

OBSTRUCTIONS IN K-THEORY, I

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0. Introduction and notation

Ever since the K -theoretic solution to the "vector fields on spheres" problem [1], the machinery of K -theory has been felt to exceed both in power and accessibility that of ordinary cohomology. For example, the solution of the Hopf invariant one problem involves delicate manipulation of secondary cohomology operations or, far more simply, a short calculation in $K(S^{2k} \cup_f B^{4k})$ of $\psi^3 \circ \psi^2 = \psi^2 \circ \psi^3$ [5]. Also, because of the generalization to the equivariant theory, many spaces (for example, homogeneous spaces as in [11]) are more readily computable in K -theory than cohomology.

Of course, relative suitability depends on the particular problem being considered, and the two theories are interdependent through, for example, Atiyah-Hirzebruch [7] and Segal [22] spectral sequences. Nevertheless, in view of the cohomological apparatus available [18], it is surprising that no systematic K -theoretic attack has hitherto been mounted on the fundamental problem of determining the existence of a (never-zero) cross-section of a vector bundle. To pose the question explicitly, let G denote (throughout this paper) a compact Lie group, acting on a compact space B . Suppose E is a complex n -dimensional G -vector-bundle over B , with associated sphere bundle projection $p: SE \rightarrow B$. We seek an equivariant cross-section $s: B \rightarrow SE$ of p ($p \circ s = 1_B$) or rather, conditions on $K_G^*(B)$, $K_G^*(SE)$ which guarantee that no such section can exist. (In general we treat $K_G^*(X)$ as bigraded, and later $K_G^*(X)$ will refer to the $I(G)$ -adic completion of $K_G^*(X)$ as an $R(G)$ -module, where $I(G)$ is the augmentation ideal of the unitary representation ring $R(G)$.)

First recall [5] that exterior multiplication by s induces an isomorphism of the exterior power bundles $\bigwedge^{ev}(E) = \bigoplus_{i \geq 0} \bigwedge^{2i}(E)$, $\bigwedge^{odd}(E) = \bigoplus_{i \geq 0} \bigwedge^{2i+1}(E)$ over B associated to E . Thus, in $K_G(B)$,

$$\lambda_{-1}(E) = \bigwedge^{ev}(E) - \bigwedge^{odd}(E) = 0,$$

so that $\lambda_{-1}(E)$, the " K -theory top Chern class", serves as a primary obstruction to the existence of s . The remarkable fact is how much information is conveyed by its vanishing. In §1 below we point out how it forces the vanishing of various alternative primary obstructions. Then in §2 and §3 we show that it completely "ties up" the algebra of $K_G^*(SE)$. Specifically, (2.2) and (3.1) together imply the following.

THEOREM 0.1. *If $\lambda_{-1}(E) = 0$ in $K_G(B)$, then as $K_G^*(B)$ -algebras*

$$K_G^*(SE) \cong K_G^*(B \times SC^n).$$

Hence SE is $K_G^*(B)$ -algebraically indistinguishable from the sphere bundle of the

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trivial G -vector-bundle $B \times \mathbb{C}^n$. However, the vanishing of $\lambda_{-1}(E)$ is by no means sufficient for the existence of a section, as the following simple example shows.

Example 0.2. Let B be a point, G be $\mathbb{Z}/2 \times \mathbb{Z}/2$, and E be the sum of the three non-trivial one-dimensional G -modules. Then E admits an H -section whenever H is a cyclic subgroup of G , so that $\lambda_{-1}(E) = 0$ in each $R(H)$ and so in $R(G) = K_G(B)$ (cf. (5.2) below), although E has no G -section.

This still leaves open questions of the compatibility of the isomorphism with K -theory operations. Apart from discussion of the relatively easy rational case in §4 below, their consideration is deferred to a sequel. A final section discusses related problems of restriction of G to closed subgroups.

The author apologizes for delay in publishing this material after having lectured on it as long ago as 1975, 1976 at M.I.T., Oxford, Warwick and Oberwolfach. Its appearance even now owes much to the encouragement of Brian Steer, who warned that the results were in danger of becoming "folklore", and who as my Oxford thesis supervisor roused my interest in the subject in the first place.

