THE PLUS-CONSTRUCTION AND FIBRATIONS

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[Received 28th April 1981]

1. Introduction

THE plus-construction functor on topological spaces was introduced primarily for its application to the classifying spaces of general linear groups of rings, in order to determine their higher algebraic K-theory [**18**]. It has subsequently been given a bordism-theoretic interpretation [**9**] as well as proving useful in the study of stable homotopy theory [**17**] and of knot complements [**15**]. Because it preserves homology groups, the key issue is its effect on homotopy groups. Here the main tool is of course the long exact sequence of homotopy groups of a fibre sequence $F \to E \xrightarrow{p} B$. In order to apply this to gain information on the plus-construction, we need to know under what conditions $F^+ \to E^+ \xrightarrow{p^+} B^+$ is also a fibre sequence. This is the question we address below.

Most theorems to date deal only with F, E, B as classifying spaces of discrete groups [**7**], [**19**]. The literature also contains one or two unproved assertions. So far as I can tell, all published results, whether proved or not, are (strict, but easy) consequences of the following. (We denote by $\mathcal{P}\pi$ the perfect radical of a discrete group π [**2**]. All homology groups are to have trivial integer coefficients, and all spaces to be of the homotopy type of connected CW-complexes.)

THEOREM 1.1. *If $F \to E \xrightarrow{p} B$ is a fibre sequence, then $F^+ \to E^+ \xrightarrow{p^+} B^+$ is also, provided that either*

(a) $\mathcal{P}\pi_1(B) = 1$, *or*

(b) $\mathcal{P}\pi_1(B)$ *acts trivially* (= *nilpotently*) *on* $H_*(F)$ *and* F^+ *is nilpotent.*

One particular application of Theorem 1.1(a) is to shorten considerably the proof of [**20**] Theorem 4.

I ought to explain the parenthetical assertion in 1.1(b).

PROPOSITION 1.2. *A perfect group P acts trivially on a group H if and only if it acts nilpotently on H.*

Proof. The action is equivalent to a homomorphism $\psi: P \to \mathrm{Aut}\,(H)$. A theorem of P. Hall [**16**] says that if $\mathrm{Im}\,\psi$ acts nilpotently on H then $\mathrm{Im}\,\psi$ must be nilpotent. In particular $\mathrm{Im}\,\psi$ is soluble and its derived series terminates at 1. However P perfect implies Im ψ perfect. So Im $\psi = 1$ and the action is trivial.

The condition that F^+ be nilpotent may be amplified somewhat. For a space X is nilpotent if and only if its fundamental group is nilpotent and acts nilpotently on the homology of the universal cover $\tilde{X}$ [**11** II2.18, 19]. When $X = F^+$ this becomes $\pi_1(F)/\mathcal{P}\pi_1(F)$ is nilpotent and acts nilpotently on $H_*(\bar{F})$ where $\bar{F}$ is the covering space of F associated to $\mathcal{P}\pi_1(F)$. This is because $H_*(\bar{F}) = H_*(\bar{F}^+)$ and $\bar{F}^+ = \widetilde{F^+}$ [**7**].

The proof of 1.1 will be shown in § 2 to hinge on the following characterisation of maps p for which p^+ is particularly well-behaved (namely nilpotent, where we recall from [**4**], [**11**] that this means that $\pi_1(p^+)$ is onto and the Moore–Postnikov decomposition of p admits a refinement by principal fibrations. Likewise, the fibre sequence $F \to E \xrightarrow{p} B$ is said to be quasi-nilpotent if the induced action of $\pi_1(B)$ on $H_*(F)$ is nilpotent.)

THEOREM 1.3. *$F^+ \to E^+ \xrightarrow{p^+} B^+$ is a nilpotent fibre sequence if and only if $F \to E \xrightarrow{p} B$ is a quasi-nilpotent fibre sequence with F^+ nilpotent.*

This is proved in § 3. In § 4 below I remark how these considerations lead to further generalisation of the classical Zeeman comparison theorem between Serre spectral sequences of fibre sequences.

