

PROJECTIVE SPACE IMMERSIONS,
BILINEAR MAPS AND STABLE HOMOTOPY GROUPS OF SPHERES

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The first two sections of this article are based on my talk at Siegen, much of which was in turn based on [10], [11]. The results in §3 were obtained after the symposium (and indeed were motivated by it).

The final section is a tabulation of known (to me) immersions and embeddings of (real) projective spaces. Aside from a modification forced by §3 below, the tables are those I circulated during the symposium. Although they exclude most negative results, which defy simple presentation (but see [6], [12], [14]), they do provide a revealing comparison with the previously published tables [18], [25] of a decade ago. However their inclusion here is not for their historical charm but in hope that they may prove of some value to other practitioners of the art.

1. Three classical problems and their (non-classical) interdependence

The three problems are

A Determine for which n, t there exists an immersion (or embedding) of P^n in \mathbb{R}^{2n-t} .

B Determine for which m, n, t there exists a non-singular bilinear map $f: \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{m+n+1-t}$.

C Provide a geometric description of elements of π_*^S , the stable homotopy groups of spheres.

Clearly A and B are well-posed (once one knows that f non-singular - a term we in future suppress - means $f^{-1}(0) = \mathbb{R}^{m+1} \vee \mathbb{R}^{n+1}$). C allows for greater variety of interpretation, as witness the symposium contributions of Eccles, Jones, Löffler, Ossa et al. (The philosophical point is that it would be nice - and even useful - to give some geometric substance to such elements which in many cases are only known as the survivors of some purely algebraic spectral sequence to the bitter end.) Certainly, some elements of π_t^S are bilinearly representable, which is to say obtainable from a bilinear map f by the Hopf construction H , where

$$Hf: S^{m+n+1} \rightarrow S^{m+n+1-t}$$

$$(x \cos \theta, y \sin \theta) \mapsto (\cos 2\theta, \sin 2\theta \cdot f(x, y) / \|f(x, y)\|),$$

$$x \in S^m, y \in S^n, -\frac{\pi}{2} < \theta \leq \pi/2.$$

K.Y. Lam [27] and L. Smith [39] observed that the Kahn-Priddy theorem gives valuable information about the image of H . In particular,

(1.1) THEOREM [27]: If $\alpha \in \pi_*^S$, then 2α is bilinearly representable.

To this encouraging result Al-Sabti and Bier soon replied

(1.2) THEOREM [5]: There exist elements of π_*^S which are not bilinearly representable.

An example, $\bar{\kappa} \in \pi_{20}^S$, is discussed further below. Much earlier H. Hopf himself, and then Ginsburg, had related \underline{B} to \underline{A} . (Recall that a map f with domain $X \times X$ is symmetric if invariant with respect to the interchange involution, i.e. $f(y,x) = f(x,y)$.)

(1.3) THEOREM [23]: If there exists a symmetric bilinear map $f: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n+1-t}$, then P^n embeds in \mathbb{R}^{2n-t} .

(1.4) THEOREM [16]: If there exists a bilinear map $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n+1-t}$ ($t < n$), then P^n immerses in \mathbb{R}^{2n-t} .

(1.3) has the only truly constructive proof of all the results here: restrict f to the diagonal and normalise to obtain an embedding in S^{2n-t} . For progress in the converse direction one has to consider a topological version of \underline{B} :-

\underline{B}' Determine for which m,n,t there exists a biskew map $g: S^m \times S^n \rightarrow S^{m+n-t}$.