1. Comparison of primary obstructions

If E admits a G -section, then tensoring with \mathbb{C} gives rise to a trivial line bundle summand of E so that we may write $E = E' \oplus 1$ where the G -bundle E' is $(n-1)$ -dimensional, and 1 denotes the trivial bundle $B \times \mathbb{C}^1$. Thus for all $j \geq n = \dim E$ we have $\lambda^j(E-1) = \lambda^j(E') = 0$, or equivalently $\gamma^j(E-j) = 0$. Further, we have seen that $\lambda_{-1}(E) = 0$. This also follows by multiplicativity:

$$\lambda_{-1}(E) = \lambda_{-1}(E') \cdot \lambda_{-1}(1) = \lambda_{-1}(E') \cdot 0 = 0.$$

Similarly,

$$\rho^k(E) = \rho^k(E') \cdot \rho^k(1) = \rho^k(E') \cdot k,$$

where ρ^k is Bott's "cannibalistic class". These various characteristic classes may be defined in the more general context of (special) λ -rings [10], where the following arguments occur.

PROPOSITION 1.1. *Let $x \in R$, a λ -ring. Then $\lambda_{-1}(x) = 0$ if and only if $\lambda^j(x-1) = \gamma^j(x-j) = 0$ whenever $j \geq \dim x$; when this is the case, $k|\rho^k(x)$ whenever $k \geq 1$.*

Proof. (i) Generalizing [12; Lemma 22], we show by induction on $\dim x$ that, for $j \geq \dim x$,

$$\lambda^j(x-1) = (-1)^j \lambda_{-1}(x).$$

If x is 1-dimensional,

$$\lambda^j(x-1) = (1+xt)(1+t)^{-1} = (1+xt) \sum_{i \geq 0} (-1)^i t^i = 1 + \sum_{j \geq 1} (-1)^j (1-x)t^j.$$

Therefore $\lambda^j(x-1) = (-1)^j (1-x) = (-1)^j \lambda_{-1}(x)$ for $j \geq 1$.

For the general case, note that by the verification principle [10; (3.2)] it suffices

to consider $x = y + a$, where a is one-dimensional, and the result holds for y by the induction hypothesis. So for $j \geq \dim x = \dim y + 1$,

$$\begin{aligned} \lambda^j(y+a-1) &= \lambda^j(y-1) + a\lambda^{j-1}(y-1) = (-1)^j \lambda_{-1}(y) + a(-1)^{j-1} \lambda_{-1}(y) \\ &= (-1)^j(1-a) \cdot \lambda_{-1}(y) = (-1)^j \lambda_{-1}(x). \end{aligned}$$

(ii) Let I_x denote the ideal of R generated by $\{\psi^k(\lambda_{-1}(x))\}_{k \geq 1}$. We prove that $\rho^k(x) \in I_x + k \cdot R$. Indeed, since $\psi^k(I_x + IR) \subset I_x + IR$ and $\rho^{kl}(x) = \psi^k(\rho^l(x)) \cdot \rho^k(x)$, it is enough to consider $\rho^p(x)$, where p is prime; again, by [10; (3;2)] we may suppose that $x = \sum x_i$, where $\dim x_i = 1$. But then

$$\begin{aligned} \rho^p(x) &= \prod \rho^p(x_i) = \prod (1 + x_i + \dots + x_i^{p-1}) \\ &= \prod \frac{1 - x_i^p}{1 - x_i} = \prod (1 - x_i)^{p-1} \pmod{p}, \end{aligned}$$

and $\prod (1 - x_i)^{p-1} = \lambda_{-1}(x)^{p-1}$, so that $\rho^p(x) \in I_x + pR$ as required.

Applications of these obstructions to non-immersions of projective spaces and lens spaces in Euclidean space may be found in [2], [11], where λ^j and λ_{-1} respectively are used.

2. The Gysin sequence

We begin with the exact Gysin sequence (2.1) of [13] for an h -orientable m -dimensional vector bundle E with orientation class $\lambda \in h^m(BE, SE)$, where $h^*()$ is a generalized (multiplicative) cohomology theory. Let $p: SE \rightarrow B$ be the projection map, $t: B \rightarrow BE$ the zero-section and $j: BE \hookrightarrow (BE, SE)$ the inclusion. We are concerned with

$$h^{i-m}(B) \xrightarrow{t^* \circ j^*(\lambda)} h^i(B) \xrightarrow{p^*} h^i(SE) \xrightarrow{\delta} h^{i-m+1}(B) \xrightarrow{t^* \circ j^*(\lambda)} h^{i+1}(B). \quad (2.1)$$

LEMMA 2.2. In (2.1) let $\Omega \in h^{m-1}(SE)$ be an element of square zero with $\delta(\Omega) = 1$. Then $h^*(SE)$ and $h^*(B)[\Omega]/(\Omega^2)$ are isomorphic as graded $h^*(B)$ -algebras.