2. Deduction of Theorem 1.1 from Theorem 1.3

In fact, the proof of 1.1(a) does not rely on 1.3 but follows directly from the fundamental properties of the plus-construction. We merely recall [**18**] that for a given X the space X^+ is characterised (up to homotopy equivalence) by $\mathcal{P}\pi_1(X^+) = 1$ and the existence of a map ("Quillenization" [**14**]) $q_X\colon X \to X^+$ whose homotopy fibre F_{q_X} is acyclic (or equivalently, which induces an isomorphism on homology with abelian local coefficients). Now because $B = B^+$ there is a commuting diagram

$$\begin{array}{ccc} F & \xrightarrow{F_{q_E}} & F_{p^+} \\ \downarrow & & \downarrow{\scriptstyle i} \\ E & \xrightarrow{q_E} & E^+ \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p^+} & \\ & B & \end{array}$$

where q_E induces $F_{q_E}\colon F \to F_{p^+}$ on the homotopy fibres. A quite general, elementary argument shows that the homotopy fibres F_{q_E} and F_{Fq_E} have the same homotopy type. We know F_{q_E} is acyclic, so that $F_{F_{q_E}}$ is too. It only remains therefore to check that $\mathscr{P}\pi_1(F_{p^+})$ is trivial. From the exact sequence

$$\pi_2(B) \xrightarrow{\partial} \pi_1(F_{p^+}) \xrightarrow{i_*} \pi_1(E^+),$$

$$i_*\mathscr{P}\pi_1(F_{p^+}) = i_*[\mathscr{P}\pi_1(F_{p^+}), \mathscr{P}\pi_1(F_{p^+})] = [i_*\mathscr{P}\pi_1(F_{p^+}), i_*\mathscr{P}\pi_1(F_{p^+})]$$

is perfect in $\pi_1(E^+)$ and accordingly trivial. So $\mathscr{P}\pi_1(F_{p^+}) \subseteq \partial\pi_2(B)$, whence

$$\mathscr{P}\pi_1(F_{p^+}) = [\mathscr{P}\pi_1(F_{p^+}), \mathscr{P}\pi_1(F_{p^+})] \subseteq \partial[\pi_2(B), \pi_2(B)] = 1$$

because $\pi_2(B)$ is abelian. This clinches the proof of 1.1(a).

This result quickly leads to its own generalisation, as follows.

LEMMA 2.1. *Let $A \to B \xrightarrow{r} C$ be a fibre sequence with $\mathscr{P}\pi_1(C) = 1$. If the square*

$$\begin{array}{ccc} D & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \longrightarrow & B \end{array}$$

is homotopy-Cartesian ($D \simeq A \times_B E$), then $F_{q^+} \simeq F_{p^+}$.

The proof consists in observing that, because of 1.1(a), the diagram of homotopy-Cartesian squares

$$\begin{array}{ccc} D & \longrightarrow & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ A & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle r} \\ PC & \longrightarrow & C \end{array}$$

(in which PC is the contractible space of paths in C) gives rise to another

$$\begin{array}{ccc} D^+ & \longrightarrow & E^+ \\ {\scriptstyle q^+}\downarrow & & \downarrow{\scriptstyle p^+} \\ A^+ & \longrightarrow & B^+ \\ \downarrow & & \downarrow{\scriptstyle r^+} \\ PC & \longrightarrow & C \end{array}$$

That is, $A^+ = F_{r^+}$ and $D^+ = F_{r^+\circ p^+}$. The desired result is immediate from the upper square. Note that the case where A is just a point ($r = \mathrm{id}_B\colon B \to B$) reduces to 1.1(a). Another special case will be needed below. If, as in § 1, $\bar{B}$ is the regular covering space of B with $\pi_1(\bar{B}) = \mathcal{P}\pi_1(B)$, then the fibre sequence $\bar{B} \to B \to K(\pi_1(B)/\mathcal{P}\pi_1(B), 1)$ satisfies the initial condition of 2.1. ($\mathcal{P}(\pi/\mathcal{P}\pi) = 1$ from [**2**] or [**3**].)

LEMMA 2.2. *If the square*

$$\begin{array}{ccc} \bar{E} & \longrightarrow & E \\ {\scriptstyle \bar{p}}\big\downarrow & & \big\downarrow{\scriptstyle \bar{p}} \\ \bar{B} & \longrightarrow & B \end{array}$$

is homotopy-Cartesian, then $F_{\bar{p}^+} \simeq F_{p^+}$.