Of course biskew means $g(-x,y) = g(x,-y) = -g(x,y)$. Evidently the normalisation of any bilinear map is biskew, and so \underline{H} may be regarded as a construction on biskew maps. Although an example exists of (m,n,t) $(=(12,27,8)$ [19]) for which there is a biskew map which is not the restriction/normalisation of a bilinear one, the loss of information through this specialisation of the problem is not felt to be large. In particular, after Hurwicz-Radon and Adams one knows that for $m = t$ the problems \underline{B} and \underline{B}' are equivalent, as they also are for $m = n < 19$ [1]. Moreover, if one adopts a thoroughly stable point of view then

K.Y. Lam has shown that the Hopf construction fails to distinguish between \underline{B} and \underline{B}' . Specifically,

(1.4) THEOREM [27]: If $\alpha \in \pi_*^S$ may be obtained by the Hopf construction on some biskew map, then α is bilinearly representable.

The proof of (1.4) uses Weierstrass approximation and so allows no control over the dimensions involved. In particular the condition $m = n$, which we impose from now on, has to be sacrificed in obtaining the bilinear map. However, the topological gain in restricting attention to biskew maps is enormous. For there is a precise fit between problems \underline{A} and \underline{B}' ($m = n$).

(1.5) THEOREM [10]: For $2t \leq n-4$ there is a one-to-one correspondence between biskew maps $S^n \times S^n \rightarrow S^{2n-t}$ and immersions $P^n \rightarrow \mathbb{R}^{2n-t}$ and, by restriction, between symmetric biskew maps and embeddings, with biskew homotopy classes corresponding to regular homotopy classes and symmetric biskew homotopy classes corresponding to isotopy classes.

The numerical condition of (1.5) is needed only when $n \leq 15$ [3], [10], but is deemed satisfied for the remainder of this section. The correspondence consists of the composition of two functions π^*, φ which we now describe. Given an n -manifold M , let $\text{Imm}(M, \mathbb{R}^{2n-t})$, $\text{Emb}(M, \mathbb{R}^{2n-t})$ denote respectively the set of (regular homotopy classes of) immersions, embeddings of M in \mathbb{R}^{2n-t} . Then the double covering projection $\pi: S^n \rightarrow P^n$ induces $\pi^*: \text{Imm}(P^n, \mathbb{R}^{2n-t}) \rightarrow \text{Imm}(S^n, \mathbb{R}^{2n-t})$. Meanwhile φ is defined through the induced tangent bundle monomorphism $a_*: \tau S^n \rightarrow (2n-t)\epsilon$ of an immersion $a: S^n \rightarrow \mathbb{R}^{2n-t}$ (ϵ being the trivial line bundle). If one adds a trivial line bundle so as to trivialise the tangent bundle and then restricts to sphere bundles the resulting map

$$S(a_* \oplus 1) : S^n \times S^n \rightarrow S^n \times S^{2n-t}$$

projects to the latter fibre to give $\varphi_a = \text{pr}_2 \circ S(a_* \oplus 1)$. This map is right-skew (indeed it comes from a right-linear map) and further biskew if $a \in \text{Im } \pi^*$.

Not only does $a \mapsto \varphi_{\pi^*a} = \varphi \circ \pi^*(a)$ give the correspondence of Theorem 1.5, but composition with the Hopf construction H yields the following links between problems A and C.

(1.6) THEOREM [11]: Let $t \in \mathbb{N}$.

(i) There exist infinitely many values of n for which

$$H \circ \varphi_{\text{Imm}(S^n, \mathbb{R}^{2n-t})} = \pi_t^S.$$

(ii) $H \circ \varphi_{\text{Emb}(S^n, \mathbb{R}^{2n-t})} = 0$.

(iii) There exist infinitely many values of n for which

$$2\pi_t^S \subset H \circ \varphi \circ \pi^* \text{Imm}(P^n, \mathbb{R}^{2n-t}).$$

(iv) There exists t' such that

$$H \circ \varphi \circ \pi^* \text{Imm}(P^n, \mathbb{R}^{2n-t'}) \neq \pi_{t'}^S.$$

(v) If t is even, then

$$2 \cdot H \circ \varphi \circ \pi^* \text{Emb}(P^n, \mathbb{R}^{2n-t}) = 0.$$

(1.7) EXAMPLE: Consider $\bar{\kappa}$, the generator of the 20-stem, of order 8. It may be represented by an immersion of S^n in \mathbb{R}^{2n-20} which can neither be regular homotopic to an embedding nor factor through P^n . On the other hand, $2\bar{\kappa}$ can be represented by some P^n immersed in \mathbb{R}^{2n-20} , however again this immersion cannot be regular homotopic to an embedding.

