

Proof. The cohomological map κ_* has been discussed above; its existence relies only on the normality of H in K. For the $\mathscr{E}$xt map, let $[\pi]$ represent an extension $K \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$. Then $\kappa_*[\pi]$ is defined to be the equivalence class of the extension $K/H \rightarrowtail G/H \twoheadrightarrow Q$. Here one needs H characteristic in K in order that H be normal in G. Also, when H is characteristic in K there is a canonical homomorphism $\hat{\kappa}: \mathrm{Out}(K) \to \mathrm{Out}(K/H)$. So $\kappa_*: \mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathrm{Hom}(Q, \mathrm{Out}(K/H))$ is given simply by composition with $\hat{\kappa}$. Verification of commutativity of the two squares is a tedious but uncomplicated exercise.

The map Δ is more interesting. It can be viewed as the composition of two of the three constructions on extensions already presented. Beginning with the standard extension

$$K/Z(K) = \mathrm{Inn}(K) \rightarrowtail \mathrm{Aut}(K) \twoheadrightarrow \mathrm{Out}(K),$$

we obtain by assumption a second extension

$$K/H \rightarrowtail \mathrm{Aut}(K)\,/\,(H/Z(K)) \twoheadrightarrow \mathrm{Out}(K),$$

and then pull it back over a given homomorphism $\psi: Q \to \mathrm{Out}(K)$ to obtain as $\Delta(\psi)$ the induced extension with quotient Q and kernel K/H. The check of commutativity of the two triangles formed is again routine. (In the case $H = Z(K)$, Rose [20] calls the values of Δ *sited extensions.*)

One is tempted to speculate on the existence of a map κ_* at the H^3 level. However this first requires a map of coefficient systems. There is in general no function $\mathrm{Aut}(Z(K)) \to \mathrm{Aut}(Z(K/H))$ such that the square

$$\begin{array}{ccc} \mathrm{Out}(K) & \to & \mathrm{Aut}(Z(K)) \\ \downarrow & & \downarrow \\ \mathrm{Out}(K/H) & \to & \mathrm{Aut}(Z(K/H)) \end{array}$$

(whose horizontal maps are given by restriction) commutes, as may be seen by reference to the example where K is the centreless alternating group A_4 and H is the four-group, a characteristic subgroup. For then $\mathrm{Aut}(Z(K))$ is trivial, while

$$\mathrm{Out}(K) \to \mathrm{Out}(K/H) \cong \mathrm{Aut}(Z(K/H))$$

is surjective and non-trivial (see, for example, [22]).

An immediate consequence of (1.2) is familiar (for example [20]).

COROLLARY 1.3. *If* $Z(K) = 1$, *then* B *and* Δ *are inverse bijections.*

In particular, every extension with kernel K *is induced from the extension* $K \rightarrowtail \mathrm{Aut}(K) \twoheadrightarrow \mathrm{Out}(K)$ *by a homomorphism into* $\mathrm{Out}(K)$.

Another sense in which B admits a partial inverse is provided by the semi-direct product construction (described in for example [21 Theorem 9.9]). This may be regarded as an injection $\mathrm{E}: \mathrm{Hom}(Q, \mathrm{Aut}(K)) \rightarrowtail \mathscr{E}\mathrm{xt}(Q, K)$, which evidently makes the triangle

$$\begin{array}{ccc} & \mathrm{E} \swarrow & \mathrm{Hom}(Q, \mathrm{Aut}(K)) \\ & & \downarrow \\ \mathscr{E}\mathrm{xt}(Q, K) & \xrightarrow{\mathrm{B}} & \mathrm{Hom}(Q, \mathrm{Out}(K)) \end{array}$$

commute. When K is abelian (that is when the usual epimorphism $\mathrm{Aut}(K) \to \mathrm{Out}(K)$ is an isomorphism), E becomes right inverse to B. Thus Γ becomes trivial (as may also be seen topologically from consideration of the universal fibration). A perhaps surprising consequence of this fact is that $\Gamma = \Gamma_K$ does *not* in general factor as

$$\mathrm{Hom}(Q, \mathrm{Out}(K)) \to \mathrm{Hom}(Q, \mathrm{Aut}(Z(K))) \xrightarrow{\Gamma_{Z(K)}} \coprod_{\alpha} H^3(Q; \{Z(K)\}_\alpha)$$

(where $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$ is induced by restriction), for the triviality of $\Gamma_{Z(K)}$ would force that of the composition Γ_K. However, after [9] (see also [11 p. 80]) one knows that for any abelian group Z and the collection **K** of groups having Z as centre,

$$\coprod_{\mathbf{K}} \mathrm{Hom}(Q, \mathrm{Out}(K)) \xrightarrow{\coprod \Gamma_K} \coprod_{\alpha} H^3(Q; \{Z\}_\alpha)$$

is a surjection.