Proof. We first show that any Ω with $\delta(\Omega) = 1$ yields an additive splitting of (2.1)—an explicit proof that $\text{Ext}_{h^*(B)}(h^*(B), h^*(B)) = 0$. For, the short exact sequence

$$0 \longrightarrow h^i(B) \xrightarrow{p^*} h^i(SE) \xrightarrow{\delta} h^{i-m+1}(B) \longrightarrow 0$$

is split by the $h^*(B)$ -module homomorphism $\theta: h^*(B) \rightarrow h^*(SE)$ of degree $m-1$ given by

$$\theta(x) = p^*(x) \cdot \Omega, \quad x \in h^*(B).$$

In other words, if $x' \in h^*(B)$ then $\theta(x'_i x) = p^*(x') \cdot \theta(x)$, while

$\delta \circ \theta(x) = x$, $\delta(\Omega) = x$. Consequently, there exists an $h^*(B)$ -module homomorphism $\sigma = \sigma_\Omega : h^*(SE) \rightarrow h^*(B)$ of degree 0, such that $\delta \circ \sigma = 0$, $\sigma \circ p^* = \text{id}$; and any $y, y' \in h^*(SE)$ may be written as $y = p^*(x_0) + p^*(x_1) \cdot \Omega$, $y' = p^*(x'_0) + p^*(x'_1) \cdot \Omega$, where $x_i, x'_i \in h^*(B)$ and $x_0 = \sigma(y)$, $x_1 = \delta(y)$, etc.

Note that if we knew that $p : SE \rightarrow B$ admitted a section $s : B \rightarrow SE$, we could choose any $\omega \in \delta^{-1}(1)$ and begin with $\Omega = \omega - p^* \circ s^*(\omega)$ to obtain $\sigma = s^*$ by the above procedure. Then certainly $\sigma(y \cdot y') = \sigma(y) \cdot \sigma(y')$, but possibly $\Omega^2 \neq 0$ (for example, consider $\Omega = \alpha_1(\zeta) \in KO^{-1}(\text{pt.})$ [6; (6.9), (11.5)] or $\Omega = \eta \in KR^{-1}(\text{pt.})$ [4; (3.7)]; see also [24] concerning $K_G^*(X, \mathbb{Z}/p)$).

Now suppose further that $\Omega^2 = 0$. Then

$$\begin{aligned} y \cdot y' &= p^*(x_0) \cdot p^*(x'_0) + p^*(x_0) \cdot p^*(x'_1) \cdot \Omega + p^*(x_1) \cdot \Omega \cdot p^*(x'_0) + p^*(x_1) \cdot \Omega \cdot p^*(x'_1) \cdot \Omega \\ &= p^*(x_0 \cdot x'_0) + \theta(x_0 \cdot x'_1 + (-1)^k x_1 \cdot x'_0), \quad k = (m-1) \cdot \dim y'. \end{aligned}$$

Thus $\sigma(y \cdot y') = x_0 \cdot x'_0 = \sigma(y) \cdot \sigma(y')$, while

$$\delta(y \cdot y') = x_0 \cdot x'_1 + (-1)^k x_1 \cdot x'_0 = \sigma(y) \cdot \delta(y') + (-1)^k \delta(y) \cdot \sigma(y').$$

Hence the homomorphism $y \mapsto \sigma(y) + \delta(y) \cdot \Omega$ from $h^*(SE)$ to $h^*(B)[\Omega]/(\Omega^2)$ is a graded $h^*(B)$ -algebra isomorphism as required.

The case of $h^*(X) = H^*(X; \mathbb{Q})$ is discussed in [18]. In view of the isomorphism $\text{ch} : K^*(X) \otimes \mathbb{Q} \rightarrow H^*(X; \mathbb{Q})$ [7], this may be regarded as a special case of our main interest, $K_G^*(X)$, which we now consider in detail.

3. The square of an element in K^1

In order that Lemma 2.2 apply to equivariant K -theory it suffices to establish the following.