2. Inductive methods for projective space immersions

The proof of (1.6) depends crucially on the commutative diagram [9]

$$\begin{array}{ccc}
 & \pi_{2n+1}(S^{2n+1-t}) & \\
 H \circ \varphi \nearrow & & \nwarrow J \\
 \text{Imm}(S^n, \mathbb{R}^{2n-t}) & \xrightarrow{\phi} & \pi_n(V_{2n-t+1, n+1}) \\
 \Omega \searrow & & \nearrow j_* \\
 & \pi_n(V_{2n-t, n}) &
 \end{array}$$

in which J is the generalised J -homomorphism, $j: V_{2n-t, n} \hookrightarrow V_{2n-t+1, n+1}$, Ω is the Smale invariant isomorphism [38], and, given an immersion $a: S^n \rightarrow \mathbb{R}^{2n-t}$, $\phi_a \in \pi_n(V_{2n-t+1, n+1})$ is related to the right-skew map $\varphi_a: S^n \times S^n \rightarrow S^{2n-t}$ by exponential correspondence. (Recall that φ_a is actually induced from a right-linear map $\text{pr}_2 \circ (a_* \oplus 1): S^n \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{2n-t+1}$.) If we begin instead with an immersion $a: P^n \rightarrow \mathbb{R}^{2n-t}$, $\phi_{a \circ \pi}$ is called the Smale invariant of a [9]; the corresponding $\varphi_{a \circ \pi}$ is biskew. When a is an embedding then by (1.5) $\varphi_{a \circ \pi}$ is symmetric, otherwise its twisting $\varphi_{a \circ \pi} \circ T$ by the interchange involution T on $S^n \times S^n$ may, by (1.5), represent a different regular homotopy class of immersions, namely those adjoint to a . In any event the element of $\pi_n(V_{2n-t+1, n+1})$ corresponding to $\varphi_{a \circ \pi} \circ T$ is called the adjoint Smale invariant of a . Thus an embedding is self-adjoint and its Smale invariants coincide. In general these two (regular homotopy) invariants carry much (sometimes all) information concerning whether or not a extends to an immersion of P^{n+1} .

This comes about in the following way. Corresponding to the decomposition of S^{n+1} into a copy of $S^n \times I$ and two copies of B^{n+1} according as its last coordinate lies in $[-\frac{1}{3}, \frac{1}{3}]$ or $[-1, -\frac{1}{3}]$, $[\frac{1}{3}, 1]$, there is a decomposition of P^{n+1} as $P^{n+1} = BH_n \cup B^{n+1}$, where H_n is the canonical line bundle over P^n .

(2.1) LEMMA [10]: An immersion of P^n in \mathbb{R}^{2n-t} extends to an immersion of BH_n in \mathbb{R}^{2n-t} if and only if its adjoint Smale invariant is zero.

This is the key lemma. Note that it involves an extension of the original immersion itself and not merely an extension up to regular homotopy. This carries the bonus that if we start with an embedding, then a neighbourhood of the zero-section of BH_n is also embedded. Thus

(2.2) COROLLARY [10]: An embedding of P^n in \mathbb{R}^{2n-t} extends to an embedding of BH_n in \mathbb{R}^{2n-t} if and only if its Smale invariant is zero.

It now remains to extend the embedding over $BH_n \cup B^{n+1}$. This can cost a dimension (and when $n+1$ is a power of 2 must cost a dimension) if one wishes to avoid creating double points. In other words, the following theorem is best possible in general.