There are other favourable circumstances in which one can say a good deal further about $\mathscr{E}\mathrm{xt}(Q, K)$. We record here two results from [3] (respectively (2.9) and (2.6)). These use the notation $\mathscr{P}G$ for the maximal perfect subgroup (perfect radical) of a group G.

PROPOSITION 1.4. *Let* Q *be a perfect group. If the (equivalence class of the) extension* $K \rightarrowtail G \twoheadrightarrow Q$ *lies in the image of* A, *then*

$$\mathscr{P}G = \mathscr{P}K \,.\, \mathscr{P}C_{K\,.\,\mathscr{P}G}(K)\,.$$

When the kernel is hypoabelian ($\mathscr{P}K = 1$), this condition simplifies to the statement that it commutes with $\mathscr{P}G$. Here one can make explicit what additional condition is sufficient as well as necessary.

PROPOSITION 1.5. *Let* Q *be perfect and* K *hypoabelian. An extension* $K \rightarrowtail G \stackrel{\pi}{\twoheadrightarrow} Q$ *lies in the image of* A *if and only if both*

a) π *is an epimorphism preserving perfect radicals (that is,* $\pi\mathscr{P}G = Q$*); and*

b) $[\mathscr{P}G, K] = 1$.

These conditions are easily verified for an extension where the kernel lies in the hypercentre of G. For then K must be nilpotent, so that condition (a) is satisfied by [3 (2.3) (iii)]. On the other hand, because G acts nilpotently on K so does $\mathscr{P}G$; by Kaluzhnin's theorem [18 (7.1.1.1)] the image of the perfect group $\mathscr{P}G$ in Aut(K) induced by conjugation is nilpotent and hence trivial. This result also admits a converse, for if K is nilpotent then the extension obtained by the construction A is easily seen to have its kernel in the hypercentre.

COROLLARY 1.6. *Let* K *be a nilpotent group and* Q *perfect. Then the set of equivalence classes of extensions with kernel* K *in the hypercentre and with quotient* Q *is in* $1-1$ *correspondence with* $H^2(Q; Z(K)) \cong \mathrm{Hom}(H_2(Q), Z(K))$.

Here $H_2(Q) = H_2(Q; \mathbf{Z})$ is just the Schur multiplier of Q. The given isomorphism is immediate from the universal coefficient theorem because Q is perfect.

2. RELATIVE COMPLETENESS AND CO-COMPLETENESS

This paragraph takes us to the point of departure for our examples.

PROPOSITION 2.1. *Suppose that groups* Q *and* K *have the property that every homomorphism from* Q *to* Out(K) *is trivial. Then every extension with kernel* K *and quotient* Q *is trivial, provided that also either*

(a) K *is centreless, or*

(b) Q *is superperfect.*

Case (a) of (2.1) is of course known and includes the example of complete groups (that is, Out(K)$=1$ too). It follows immediately from (1.3).

Case (b) requires a little more attention. Recall that Q superperfect means that its first and second homology groups with trivial integer coefficients both vanish. By means of the universal coefficient/Künneth formulae, the first and second cohomology with arbitrary trivial coefficients are also zero. So both the H^2 and Hom sets in the exact sequence of (1.1) are singletons.

The following terminology is suggested by (2.1). Let **Q** be a class of groups. Then K is *complete relative to* **Q** if every extension with kernel K and quotient in **Q** is trivial. The next result is widely known, but is included for the sake of completeness (sic).