LEMMA 3.1. *Let G be a compact Lie group, and X a locally compact G -space. If $x \in K_G^1(X)$, then $x^2 = 0$ in $K_G^0(X)$.*

Proof. Since $K_G^*(X) \cong K_G^*(X^+)$ where X^+ is compact [22; (2.8)], one may as well assume that X is already a compact G -space. We now describe three proofs (in chronological order).

Firstly, because $K_G^*(X)$ is a graded anti-commutative ring, we have $2x^2 = 0$. So we need only prove the result at the prime (2) in the representation ring $R(G)$. However, (2) contains the augmentation ideal $I(G)$ of $R(G)$, whence

$$K_G^*(X)_{(2)} = (1 + I(G))^{-1} K_G^*(X)_{(2)}.$$

Now the argument of [9; (2.1)] shows that $K_G^*(X) \hat{\rightarrow} \varprojlim K^*(X \times_G E_G^n)$ is always injective (where E_G^n is the join of n copies of G), and of course remains so after localization. Moreover, all elements of $K^1(X \times_G E_G^n)$ have square zero after [14; (4.7)]; it follows that

$$x^2 \in \text{Ker} [K_G^*(X)_{(2)} \longrightarrow K_G^*(X) \hat{\rightarrow}].$$

which coincides with the trivial

$$\text{Ker} [K_G^*(X)_{(2)} \longrightarrow (1 + I(G))^{-1} K_G^*(X)_{(2)}],$$

provided that Krull's theorem [8; (10.17)] applies. This establishes the result whenever $K_G^*(X)$ is finite over $R(G)$, as, for example, when G is a compact G -CW complex or even a compact G -manifold.

Passage to the general compact case occurs through the following device of G. Segal. Given a unitary representation $\rho: G \rightarrow U(n)$, denote by $U(\rho)$ the compact G -manifold $U(n)$ whose G -action is given by

$$(g, A) \mapsto \rho(g)A\rho(g^{-1}).$$

LEMMA 3.2. *Let G be a compact Lie group, X be a compact G -space, and $x \in K_G^1(X)$. Then there exists a representation $\rho: G \rightarrow U(n)$, an element $\theta \in K_G^1(U(\rho))$ and a G -map $f: X \rightarrow U(\rho)$ such that $x = f^*\theta$.*

Proof. Represent x by a G -isomorphism $(\text{id}, x): X \times M \rightarrow X \times M$ where M is a complex G -module, associated to a unitary representation ρ . Then x defines $f: X \rightarrow U(\rho)$ by $f(x)(m) = x(x, m)$ and θ corresponds to the G -isomorphism $U(n) \times M \rightarrow U(n) \times M, (A, m) \mapsto (A, A(m))$.

Of course, (3.2) means that (3.1) follows from a computation of $K_G^*(U(\rho))$. Minami [19; (1.1)] has shown that $K_G^*(U(\rho))$ is an exterior algebra over $R(G)$, so that (3.1) is immediate. Alternatively, one can use (3.1) to make deductions in the reverse direction.

The third proof is along lines kindly suggested by M. C. Crabb. Let \mathbb{R}, H and N denote the $\mathbb{Z}/2$ -modules corresponding respectively to the trivial and non-trivial one-dimensional representations and the regular representation (so that $N = \mathbb{R} + H$). Trivially extend them to $(G \times \mathbb{Z}/2)$ -modules. Then, in the notation of [3], the map $K_G^{-1}(X) \rightarrow K_G^{-2}(X), x \mapsto x^2$, has the factorization (with $\Delta: X \rightarrow X \times X$ as in [5; p. 86])

$$\begin{array}{ccc} K_G^{-1}(X) = K_G(X \times \mathbb{R}) & \xrightarrow{\boxtimes 2} & K_{G \times \mathbb{Z}/2}((X \times \mathbb{R})^2) \\ \downarrow x \mapsto x^2 & & \downarrow (\Delta \times 1)^2 \\ & & K_{G \times \mathbb{Z}/2}(X \times N) \\ & & \downarrow \cong \\ K_G^{-2}(X) = K_G(X \times \mathbb{R} \times \mathbb{R}) & \longleftarrow & K_{G \times \mathbb{Z}/2}(X \times \mathbb{R} \times H). \end{array}$$

The last homomorphism is that of group restriction and is known to be trivial [17; p. V-57]. (In fact $K_{G \times \mathbb{Z}/2}(X \times \mathbb{R} \times H) = K_G(X \times \mathbb{R}) = K_G^{-1}(X)$.)