(2.3) THEOREM [10]: An embedding of P^n in \mathbb{R}^{2n-t} with trivial Smale invariant extends to an immersion of P^{n+1} in \mathbb{R}^{2n-t} and embedding of P^{n+1} in \mathbb{R}^{2n-t+1} .

Theorem 3.1 below is an application of this result. Predictably, relaxation of conditions on the original immersion weakens the conclusion about its extension.

(2.4) THEOREM [10]: An immersion of P^n in \mathbb{R}^{2n-t} with trivial Smale and adjoint Smale invariants extends to an immersion of P^{n+1} in \mathbb{R}^{2n-t+1} .

3. Embeddings and immersions of even-dimensional spaces

As an application of the methods discussed in the previous section we establish general embedding and immersion results for even-dimensional projective spaces.

Inevitably we need a few number-theoretic functions, even for statements of results. Discussion of their properties is deferred until after (3.2), while an early glance at (3.1), (3.6) may help to motivate their digestion by the reader. It is also helpful to adopt two notations for the value of functions $\beta, \gamma: \{0, 1, 2, 3, \dots\} \rightarrow \{0, 1, 2, 3, \dots\}$ at n , namely β_n if β is bounded above, and $\gamma(n)$ if γ is not so bounded. Suppose

$$n = \sum_{i=0}^{\tau(n)} u_i 2^i, \quad u_i \in \{0, 1\} \quad u_{\tau(n)} = 1.$$

Then define

$$\kappa_n = u_0 + (u_1 + u_2)^2 - \max(u_0, u_1, u_2),$$

$$\nu(n) = \max\{i \geq 0 \mid u_0 = \dots = u_{i-1} = 0\},$$

$$\alpha(n) = \sum_{i \geq 0} u_i,$$

$$\rho(n) = u_1 + \sum_{i \geq 3} u_i 2^{i-3} = u_1 + \left\lfloor \frac{n}{8} \right\rfloor,$$

$$\phi(n) = \min(\alpha(n), \rho(n)),$$

$$\theta(n) = \min(\alpha(n) + \kappa_n, \rho(n)).$$

The main result of this section applies our inductive techniques to the odd-dimensional embeddings e_S and immersions i_M of Steer [41], [8] and Milgram [33] respectively.

(3.1) THEOREM: Let $n \geq 32$ be even. Then

$$e_S: P^{n-1} \rightarrow \mathbb{R}^{2n-1-\alpha(n-1)-\kappa_{n-1}}$$

extends to an immersion

$$P^n \rightarrow \mathbb{R}^{2n-\alpha(n)-\kappa_n}$$

and embedding

$$P^n \rightarrow \mathbb{R}^{2n+1-\alpha(n)-\kappa_n} .$$

Apart from the case $n \equiv 0 \pmod{8}$, when an improvement of one dimension is effected, the Euclidean space dimensions are those obtained by restriction of the embedding/immersion of P^{n+1} established in [41], [33]. (For odd m , Milgram's $k(m) = \kappa_m$.) Because of the geometric nature of our induction technique, we are able to avoid getting immersed (sic) in the algebraic machinations of [33], [41], beyond checking the following (by means of [33] §3(4)).

(3.2) LEMMA: Milgram's immersions i_M are regular homotopic under restriction (of projective spaces) and inclusion (of Euclidean spaces).