PROPOSITION 2.2. *A group K is complete if and only if it is complete relative to all groups.*

Proof. It remains to establish the sufficiency of the relative condition. We first show that K is centreless. This can be done by a little homological algebra applied to H^2 in the exact sequence of (1.1). More directly, let Z_1 and Z_2 be two copies of $Z = Z(K)$, equipped with isomorphisms $\theta_j: Z \to Z_j$, $j = 1, 2$. In the group $K \times (Z_1 * Z_2)$, let $\bar{Z}$ denote the normal closure of the subgroup generated by all elements of the form

$$(z, 1)\,[(1, \theta_1(z)), (1, \theta_2(z))]$$

with $z \in Z$, and let G be the quotient $(K \times (Z_1 * Z_2))/\bar{Z}$. Evidently K is normal in $K \times (Z_1 * Z_2)$ and so in G. From the triviality of the extension $K \overset{\iota}{\rightarrowtail} G \twoheadrightarrow G/\iota(K)$, there is a left inverse $\rho: G \to K$ to ι. Thus the triviality of each $[\iota K, (1, \theta_j(z))]$ in G implies that of $[K, \rho(1, \theta_j(z))]$ in K, making each $\rho(1, \theta_j(z)) \in Z$. Then any $z \in Z$ satisfies

$$\begin{aligned} z = \rho\iota(z) = \rho(z, 1) &= \rho[(1, \theta_2(z)), (1, \theta_1(z)] \\ &= [\rho(1, \theta_2(z)), \rho(1, \theta_1(z))] \\ &\in [Z, Z] = 1\,. \end{aligned}$$

Hence K is indeed centreless, so that (1.3) applies. In particular the set $\mathrm{Hom}(\mathrm{Out}(K), \mathrm{Out}(K))$ must be a singleton, forcing $\mathrm{Out}(K) = 1$.

The literature contains various results which may be expressed as examples of relative completeness (such as [21 Exercises 536, 537]). It is sometimes convenient to dualise this phraseology. Thus, for a class **C** of groups, we say that Q is *co-complete relative to* **C** if $\mathscr{E}\mathrm{xt}(Q, K)$ is trivial whenever $K \in \mathbf{C}$. The asymmetry between these two concepts is highlighted by the absence of a counterpart to (2.2); that is, there are no non-trivial (absolutely) co-complete groups. To see this, consider the left regular representation of Q regarded as a non-trivial homomorphism from Q to the automorphism group of the free abelian group $\mathrm{Fr}(Q)_{ab}$ generated by the elements of Q. The semi-direct product which results (via E) is then a non-trivial element of $\mathscr{E}\mathrm{xt}(Q, \mathrm{Fr}(Q)_{ab})$.

This example is quite suggestive inasmuch as, in order to find a group relative to which the quotient Q is not co-complete, we have passed to a group which is large in comparison with Q. One might therefore speculate on the existence of quotient groups which are co-complete relative to all groups of a certain size. Examples of such quotients are presented in the next paragraph.

3. Examples

In view of (2.1), our examples are of superperfect groups Q whose homomorphic images of sufficiently small cardinality, say $\leqslant \alpha$, are all trivial. For this purpose it is worth recalling that an abelian group with a generating set of cardinality β has automorphism group of order at most 2^{β}. We feature three types of example.

I. *The acyclic groups considered by de la Harpe and McDuff*

Acyclic groups have the same homology (with trivial integer coefficients) as the trivial group and so are certainly superperfect. On the other hand, the acyclic groups discussed in [12] have the further property that any countable image is trivial. Hence they are *co-complete relative to all* K *with* $\operatorname{Out}(K)$ *countable*, and in particular relative to all finitely generated groups.

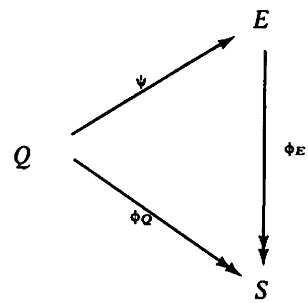
II. *The universal central extension over a simple group*

Let S be a non-abelian simple group. Being perfect, S admits a universal central extension Q [14], [17] (that is, an initial object in the category of all equivalence classes of extensions with central kernel and quotient S). Now Q is well-known to be superperfect — indeed, it is the unique superperfect central extension over S —, so we consider its possible images.

Proposition 3.1. *The non-trivial homomorphic images of* Q *are precisely the perfect central extensions of* S.

Since any image of Q is also perfect, clearly not all central extensions over S need be obtained in this way. For example, take the direct product of such an extension with an abelian group. However, if E has quotient S and central kernel then by [2 (1.6)b)] so does its maximal perfect subgroup $\mathscr{P}E$. So every central extension contains a preferred perfect central extension. In fact it contains a unique perfect central extension, because writing $\mathscr{P}E = P \,.\, C$ with P perfect and C central forces P to be normal in $\mathscr{P}E$ and the quotient $\mathscr{P}E/P$ abelian, hence trivial.