Deduction of 1.1(b) from 1.3 is now immediate. For the hypotheses of 1.1(b) ensure that the induced fibre sequence $F \to \bar{E} \xrightarrow{p} \bar{B}$ satisfies the necessary (and sufficient) conditions of 1.3. This forces the conclusion that $F^+ \simeq F_{\bar{p}^+} \simeq F_{p^+}$.

3. The fibre-wise plus-construction and proof of 1.3

A crucial lemma here is the following.

LEMMA 3.1. $F_f \to X \xrightarrow{f} Y$ *is nilpotent if and only if it is quasi-nilpotent and* F_f *is nilpotent.*

The "only if" part is due to Bousfield [**5**, 7.2] and the "if" statement to Hilton [**10**, 2.2]. An alternative proof of the latter is worth noting. Observe that after [**6**, 4.2] such a fibre sequence is induced from a universal example of the form $F_f \to \mathrm{Baut}^\circ_G F_f \xrightarrow{g} \mathrm{Baut}_G G_f$ where according to [**6**, 3.4, 3.5] both total space and base are nilpotent. This makes g, and hence its pull-back *f*, nilpotent [**4**, II4.5].

We turn now to the proof of 1.3. The necessity argument is the easier. We have just seen that if $F^+ \to E^+ \xrightarrow{p^+} B^+$ is nilpotent then so is the fibre F^+. So too is its pull-back $F^+ \to B \times_{B^+} E^+ \to B$ induced by $q_B : B \to B^+$. Then the maps p: $E \to B$ and q_E: $E \to E^+$ determine (by functoriality of the plus-construction) a map of fibre sequences

$$\begin{array}{ccc} F & \xrightarrow{q_F} & F^+ \\ & \searrow \qquad \swarrow & \\ & E \to B \times_{B^+} E^+ & \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & B & \end{array}$$

Hence the nilpotent action of $\pi_1(B)$ on $H_*(F^+)$ is equivalent through the homology isomorphism $H_*(q_F)$ to a nilpotent action of $\pi_1(B)$ on $H_*(F)$.

In order to prove the converse, it is convenient to introduce the *fibre-wise plus-construction.* We establish its existence in the following way. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre sequence. Now $q_F\colon F \to F^+$ may be assumed to be a cofibration (if, for example, the inclusion map associated with the adjunction of 2- and 3-cells to F) which enjoys the property of being initial in the category of maps $f\colon F \to X$, X arbitrary, such that $\operatorname{Ker} \pi_1(f) \supseteq \mathscr{P}\pi_1(F)$ (whose morphisms are commuting triangles, apex F). Thus $p \circ i\colon F \to B$ factors through F^+. This defines a unique map p_1 from the push-out $E_1 = E \cup_F F^+$ to B such that $p_1 \circ p' = p$, where $q'\colon E \to E_1$ is the push-out of q_F. A straightforward comparison of homology exact sequences of the pairs $(F^+, q_F(F))$, $(E_1, q'(E))$ (using exicision) shows that q' induces an isomorphism on homology with arbitrary abelian local coefficients, which is equivalent to $F_{q'}$ being acyclic [**3**], [**8**]. Then as in § 2 above Fq' also has acyclic fibre, leaving only the triviality of $\mathscr{P}\pi_1(F_{p_1})$ to be proved. Now $H_1(F_{Fq'}) = 0$ means that $\pi_1(F_{Fq'})$ is perfect, implying that its image, the kernel of $\pi_1(Fq')$, is also perfect. It follows [**2**], [**13**] that $\mathscr{P}\pi_1(F_{p_1}) = (Fq')_*\mathscr{P}\pi_1(F)$. We deduce that (for $i_1\colon F_{p_1} \to E_1$)

$$\begin{aligned} i_{1*}\mathscr{P}\pi_1(F_{p_1}) &= i_{1*}(Fq')_*\mathscr{P}\pi_1(F) \\ &= q'_* i_* \mathscr{P}\pi_1(F) \\ &= i'_* q_{f*}\mathscr{P}\pi_1(F) \subseteq i'_*\mathscr{P}\pi_1(F^+) = 1. \end{aligned}$$

Thus $\mathscr{P}\pi_1(F_{p_1}) \subseteq \operatorname{Ker} \pi_1(i_1) = \partial\pi_2(B)$, forcing $\mathscr{P}\pi_1(F_{p_1})$ to be both perfect and abelian, a task either impossible or trivial. (Bearing in mind that the characteristic properties of F^+ given in § 2 are shared by a unique homotopy type) we have thus confirmed the existence of a map from the fibre sequence $F \to E \xrightarrow{p} B$ to another of the form $F^+ \to E_1 \xrightarrow{p_1} B$.