Given this, it is easy enough to see that $i_M: P^{n-1} \rightarrow \mathbb{R}^{2n-\alpha(n)-\kappa_n}$ has trivial Smale invariant ((3.7) below); the crucial lemma is that e_S does too. To decide in which Euclidean space $i_M(P^{n-1})$ and $e_S(P^{n-1})$ have the same Smale invariant we need to know something about the function θ . First note that ρ satisfies the iterative condition:-

(3.3) For $n \geq 8$,

$$\rho(n) = \rho(2^{\tau(n)} - 1) + \rho(n - 2^{\tau(n)}).$$

Further,

(3.4) For m odd,

$$\rho(m) - \alpha(m) \leq 0 \quad \text{if } m < 32,$$

$$0 \leq \rho(m) - \alpha(m) \leq \kappa_m \quad \text{if } 32 < m < 64, \quad m \equiv 7 \pmod{8},$$

$$\rho(m) - \alpha(m) \geq \kappa_m \quad \text{otherwise.}$$

From (3.3) and (3.4) it is simple enough to verify the following valuable observation, which says that θ is a good function for the purposes of Lemma 3.6. (Indeed, thanks to (3.3) there is none better.)

(3.5) LEMMA: Let m > 8 be odd. Then

$$\theta(m) \leq \theta(2^{\tau(m)} - 1) + \phi(m - 2^{\tau(m)}).$$

(3.6) LEMMA: Let m be odd. Then e_S and $i_M: P^m \rightarrow \mathbb{R}^{2m+1-\theta(m)}$ have the same Smale invariants.

Proof. By induction on m. For $m = 1, 5$ $\theta(m) = 0$ and the Smale invariants lie in the trivial group $\pi_m(V_{2m+1, m})$. For $m = 3, 7$ $\theta(m) = 1$ and, after [10] (3.4), the Smale invariants are the non-trivial element of $\pi_m(V_{2m, m}) \simeq \mathbb{Z}/2$. If $m \geq 8$, then $e_S \circ \pi: S^m \rightarrow \mathbb{R}^{2m+1-\alpha(m)-\kappa_m}$ restricts on $S^{2^{\tau}-1} \times S^{m-2^{\tau}} \subset S^m$ (τ being $\tau(m)$ and $\pi: S^m \rightarrow P^m$ the double covering projection) to the product $e_1 \circ \pi \times e_2 \circ \pi$, where (up to regular homotopy) e_1, e_2 are embeddings

$$e_1: P^{2^{\tau}-1} \rightarrow \mathbb{R}^{2(2^{\tau}-1)-\tau-3},$$

$$e_2: P^{m-2^{\tau}} \rightarrow \mathbb{R}^{2(m-2^{\tau})-\alpha(m-2^{\tau})+1}.$$

Likewise $i_M \circ \pi$ restricts to $i_1 \circ \pi \times i_2 \circ \pi$ in $\mathbb{R}^{2(2^\tau-1)-\tau-4} \times \mathbb{R}^{2(m-2^\tau)-\alpha(m-2^\tau)}$. By inductive hypothesis (and Lemma 3.2), $e_1 \circ \pi$ and $i_1 \circ \pi$ are regular homotopic in $\mathbb{R}^{2(2^\tau-1)+1-\theta(2^\tau-1)}$, as are $e_2 \circ \pi$ and $i_2 \circ \pi$ in $\mathbb{R}^{2(m-2^\tau)+1-\phi(m-2^\tau)}$. It follows from Lemma 3.5 that the product immersions $e_1 \circ \pi \times e_2 \circ \pi$ and $i_1 \circ \pi \times i_2 \circ \pi$ are regular homotopic in $\mathbb{R}^{2m-\theta(m)}$. Hence $e_S \circ \pi$ and $i_M \circ \pi$ are regular homotopic in $\mathbb{R}^{2m+1-\theta(m)}$ when restricted to $S^{2^\tau-1} \times S^{m-2^\tau} \times I$ (the $\mathbb{Z}/2 \times \mathbb{Z}/2$ -cover of the disc bundle $\text{pr}_1^*(H) \otimes \text{pr}_2^*(H)$ over $P^{2^\tau-1} \times P^{m-2^\tau}$).