Of course the assertion, that any perfect central extension of (arbitrary) S is a homomorphic image of a superperfect one, generalises to the well-known result [11 p. 213] that any stem extension is an image of a stem cover. However this case admits an easy direct proof. For, given a central extension $D \overset{\iota}{\rightarrowtail} E \overset{\phi_E}{\twoheadrightarrow} S$ with E perfect, then the commuting triangle (which exists by universality)



results in ψ being an epimorphism. For, any commutator $[e_1, e_2]$ in E is the image of $[q_1, q_2]$ in Q where $\phi_Q(q_i) = \phi_E(e_i)$, $i = 1, 2$. This is because $e_i \in \psi(q_i) \,.\, \iota(D)$ and

$$[\psi(q_1) \,.\, \iota(D), \psi(q_2) \,.\, \iota(D)] = [\psi(q_1), \psi(q_2)] \,.$$

It is the converse argument which uses the simplicity of S. Since all non-trivial quotients of Q are non-abelian, it suffices to check the following lemma.

LEMMA 3.2. *Let* E *be a central extension over* S. *Then*

(i) $S = E/Z(E)$, *and*

(ii) *every normal proper subgroup of* E *is central or contains* $[E, E]$. *Hence every non-abelian quotient of* E *is also a central extension over* S.

Proof. (i) Certainly $E/Z(E)$ is a quotient of S; it cannot be trivial since S is non-abelian.

(ii) Any normal subgroup N of E induces the normal subgroup $N \,.\, Z(E)/Z(E)$ of S. If N is non-central then this subgroup is non-trivial, hence S. Taking derived groups of the equation $E = N \,.\, Z(E)$ gives

$$[E, E] = [N, N] \leqslant N.$$

From (2.1) and (3.1) in combination we conclude immediately that the universal central extension Q over the non-abelian simple group S is *co-complete relative to all groups* K *such that no central extension over* S *is a subgroup of* $\operatorname{Out}(K)$.

For an important class of examples of this phenomenon, let F be any field. The Steinberg group $St_n(F)$ ($n \geqslant 3$, with $n = \infty$ representing $St(F)$, and with the groups $St_3(\mathbf{F}_2)$, $St_3(\mathbf{F}_4)$, $St_4(\mathbf{F}_2)$ excluded) is superperfect, being the universal central extension of the group $E_n(F)$ generated by elementary $n \times n$-matrices [17 p. 48]. Although this group is not simple (except for $E(F)$, by [1 V (2.1)]), its central quotient $PSL_n(F)$, the projective special linear group over F, is simple [1 V (4.1), (4.5)]. Hence (with the usual three exclusions), the Steinberg groups of a field are co-complete relative to all K whose $\operatorname{Out}(K)$ fails to contain any central extension of the corresponding projective special linear group. So, for example, $St_n(\mathbf{F}_q)$ must be co-complete relative to all K with

$$|\operatorname{Out}(K)| < |PSL_n(\mathbf{F}_q)| = (q^n-1)(q^n-q)\dots(q^n-q^{n-1})/(q-1)(n, q-1).$$

III. *McLain groups*

First we recall the definition from [16], [19]. Let Λ be a linearly ordered set, F a field, and V a vector space over F with basis elements v_λ indexed by Λ. Then the McLain group $M(\Lambda, F)$ is the subgroup of the group of all linear transformations of V generated by elements of form $1 + ae_{\lambda\mu}$ where $a \in F$ and $\lambda, \mu \in \Lambda$ with $\lambda < \mu$. Here $e_{\lambda\mu}$ takes v_λ to v_μ and annihilates all other basis elements. For our purposes, it is more convenient to give an alternative description of $M(\Lambda, F)$ by means of a group presentation.