4. Further obstructions

We have seen that the splitting $\sigma_\Omega: K_G^*(SE) \rightarrow K_G^*(B)$, induced from the vanishing of $\lambda_{-1}(E) \in K_G^0(B)$ and choice of $\Omega \in K_G^1(SE)$, respects the ring structures (indeed,

$K_G^*(B)$ -algebra structures) of $K_G^*(B)$, $K_G^*(SE)$. It remains to investigate the effect on their λ -ring structures. We record here only the first step in that process.

Suppose that the splitting σ_Ω has been defined, along with the associated splitting $\theta_\Omega: K_G^*(B) \rightarrow K_G^{1-\varepsilon}(SE)$, $\varepsilon = 0, 1$, as in §2 above. We can now define, for $k = 0, 1, 2, \dots$,

$$\bar{\lambda}_\Omega^k = \sigma_\Omega \circ \lambda^k \circ \theta_\Omega: K_G^1(B) \rightarrow K_G^0(B).$$

Immediately, $\bar{\lambda}_\Omega^0(y) = 1$ for all y in $K_G^1(B)$. The following lemma justifies this definition.

LEMMA 4.1. $\bar{\lambda}_\Omega^k = 0$ for all $k \geq 1$ if and only if $\sigma_\Omega \circ \lambda^l = \lambda^l \circ \sigma_\Omega$ for all $l \geq 0$.

Proof. Certainly, if $\sigma_\Omega \circ \lambda^l = \lambda^l \circ \sigma_\Omega$ always holds, then for $k \geq 1$ we have $\bar{\lambda}_\Omega^k = \lambda^k \circ \sigma_\Omega \circ \theta_\Omega = 0$. Conversely, if $\bar{\lambda}_\Omega^k = 0$ whenever $1 \leq k \leq l$, then, for any $x \in K_G^0(SE)$ (with notation as in §2),

$$\begin{aligned} \sigma_\Omega \circ \lambda^l(x) &= \sigma_\Omega \circ \lambda^l(p^* \circ \sigma_\Omega(x) + \theta_\Omega \circ \delta(x)) \\ &= \sum_{k=0}^l (\sigma_\Omega \circ \lambda^{l-k} \circ p^* \circ \sigma_\Omega(x)) \cdot (\bar{\lambda}_\Omega^k \circ \delta(x)) \\ &= \sigma_\Omega \circ p^* \circ \lambda^l \circ \sigma_\Omega(x) = \lambda^l \circ \sigma_\Omega(x). \end{aligned}$$

Predictably enough, the rational, non-equivariant case is the easiest to handle. We maintain the notation of §2 above.

PROPOSITION 4.2. If $\lambda_{-1}(E) = 0$ in $K(B) \otimes \mathbb{Q}$ then there exists $\Omega \in K^1(B) \otimes \mathbb{Q}$ such that, for all $k \geq 0$,

$$\sigma_\Omega \circ \lambda^k = \lambda^k \circ \sigma_\Omega: K(SE) \otimes \mathbb{Q} \longrightarrow K(B) \otimes \mathbb{Q}.$$

Proof. Let $\mu(E) \in H^{ev}(B; \mathbb{Q})$ denote Adams' Bernoulli character $(\text{td}(E^*))^{-1}$ (cf. [15; p. 92]), so that [12; Lemme 18], [15; p. 182]

$$\text{ch}(\lambda_{-1}(E)) = (-1)^n \mu(E) c_n(E) \in H^{ev}(B; \mathbb{Q})$$

and the invertibility of $\mu(E)$ implies that $c_n(E) \in H^{2n}(B; \mathbb{Q})$ vanishes along with $\lambda_{-1}(E)$. The following commutative diagram in which the vertical homomorphisms are isomorphisms by [7; Theorem 2.4] and the horizontal sequences are exact, ensues. (Φ, Φ_H are the Thom isomorphisms for K -theory and cohomology.)

$$\begin{array}{ccccccc} K(B) \times \mathbb{Q} & \xrightarrow{p^* \times 1} & K(SE) \times \mathbb{Q} & \xrightarrow{\delta' \times 1} & K^1(BE, SE) \times \mathbb{Q} & \xleftarrow{\sim} & K^1(B) \times \mathbb{Q} \\ \wr \downarrow \text{ch} & & \wr \downarrow \text{ch} & & \wr \downarrow \text{ch} & & \wr \downarrow \text{ch} \\ \sum_r H^{2r}(B; \mathbb{Q}) & \xrightarrow{p^*} & \sum_r H^{2r}(SE; \mathbb{Q}) & \xrightarrow{\delta_H} & \sum_r H^{2r+1}(BE, SE; \mathbb{Q}) & \xleftarrow{\sim} & \sum_r H^{2r-(2n-1)}(B; \mathbb{Q}) \\ & & & & & & \mu(E) \cdot \Phi_H \end{array}$$