To complete the proof we must consider the obstruction to extending a regular homotopy (of $\mathbb{Z}/2 \times \mathbb{Z}/2$ -invariant immersions) from $S^{2^\tau-1} \times S^{m-2^\tau} \times I$ to $(S^{2^\tau-1} \times B^{m-2^\tau+1} \times O) \cup (S^{2^\tau-1} \times S^{m-2^\tau} \times I)$. After [22] (5.9) this is equivalent to the obstruction to a linear homotopy between certain pairs of vector bundle monomorphisms, viz. a difference element of

$$\begin{aligned} & \text{Mono}(\tau(S^{2^\tau-1} \times B^{m-2^\tau+1}), (S^{2^\tau-1} \times B^{m-2^\tau+1}) \times \mathbb{R}^{2m+1-\theta(m)}) \\ = & \text{Mono}(\text{pr}_1^* \tau(S^{2^\tau-1}) \times \mathbb{R}^{m-2^\tau+1}, \text{pr}_1^*(S^{2^\tau-1} \times \mathbb{R}^{m+2^\tau-\theta(m)}) \times \mathbb{R}^{m-2^\tau+1}) \\ \cong & \text{Mono}(\tau(S^{2^\tau-1}), S^{2^\tau-1} \times \mathbb{R}^{m+2^\tau-\theta(m)}) \end{aligned}$$

by stability and contractibility of $B^{m-2^\tau+1}$. This last difference element is that between two Smale invariants (of projective space immersions) in $\pi_{2^\tau-1}^{(V_{2^\tau+m-\theta(m)}, 2^\tau-1)}$. Since $m-\theta(m) \geq 2^\tau-2$ the obstruction vanishes (with occasional help from [10] (3.4)). Similarly, the obstruction to extending the regular homotopy over $(S^{2^\tau-1} \times S^{m-2^\tau} \times I) \cup (B^{2^\tau} \times S^{m-2^\tau} \times 1)$ also vanishes, whence $e_S \circ \pi$ and $i_M \circ \pi$ are regular homotopic over

$$S^m = (S^{2^\tau-1} \times B^{m-2^\tau+1} \times O) \cup (S^{2^\tau-1} \times S^{m-2^\tau} \times I) \cup (B^{2^\tau} \times S^{m-2^\tau} \times 1),$$

which is to say, e_S and i_M have the same Smale invariant.

An immediate consequence of this lemma is that the Smale and adjoint Smale invariants of $i_M: P^m \rightarrow \mathbb{R}^{2m+1-\theta(m)}$ are equal. However, more than this is true. For the bilinear map [33] Corollary 7 used to construct i_M has the additional property ([33] Theorem 6(i)) that the interchange involution on $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1}$ commutes with a sign change on all components of the range $\mathbb{R}^{2m-\alpha(m)-\kappa_m+1}$ of i_M other than the first. Thus the induced axial map $P^m \times P^m \rightarrow P^{2m-\alpha(m)-\kappa_m}$ is symmaxial provided $\alpha(m) + \kappa_m$ is even; otherwise its inclusion into $P^{2m-\alpha(m)-\kappa_m+1}$ is symmaxial. Equivalently $i_M: P^m \rightarrow \mathbb{R}^{2m-\alpha(m)-\kappa_m+\epsilon}$ is self-adjoint where $\epsilon = 0$ or 1 according as $\alpha(m) + \kappa_m \equiv 0$ or $1 \pmod{2}$.

The theorem now follows immediately from Theorem 2.3, Lemma 3.6 and the following lemma.

(3.7) LEMMA: Let n be even. Then $i_M: P^{n-1} \rightarrow \mathbb{R}^{2n-\alpha(n)-\kappa_n}$ has trivial Smale invariant.

This is in turn a consequence of the following much more general result. It seems worth providing a proof here as the literature contains several statements of the assertion - all incorrect!

(3.8) PROPOSITION: Milgram's system of non-singular bilinear maps includes

$$\mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{n+m+1-\alpha(n)+\alpha(n-m)-(2^d-d-1)},$$

where $n \geq m$ and $d = \min(3, \nu(n+1), \nu(m+1))$.