LEMMA 3.3. *The group* $M(\Lambda, F)$ *has presentation given by:*

generators

$$1 + ae_{\lambda\mu}, \quad a \in F; \lambda, \mu \in \Lambda \quad \text{with} \quad \lambda < \mu$$

relations

$$(1+ae_{\lambda\mu})(1+be_{\lambda\mu}) = 1 + (a+b)e_{\lambda\mu} \qquad \text{(i)},$$

$$[1+ae_{\lambda\mu}, 1+be_{\zeta\eta}] = \begin{cases} 1 & \mu \neq \zeta, \lambda \neq \eta \qquad \text{(ii)}, \\ 1 + abe_{\lambda\eta} & \mu = \zeta \qquad \text{(iii)}. \end{cases}$$

Proof. The claimed relations follow quickly from the definitions, since $e_{\lambda\mu}e_{\zeta\eta} = e_{\lambda\eta}$ when $\mu = \zeta$ and is zero otherwise. To see that they imply all others, observe that any product which is not made trivial by these relations alone may be rewritten by means of (i), (ii), (iii) in the form

$$(1+ae_{\lambda_0\mu_0}) \prod_{\lambda>\lambda_0} (1+b_\lambda e_{\lambda\mu_0}) \prod_{\substack{\lambda \\ \mu<\mu_0}} (1+c_{\lambda\mu}e_{\lambda\mu})$$

for some $\lambda_0, \mu_0 \in \Lambda$ with $\lambda_0 < \mu_0$ and non-zero $a \in F$. However, the transformation represented by this product sends the basis element v_{λ_0} to a linear combination in which a appears as the coefficient of v_{μ_0}. Hence it is non-trivial. Thus $M(\Lambda, F)$ admits no relation which is not already a consequence of the given three types.

Despite obvious similarities with the Steinberg groups of II above, these groups are not accommodated by that discussion, for they are well-known to have trivial centre so long as Λ does not have a first and last element. Again, they are not perfect in general, unless Λ is dense. However, there is then the following further, somewhat surprising, similarity.

PROPOSITION 3.4. *If* Λ *is dense, then* $M(\Lambda, F)$ *is superperfect.*

The proof is deferred to the next section. An alternative (contemporaneous) proof, concentrating on the linear order structure of Λ, is to be found in [4].

PROPOSITION 3.5. *If* Λ *is dense, then the order of a non-trivial homomorphic image of* $M(\Lambda, F)$ *cannot be less than the cardinality of* F *or of every interval of* Λ.

Proof. Let π be an epimorphism from $M(\Lambda, F)$ onto a group of order less than card(F). Given an arbitrary generator $1 + ae_{\lambda\mu}$ of $M(\Lambda, F)$, take any ν in the interval (λ, μ) and consider the set $\{\pi(1+be_{\lambda\nu})\}_{b\in F}$. Since its cardinality is less than that of F, there exist distinct b_1, b_2 in F with $\pi(1+b_1e_{\lambda\nu}) = \pi(1+b_2e_{\lambda\nu})$. Then $1 + (b_1-b_2)e_{\lambda\nu}$ lies in Kerπ, whence so does

$$1 + ae_{\lambda\mu} = [1 + (b_1-b_2)e_{\lambda\nu}, 1 + (b_1-b_2)^{-1}ae_{\nu\mu}] .$$

The argument on the cardinality of intervals of Λ is similar (cf. [24 Lemma 1 (b)]).

The immediate conclusion of this discussion is that, for dense Λ, $M(\Lambda, F)$ *is co-complete relative to all groups* K *whose* Out(K) *has order less than the cardinality either of* F *or of every interval of* Λ.

4. The group of a category and ring

The proof of the superperfectness of $M(\Lambda, F)$ originally obtained for this paper so closely resembled in spirit those in [17], [2] showing the Steinberg groups to be superperfect that I felt that the two classes of groups ought to admit a mutual generalisation allowing a single proof of both facts. A suitable construction for this purpose appears below; it may well be of wider interest.

Let $\mathscr{C}$ denote a small category and A an associative ring with identity 1. Then $G = G_{\mathscr{C},A}$ is to be the group with the following presentation. Its generating set is $\mathrm{Mor}(\mathscr{C}) \times A$, whose elements are typically written f^a. However, if f is an identity morphism $1_X (X \in \mathrm{Ob}(\mathscr{C}))$ then in G $1_X^a = 1$ for any $a \in G$. The remaining relations take the following form (for arbitrary non-identity morphisms f, g, h and ring elements a, b).

$$f^a f^b = f^{a+b} \qquad \text{(i)},$$

and, if the composition $h \circ g$ is not defined,

$$[g^a, h^b] = \begin{cases} (g \circ h)^{ab} & \text{if defined} \qquad \text{(ii)}, \\ 1 & \text{otherwise} \qquad \text{(iii)}. \end{cases}$$

(From (i) and (ii) we may deduce that if $h \circ g$, but not $g \circ h$, is defined, then $[g^a, h^b] = (h \circ g)^{-ba}$. Finally, if $h \circ g$ and $g \circ h$ are both defined, then no immediate inference is available.)