It is also possible to obtain uniqueness results here. The strongest such applies if for each point b of B the inclusion $b \hookrightarrow B$ is a cofibration (as, for example, when B is a CW-complex). It may then be shown that the inclusion j of E_1 in the total space of the mapping-path fibration $p_0\colon E_0 \to B$ associated to p_1 is also a cofibration. Now from choice of q_F as a cofibration one ends up with a commuting triangle

$$\begin{array}{ccc} E & \xrightarrow{q} & E_0 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_0} & \\ & B & \end{array}$$

in which q is a cofibration, p_0 a fibration and $Fq\colon F \to F_{p_0}$ is equivalent to the plus-construction. Application of [**1**, 2.3] ultimately reveals any two such fibrations p_0 to be ex-homotopy equivalent (i.e. under E and over B) to each other. The argument draws upon the corresponding property for the fibres [**3**], the universal property for $E \cup_F F^+$ and the fact that j is both a cofibration and a homotopy equivalence, in order to construct the ex-homotopy equivalence.

To resume the proof of 1.3, we apply the fibre-wise plus-construction to a quasi-nilpotent fibre sequence $F \to E \xrightarrow{p} B$ where F^+ is nilpotent, and obtain a fibre sequence $F^+ \to E_1 \xrightarrow{p_1} B$. Now $\pi_1(B)$ acts nilpotently on $H_*(F) = H_*(F^+)$ so that p_1 is quasi-nilpotent and thence, by 3.1, nilpotent. Of course $(F^+)^+ = F^+$, while by construction $E_1^+ = E^+$. It therefore suffices to establish the result in the special case where $F \to E \xrightarrow{p} B$ is a nilpotent fibre sequence. (This is indeed a special case because, after 3.1 again, it implies p is quasi-nilpotent and that F is nilpotent, so that $F^+ = F$ is too). We prove first that $p^+\colon E^+ \to B^+$ is nilpotent as required, and then show it has the correct homotopy fibre.

Certainly $\pi_1(p^+)$ is onto, because $p^+ \circ q_E = q_B \circ p$ and both p and q_B induce epimorphisms of fundamental groups. Now we are given that the Moore–Postnikov system of p refines to principal fibrations, and seek to establish that that for p^+ does too. This amounts to showing that if $X_i \to X_{i-1} \to K(A_i, n_i)$, $n_i \geq 2$, is a fibre sequence, then so is $X_i^+ \to X_{i-1}^+ \to K(A_i, n_i)$. However this is an immediate application of 1.1(a).

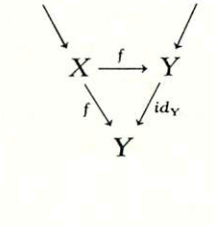
Lastly, we check the homotopy fibre of the nilpotent map $p^+\colon E^+ \to B^+$. From 3.1 both F and F_{p^+} are nilpotent, while by 4.2 below $Fq_E\colon F \to F_{p^+}$ is an integral homology equivalence. From 4.1 below this is enough to guarantee that Fq_E is a homotopy equivalence, as required to complete the proof of 1.3.

4. Further generalisations of the Zeeman comparison theorem

This section discusses the generalisations in [**12**] of theorems which enable one to study the homological effect of a map of fibre sequences. In the first instance, the following direct application of [**12**] generalises Dror's generalised Whitehead theorem [**5** 4.3] where the special case of X, Y nilpotent is proved.

PROPOSITION 4.1. *A nilpotent, integral homology equivalence $f\colon X \to Y$ is a homotopy equivalence.*

Proof. Comparison of the map of quasi-nilpotent fibre sequences

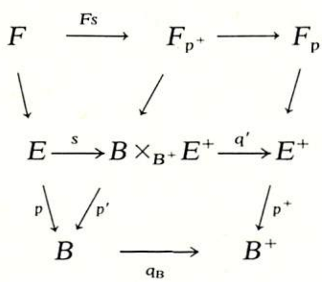


shows that Ff is also an integral homology equivalence. This makes $\tilde{H}_n(F_f)=0$ for all n. In particular, for $n=1$, it forces $\mathcal{P}\pi_1(F_f)=\pi_1(F_f)$. However from 3.1 F_f, and thereby $\pi_1(F_f)$, is nilpotent, making $\mathcal{P}\pi_1(F_f)$ trivial. Thus F_f is simply-connected as well, and hence, by the original Whitehead theorem, contractible.