There is a similar diagram for $H^{\text{odd}}(SE; \mathbb{Q})$ and $K^1(SE) \times \mathbb{Q}$. Choosing $\Omega_H \in H^{2n-1}(SE; \mathbb{Q})$ such that $\delta'_H(\Omega_H) = \Phi_H(1)$, define $\Omega \in K^1(SE) \otimes \mathbb{Q}$ by $\Omega = \text{ch}^{-1}(p^*(\mu(E)) \cdot \Omega_H)$. Then

$$\delta'(\Omega) = \text{ch}^{-1} \circ \delta'_H(p^*(\mu(E)) \cdot \Omega_H) = \text{ch}^{-1}(\mu(E) \cdot \Phi_H(1)) = \Phi(1) \otimes 1$$

as required for σ_Ω to be defined. Further, for any $y \in K^1(B) \otimes \mathbb{Q}$ we have from [1; Theorem 5.1]

$$\begin{aligned} \text{ch} \circ \psi^k(p^*(y) \cdot \Omega) &= \sum_r k^r \text{ch}_{2r}(p^*(y) \cdot \Omega) \\ &= \sum_r k^r \sum_{i=0}^r p^*(\text{ch}_{2i+1}(y)) \cdot p^*(\mu_{2r-2i-2n}(E)) \cdot \Omega_H \\ &= \text{ch} \left(p^* \circ \text{ch}^{-1} \left(\mu(E)^{-1} \cdot \sum_{i,j} k^{i+j} \text{ch}_{2i+1}(y) \cdot \mu_{2j-2n}(E) \right) \cdot \Omega \right); \end{aligned}$$

whence $\sigma_\Omega \circ \psi^k \circ \theta_\Omega(y) = 0$. The formula [10; p. 265]

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \dots + (-1)^k k \lambda^k(x) = 0$$

now yields (by the multiplicativity of σ_Ω) that $\lambda_\Omega^k(y) = \sigma_\Omega \circ \lambda^k \circ \theta_\Omega(y)$ has torsion and therefore vanishes in $K(B) \otimes \mathbb{Q}$, $k \geq 1$. So the lemma gives the result.

Note that the proof above relies on the surjectivity of ch and is thus of no avail in the integral case for torsion-free B . Likewise it does not seem to be accessible to induction techniques (such as those of the next section) which might lead to the corresponding rational, equivariant case in general. Of course, when G acts freely on B (so that $K_G^*(B) = K^*(B/G)$, $K_G^*(SE) = K^*(SE/G)$) we are reduced to the non-equivariant situation. In particular, one may replace B by $B \times E_G$ (where E_G is a universal G -space) and apply [9].

COROLLARY 4.3. *Suppose $K_G^*(B)$ is finite over $R(G)$ and that $\lambda_{-1}(E) = 0$ in $K_G^*(B) \otimes \mathbb{Q}$. Then there exists $\Omega \in K_G^1(B) \otimes \mathbb{Q}$ such that, for all $k \geq 0$,*

$$\sigma_\Omega \circ \lambda^k = \lambda^k \circ \sigma_\Omega : K_G(SE) \otimes \mathbb{Q} \longrightarrow K_G(B) \otimes \mathbb{Q}.$$

5. Restriction to subgroups

We begin by stating the generalizations to the compact case of useful results on restriction of finite group representations due respectively to G. Segal [20] and S. Jackowski [16]. Firstly, following [20], natural (in X) $K_G(X)$ -module induction homomorphisms $K_H(X) \rightarrow K_G(X)$ (where H is a closed subgroup of G) are defined. If \mathcal{H} is the family of hyperelementary subgroups of G [21] then the image of their sum

$$\bigoplus_{H \in \mathcal{H}} K_H(X) \xrightarrow{\phi} K_G(X)$$

is a submodule (ideal) of $K_G(X)$ containing the image of

$$\bigoplus_{H \in \mathcal{H}} R(H) \longrightarrow R(G) \longrightarrow K_G(X),$$

and in particular the identity element. Thus ϕ is an epimorphism, indeed split (Frobenius reciprocity again) by the restriction homomorphism.