Examples

1. For arbitrary A and integer $n \geqslant 3$ let $\mathscr{C}$ have object set $\{1, \ldots, n\}$ and, whenever $i \neq j$, morphisms $x_{ij} : j \mapsto i$ and $x_{ji} : i \mapsto j$. Then the group G is the Steinberg group $\mathrm{St}_n(A)$ or, when $n = \infty$, $\mathrm{St}(A)$; our notation for elements is just the usual one.

2. Given a linearly ordered set Λ, let $\mathrm{Ob}(\mathscr{C}) = \Lambda$ and let the non-identity morphisms be $e_{\lambda\mu} : \mu \mapsto \lambda$ corresponding to the relation $\lambda < \mu$. It is clear that relations (i)-(iii) above are just (i)-(iii) of Lemma 3.3 with the element $1 + ae_{\lambda\mu}$ now rewritten as $e_{\lambda\mu}^a$. So we obtain $M(\Lambda, F)$ when the ring A is a field F.

3. Let $\mathscr{C}$ be the category with one object and a morphism f having

exactly n distinct non-identity iterates. Then $G_{\mathscr{C},\mathbf{Z}}$ is the free group on the n (possibly infinite) generators.

4. Suppose now that $\mathscr{C}$ has two distinct objects X and Y, with n morphisms from X to Y. In this case $G_{\mathscr{C},\mathbf{Z}}$ is the free abelian group on n (again, possibly infinite) generators. Alternatively, let $\mathscr{C}$ have an initial object and n others, with only the morphisms required by this property. This provides an example where the category can be varied without changing the resulting group.

5. Of course, 3. and 4. above coincide when $n = 1$. Taking this case and an arbitrary ring A gives us for G the additive group A_+ of A.

An interesting question suggested by the above might be to classify all groups which can arise from our construction. Incidentally, although we do not exploit this fact, the construction induces a bifunctor from the product of the category of small categories (suitably restricted so that its morphisms are those functors for which every identity morphism has a unique preimage) and that of rings to the category of groups.

We now relate the (homology of) group-theoretical properties of perfectness and superperfectness of G to combinatorial properties of $\mathscr{C}$. To do this, consider the following possible conditions on non-identity morphisms of $\mathscr{C}$.

(α) Every f has its domain and codomain distinct.

(β) Every f is a non-trivial composite; that is, $f = g \circ h$ for some g and h.

(γ) Given f and k with $k \circ f$ undefined, then $f = g \circ h$ for some g and h such that $k \circ h$ and $g \circ k$ are undefined.

In the presence of (α), condition (γ) evidently implies (β) as the case $k = f$.

(δ) Any commuting square

$$\begin{array}{ccc} H & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{r} & Y \end{array}$$

(with distinct vertices) admits a diagonal $s: Z \to W$ (or $W \to Z$) such that the two triangles formed commute.

Observe that Example 1. satisfies all these conditions, although (γ) forces n to be at least 5. Of course Example 2. satisfies (α) and (so long as Λ has cardinality exceeding 3) (δ), while (β) and (γ) are each equivalent to Λ being a dense ordering.

LEMMA 4.1. *For arbitrary* A, *the group* $G_{\mathscr{C},A}$ *is perfect if and only if* $\mathscr{C}$ *satisfies* (α) *and* (β).

Proof. The two conditions imply that

$$f^a = (g \circ h)^a = [g^1, h^a] \,.$$

Thus every element is a commutator. Conversely, given f which cannot be expressed as a composite $g \circ h$ where $h \circ g$ is undefined, then the homomorphism

$$G \to A_+ , \quad k^a \mapsto \begin{cases} 1 & k \neq f, \\ a & k = f, \end{cases}$$

has non-trivial abelian image, so that G cannot be perfect.