Note that application of [**12** (3.2, (3.4)] to f, Ff extends this result to mod $\mathcal{C}$ isomorphisms up to dimension n. The next lemma can be very useful, as in § 3 above, for setting up the conditions of 4.1.

LEMMA 4.2. *If the action of $\mathcal{P}\pi_1(B)$ on $H_*(F)$ resulting from the fibre sequence $F\to E\xrightarrow{p} B$ is nilpotent, then Fq_{E*}: $H_*(F)\to H_*(F_{p^+})$ is an isomorphism.*

Proof. By 3.2 we may as well assume p has been pulled-back via the covering of B associated to $\mathcal{P}\pi_1(B)$, or equally, that $\pi_1(B)$ is perfect. Then $\pi_1(B^+)=1$ and thus p^+ is quasi-nilpotent (in fact, orientable). So too therefore is its pull-back p': $B\times_{B^+}E^+\to B$. From functoriality of the plus-construction, there is a map s: $E\to B\times_{B^+}E^+$ over B (i.e. $p=p'\circ s$ and $q_E=q'\circ s$ where q' is the pull-back of q_B).



Because q_B and q' have the same homotopy fibre, which is known to be acyclic, q' must induce an isomorphism on homology (with abelian local coefficients), just as q_E does. Thus s induces an integral homology isomorphism. Since p, p' are quasi-nilpotent we conclude, by [**12**], that $Fs=Fq_E$ is an integral homology equivalence as well.

Although 4.2 is proved using [**12**], it can in turn be made to generalise further the comparison theorems appearing there. As the statement of these results involves refinements (such as homology equivalences, mod $\mathcal{C}$, of the first n homology groups only) which do not concern us here, I merely state the way in which certain conditions of [**12**] on the fundamental groups of the bases may be relaxed. These conditions require that the map $(F\to E\xrightarrow{p} B)\to(F'\to E'\xrightarrow{p'} B')$ of quasi-nilpotent fibre sequences induce an isomorphism $\pi_1(B)\cong\pi_1(B')$, or else that $\pi_1(B)$ and $\pi_1(B')$ both be nilpotent. On the other hand, 4.2 assures us that we may apply the plus-construction to consider instead the map

$(F_{p^+} \to E^+ \xrightarrow{p^+} B^+) \to (F_{p'^+} \to E'^+ \xrightarrow{p'^+} B'^+)$ without having changed any homology groups (or, therefore, quasi-nilpotence). The effect is of course a simplification of fundamental groups. We deduce that

4.3. *The conclusions of* [**12**] Theorems 3.1, 3.2, 3.5 *hold if either*
(a) *The condition* "$\pi_1(B)$, $\pi_1(B')$ nilpotent" *is weakened to* "$\pi_1(B)/\mathcal{P}\pi_1(B)$, $\pi_1(B')/\mathcal{P}\pi_1(B')$ nilpotent", *or*
(b) *the condition* "$\pi_1(B) \to \pi_1(B')$ an isomorphism" *is weakened to* "$\pi_1(B)/\mathcal{P}\pi_1(B) \to \pi_1(B')/\mathcal{P}\pi_1(B')$ an isomorphism".

Note that homological hypotheses which ensure the weakened condition (b) are to be found in [**5** 7.4].

Finally we remark that, because B^+ is simply-connected if $H_1(B)=0$, 4.2 implies the following generalisation of the Serre homology sequence for orientable fibre sequence.

PROPOSITION 4.4. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a quasi-nilpotent fibre sequence in which $\tilde{H}_i(F) = \tilde{H}_j(B) = 0$ whenever $i < s$ and $j < t$. Then there is an exact sequence*

$$H_{s+t-1}(F) \to H_{s+t-1}(E) \to H_{s+t-1}(B) \to H_{s+t-2}(F) \to \cdots \to H_0(B).$$

However, by 1.2 this is a joke.

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