LEMMA 5.1. *Restriction to the family \mathcal{H} of hyper elementary subgroups of G embeds $K_G(X)$ as a natural direct summand of $\bigoplus_{H \in \mathcal{H}} K_H(X)$.*

The second lemma uses the following terminology. A G -module M is finite-dimensional unitary representation space, whose associated isotropy family is to be the family (that is, set closed under inclusion and conjugation) of isotropy subgroups of G on SM , the unit-norm vectors of M .

LEMMA 5.2. *Let M be a G -module with associated isotropy family \mathcal{F} and $(\lambda_{-1}(M))$ the principal ideal of $R(G)$ generated by $\lambda_{-1}(M)$; let the radical of $(\lambda_{-1}(M))$ be $\mathcal{A}(\lambda_{-1}(M))$. Then*

$$(\lambda_{-1}(M)) \subset \text{Ker} \left[R(G) \longrightarrow \bigoplus_{H \in \mathcal{F}} R(H) \right] \subset \mathcal{A}(\lambda_{-1}(M)).$$

Proof. The exact Gysin sequence for the G -sphere bundle SM over a point includes

$$R(G) \xrightarrow{\lambda_{-1}(M)} R(G) \longrightarrow K_G(SM) \longrightarrow 0.$$

On the other hand, restriction to an orbit G/H ($H \in \mathcal{F}$) in SM induces $K_G(SM) \rightarrow K_G(G/H) = R(H)$, and by [22; (5.1)] the elements of $\text{Ker} \left[K_G(SM) \rightarrow \bigoplus_{H \in \mathcal{F}} R(H) \right]$ are nilpotent.

It is now possible to deduce various results of the genre [9; (5.1)] and [16; (4.3)], the following pair being perhaps the most significant.

PROPOSITION 5.3. *Let $f: Y \rightarrow X$ be a G -map between compact G -spaces. If for every cyclic subgroup H of G*

$$f_H^*: K_H^*(X) \rightarrow K_H^*(Y)$$

is an isomorphism, then so is

$$f_G^*: K_G^*(X) \rightarrow K_G^*(Y).$$

Proof. Because of (5.1) above we may assume that G is hyper elementary. Thus the component G^0 of the identity is a torus and all maximal cyclic subgroups of G are of finite index in G and of the form $\pi^{-1}(\langle g_i \rangle)$, where $\pi: G \rightarrow G/G^0$ denotes projection to the group of components of G and g_1, \dots, g_k are generators of representatives of conjugacy classes of maximal cyclic subgroups of G/G^0 . It follows that the G -module

M_i induced from the trivial 1-dimensional $\pi^{-1}(\langle g_i \rangle)$ -module has as its isotropy groups all sub-groups of $\pi^{-1}(\langle g_i \rangle)$ and its conjugates (cf. [23: p. 29]). Its dimension is $(G : \pi^{-1}(\langle g_i \rangle)) = (G/G^0 : \langle g_i \rangle)$. This makes $M = \bigoplus_{1 \leq i \leq k} M_i$ a G -module whose associated isotropy family comprises the cyclic subgroups of G . By (5.2), $\lambda_{-1}(M) = 0$, so that f induces the following homomorphism of Gysin sequences of the G -vector bundles $X \times M \rightarrow X$, $Y \times M \rightarrow Y$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_G^*(X) & \longrightarrow & K_G^*(X \times SM) & \longrightarrow & K_G^*(X) \longrightarrow 0 \\ & & \downarrow f_G^* & & \downarrow (f \times 1_{SM})_G^* & & \downarrow f_G^* \\ 0 & \longrightarrow & K_G^*(Y) & \longrightarrow & K_G^*(Y \times SM) & \longrightarrow & K_G^*(Y) \longrightarrow 0. \end{array}$$

Now $f \times 1_{SM}$ induces an isomorphism of E^2 -terms of Segal's spectral sequences associated to the fibrations $X \times SM \rightarrow SM/G$, $Y \times SM \rightarrow SM/G$ and hence, as in [16: (1.4)], ultimately of $K_G^*(X \times SM)$ and $K_G^*(Y \times SM)$. The result follows.