THEOREM 4.2. *For arbitrary* A, *if* $\mathscr{C}$ *satisfies conditions* (α), (γ) *and* (δ) *above, then* $G = G_{\mathscr{C},A}$ *is a superperfect group.*

Proof. By (4.1), G is perfect. To show that G is superperfect, we must establish that every central extension over G splits. Now notice that any splitting σ of a given central extension $K \rightarrowtail E \overset{\phi}{\twoheadrightarrow} G$ must have

$$\sigma(f^a) = [\bar{g}^1, \bar{h}^a]$$

whenever $f = g \circ h$ (since by (α) $h \circ g$ is undefined), where $\bar{h}^a$ denotes the K-coset $\phi^{-1}(h^a)$ in E. By virtue of the centrality of K, the right-hand expression determines a single element of E, which has yet to be shown independent of the particular factorisation of f. So a candidate for σ is defined by making a specific choice of factorisation for each morphism and having the above expression determine the value of each $\sigma(f^a)$. In the course of the proof it emerges that any choice would suffice, so that no real decision is actually required.

The proof uses the following facts [2 p. 68].

LEMMA 4.3. *In any group* E, *for* $u, v, w \in E$,

$$\begin{aligned} &[[u, v], w] \\ &= [u, v]\,[w, v]\,[v, wu] && \text{(a)}, \\ &= [[u, v], [w, v]]\,[[w, v], u] && \textit{if} \quad [u, w] \in Z(E) \quad \text{(b)}, \\ &= 1 && \textit{if also} \quad [v, w] \in Z(E) \quad \text{(c)}. \end{aligned}$$

Here (a) is obtained by outright multiplication, and (b) by substitution for $[v, wu] = [v, uw]$ in (a). Applying condition (γ) we deduce straightaway from (c) that

$$[\bar{f}^a, \bar{k}^b] = 1 \textit{ whenever } f \circ k \textit{ and } k \circ f \textit{ are not defined}. \qquad \text{(d)}.$$

We are now in a position to prove that σ is a homomorphism by checking that it respects relations of types (i), (ii) and (iii).

(i). Let us first show that

$$\sigma(f^a)\sigma(f^b) = \sigma(f^{a+b}).$$

Suppose that $f = g \circ h$; by rearranging (a) we may rewrite the left-hand expression as

$$\begin{aligned} &[\bar{g}^1, \bar{h}^a\bar{h}^b][\bar{h}^a, [\bar{g}^1, \bar{h}^b]^{-1}] \\ &= [\bar{g}^1, \bar{h}^{a+b}][\bar{h}^a, \bar{f}^{-b}], \end{aligned}$$

which from (d) equals $[\bar{g}^1, \bar{h}^{a+b}]$ as required. Since

$$\sigma(1) = [\bar{g}^1, \bar{h}^0] = 1$$

because $\bar{h}^0 = K$ is central, note that this forces $\sigma(f^{-a})$ to be inverse to $\sigma(f^a)$.

(ii). This is immediate from (d) and the fact that $\sigma(1) = 1$.

(iii). We require (δ), whose notation and assumptions we adopt. Then, from (b) and (d),

$$\begin{aligned} [\bar{g}^a, \bar{h}^b] &= [[\bar{r}^1, \bar{s}^a], \bar{h}^b] \\ &= [[\bar{r}^1, \bar{s}^a], [\bar{h}^b, \bar{s}^a]]\,[[\bar{h}^b, \bar{s}^a], \bar{r}^1] \\ &= [\bar{g}^a, \overline{(s \circ h)}^{-ab}]\,[\overline{(s \circ h)}^{-ab}, \bar{r}^1] \\ &= [\bar{r}^1, \overline{(s \circ h)}^{-ab}]^{-1} \\ &= [\bar{r}^1, \bar{t}^{ab}], \end{aligned}$$

the last equality following from the arguments in (i). The case $a = 1$ shows σ to be independent of the choice of factorisation of $f = g \circ h$ and so well-defined. This is immediate for two factorisations through distinct objects as in (δ). Otherwise one has to use (γ) with $k = g$ in order to construct such a pair of factorisations. This construction combines with σ being well-defined, to yield (iii).

Hence σ is a homomorphism after all, and the proof is complete.

Appendix — Homotopy groups of function spaces

The aim here is to determine the homotopy type of certain function spaces which are needed in § 1 above. (All function spaces are to have the compact-open topology.) The literature on this topic is bedevilled by the requirement of local compactness of X for exponential correspondence between maps $W \times X \to Y$ and $W \to Y^X$. As a result much of it seems to divide into two camps: those who state the facts we need in unnecessary speciality, and those who, presumably having missed the point entirely, claim undue generality.

Fortunately, it is possible to vary the local compactness assumption a little (just enough): in the present context of studying homotopy groups, the space W will be a sphere, thus compact. Now exponential correspondence still holds whenever $W \times X$ is a k-space (compactly generated space), and for compact W this occurs whenever X itself is a k-space [8].

PROPOSITION A.1. *Suppose that, for some* n, Y *has* $\pi_j(Y) = 0$ *for* $j > n$ *and* $\pi_1(Y)$ *acting nilpotently on* $\pi_j(Y)$ *for* $n - m < j \leqslant n$, *with* X *an* m*-connected* k*-space. Then, for all* $i \geqslant n - m$,

$$\pi_i((Y, y_0)^{(X, x_0)}) = 0 .$$

Proof. By the preceding remarks, we consider maps

$$S^i \times (X, x_0) \to (Y, y_0) .$$

Let $g: (X, x_0) \to (Y, y_0)$ lie in the relevant path-component of the function space. Then all maps $S^i \times X \to Y$ under investigation have to restrict to $* \vee g: S^i \vee X \to Y$. Obstructions to deforming an arbitrary extension $f: S^i \times X \to Y$ of $(* \vee g)$ to $g \circ pr_2: S^i \times X \to Y$ lie in the cohomology groups

$$H^q(S^i \times X, S^i \vee X; \{\pi_q(Y)\}) \cong \tilde{H}^{q-i}(X; \{\pi_q(Y)\}) .$$

Thus the only possible non-trivial obstructions lie in dimensions $q = i + 1, \dots, n$. So when $m = 0$, $i \geqslant n$ suffices. More generally, let P_{n-m} be the first (lowest) space in the Postnikov tower for Y which is $(n-m)$-

equivalent to Y; then applying the special case shows that obstructions to the deformation $S^i \times X \to P_{n-m}$ vanish whenever $i \geqslant n - m$. Now $Y \to P_{n-m}$ is a nilpotent map [5] (by hypothesis). The obstructions to lifting this deformation to Y thus lie in groups of the form $\tilde{H}^{q-i}(X; G)$ where the coefficients are trivial and $n - m < q \leqslant n$. Finally, the assumption of m-connectivity of X ensures that these groups vanish for $q - i \leqslant m$, that is, $q \leqslant i + m = n$.

Since CW-complexes are compactly generated, we may certainly apply the above result; better, if X, Y merely have the homotopy type of CW-complexes then so does the function space, while the homotopy type of the function space is an invariant of that of X, Y. When combined with the Whitehead theorem, these remarks force the following.

COROLLARY A.2. *Let X, Y have the homotopy type of CW-complexes. Suppose that for some $n \geqslant 1$, $\pi_j(Y) = 0$ whenever $j > n$ and $\pi_1(Y)$ acts nilpotently on $\pi_j(Y)$ for $2 \leqslant j \leqslant n$. If X is $(n-1)$-connected, then each path-component of $(Y, y_0)^{(X, x_0)}$ is contractible.*

COROLLARY A.3. *For any group K and $n \geqslant 1$ (K abelian if $n > 1$), let $X = \mathscr{K}(K, n)$ be an Eilenberg-Maclane space ($\simeq$ CW-complex). Then the path-component of 1_X in $\mathscr{E}(X; x_0)$ is contractible.*

In particular, for $n = 1$, this applies to the homotopy exact sequence of the homotopy fibration $\mathscr{E}(X; x_0) \to \mathscr{E}(X) \to X$ to show the triviality of the groups $\pi_i(\mathscr{E}(X)), i \geqslant 2$. The groups of path-components of $\mathscr{E}(X; x_0)$ and $\mathscr{E}(X)$ have been recorded in [6 p. 42], while $\pi_1(\mathscr{E}(X), 1_X)$ has been determined in [10] (albeit under conditions which our remarks have shown to be unnecessary). So in summary we have the following.

PROPOSITION A.4. *If K is a discrete group and $X = \mathscr{K}(K, 1)$, then $\mathscr{E}(X; x_0)$ has the homotopy type of the discrete group* $\mathrm{Aut}(K)$ *and $\mathscr{E}(X)$ has the homotopy type of* $\mathrm{Out}(K) \times \mathscr{K}(Z(K), 1)$.

The result needed for (1.1) above is obtained by taking the classifying space of the topological monoid $\mathscr{E}(X)$.

COROLLARY A.5. *With K and X as in* (A.4), *there is a fibration*

$$\mathscr{K}(Z(K), 2) \to B\mathscr{E}(X) \to \mathrm{BOut}(K).$$

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