PROPOSITION 5.4. *Let $f: Y \rightarrow X$ be a G -map between compact G -CW-complexes. Suppose that, for every closed subgroup H of G which projects to a p -group in G/G^0 , the map*

$$f_H^* : K_H^*(X) \rightarrow K_H^*(Y)$$

is an isomorphism. Then so is

$$f_G^* : K_G^*(X) \rightarrow K_G^*(Y).$$

The proof parallels that of (5.3), save that one chooses M to be the sum of G -modules (G hyperelementary) induced from the $\pi^{-1}(P_i)$ -trivial modules, where, for each prime divisor p_i of $(G : G^0)$, P_i is a Sylow p_i -subgroup of G/G^0 . By [21: (3.10)], $\lambda_{-1}(M)$ vanishes in the $I(G)$ -adic completion of $R(G)$. The restriction to compact G -CW-complexes is to allow all relevant modules over representation rings to be finite, in order that the completion functor be exact.

References

1. J. F. Adams, "Vector fields on spheres", *Ann. of Math.* (2), 75 (1962), 603-632.
2. M. F. Atiyah, "Immersions and embeddings of manifolds", *Topology*, 1 (1962), 125-132.
3. M. F. Atiyah, "Power operations in K -theory", *Quart. J. Math. Oxford* (2), 17 (1966), 165-193.
4. M. F. Atiyah, " K -theory and reality", *Quart. J. Math. Oxford* (2), 17 (1966), 367-386.
5. M. F. Atiyah, *K-theory* (Benjamin, New York, 1967).
6. M. F. Atiyah, R. Bott and A. Shapiro, "Clifford modules", *Topology*, 3 (1964), 3-38.
7. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proceedings of Symposia in Pure Mathematics 3 (American Mathematical Society, Providence, 1961).
8. M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra* (Addison-Wesley, Reading, Mass., 1969).
9. M. F. Atiyah and G. B. Segal, "Equivariant K -theory and completion", *J. Differential Geom.*, 3 (1969), 1-18.
10. M. F. Atiyah and D. O. Tall, "Group representations, λ -rings and the J -homomorphism", *Topology*, 8 (1969), 253-297.
11. A. J. Berrick, "Nonimmersion of lens spaces with 2-torsion", *Trans. Amer. Math. Soc.*, 224 (1976), 399-406.

12. A. Borel and J.-P. Serre. "La theoreme de Riemann-Roch", *Bull. Soc. Math. France*, 86 (1958), 97-136.
13. A. Dold. "Relations between ordinary and extraordinary cohomology theories", J. F. Adams. *Algebraic topology—a student's guide*, London Mathematical Society Lecture Notes 4 (Cambridge University Press, Cambridge, 1972), pp. 167-177.
14. S. Gitler and K. Y. Lam, "The K -theory of Stiefel manifolds", *Lecture Notes in Mathematics* 168 (Springer, Berlin, 1970), pp. 35-66.
15. F. Hirzebruch, *Topological methods in algebraic geometry* (Springer, Berlin, 1966).
16. S. Jackowski, "Equivariant K -theory and cyclic subgroups", C. Kosniowski, *Transformation groups*, London Mathematical Society Lecture Notes 26 (Cambridge University Press, Cambridge, 1977), pp. 76-91.
17. M. Karoubi, "Sur la K -théorie équivariante", *Lecture Notes in Mathematics* 136 (Springer, Berlin, 1970), pp. 187-253.
18. S. D. Liao, "On the theory of obstructions of fiber bundles", *Ann. of Math.* (2), 60 (1954), 146-191.
19. H. Minami, "Equivariant K -ring of G -manifold ($U(n)$, ad_p), II", *Osaka J. Math.*, 10 (1973), 571-583.
20. G. Segal, "Equivariant K -theory", D. Phil. Thesis, Oxford University, 1967.
21. G. Segal, "The representation ring of a compact Lie group", *Publications Mathématiques* 34 (Institut des Hautes Études Scientifiques, Paris, 1968), pp. 113-128.
22. G. Segal, "Equivariant K -theory", *Publications Mathématiques* 34 (Institut des Hautes Études Scientifiques, Paris, 1968), pp. 129-151.
23. J.-P. Serre, *Linear representations of finite groups* (Springer, Berlin, 1977).
24. V. P. Snaith. "Dyer-Lashof operations in K -theory", *Lecture Notes in Mathematics* 496 (Springer, Berlin, 1975), pp. 103-294.

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