Proof. Let F be the division algebra of dimension 2^d over \mathbb{R} , and choose $k > \tau(n)$. Milgram constructs (cf. [26] (7.1))

$$F^{2^k} \times F^{2^k} \rightarrow \mathbb{R}^{2^{k+d+1}-k-2^d}.$$

The inclusions

$$\mathbb{R}^{n+1} = F_{2^k - \frac{n+1}{2^d} + 1} \oplus \dots \oplus F_{2^k} \subset F_1 \oplus \dots \oplus F_{2^k}$$

$$\mathbb{R}^{m+1} = F_{2^k - \frac{n+1}{2^d} + 1} \oplus \dots \oplus F_{2^k - \frac{n-m}{2^d}} \subset F_1 \oplus \dots \oplus F_{2^k}$$

induce a restriction to $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1}$ whose first

$$2^d \cdot 2(2^k - \frac{n+1}{2^d}) - \alpha(2(2^k - \frac{n+1}{2^d}))$$

and last

$$2^d \cdot \frac{n-m}{2^d} - \alpha(\frac{n-m}{2^d})$$

components vanish, after [33] Lemma 11 and the correct version ([41]

Lemma 4.2(i)) of [33] Lemma 9. (In other words, in this last reference replace $V^{i-m+2,k}$ by $V^{m,k}$. This allows the induction when $2^{k-1} < m$. On the other hand, application of the original version leads eventually, via (3.9), (3.10), to a contradiction.)

Here is a construction which suggests an easy recapture of Steer's embedding of P^n in $\mathbb{R}^{2n-\alpha(n)-\kappa_n+1}$, by combining with (3.8) for $p = n-m-1$ as the mod 8 residue of n , and using the $m \equiv 7 \pmod{8}$ case as inductive hypothesis. (However, an analogue of (3.6) is needed for iteration of the $7 \pmod{8}$ case.)

(3.9) PROPOSITION: Suppose an embedding of $(q+1)H_p$ in \mathbb{R}^M restricts to an embedding of P^p in $\mathbb{R}^{M-(2q+1-N)}$. If P^q immerses in \mathbb{R}^N then P^{p+q+1} immerses in \mathbb{R}^{M+N} , embeds in \mathbb{R}^{M+N+1} , provided $M+N-2 > p+q+\max(p,q)$.

Proof. We have an embedding of P^p in $\mathbb{R}^{M-(2q+1-N)}$ with normal bundle v_1 such that $(q+1)H_p \subset v_1 \oplus (2q+1-N)\epsilon$. By [22] and stability any embedding of P^q in \mathbb{R}^{2q+1} has $(2q+1-N)\epsilon$ as a subbundle of its normal bundle v_2 . Thus $\text{pr}_2^*(q+1)H_p$ is a subbundle of the normal bundle $\text{pr}_1^*v_1 \oplus \text{pr}_2^*v_2$ of the product embedding $P^p \times P^q \rightarrow \mathbb{R}^{M-(2q+1-N)} \times \mathbb{R}^{2q+1}$. So the disc bundle $\text{pr}_1^*B((q+1)H_p)$ embeds in \mathbb{R}^{M+N} . After [21] the connectivity of the embedding restricted to $\text{pr}_1^*S((q+1)H_p) = \text{pr}_2^*S((p+1)H_q)$ ensures that it extends from (a neighbourhood of) the sphere bundle to an embedding over $\text{pr}_2^*B((p+1)H_q)$ (cf. [41] Proposition 3.1). The union of these two embeddings is then an immersion of

$$P^{p+q+1} = \text{pr}_1^*B((q+1)H_p) \cup \text{pr}_2^*B((p+1)H_q)$$

in \mathbb{R}^{M+N} which regularly deforms to an embedding in \mathbb{R}^{M+N+1} .

This result considerably strengthens [31] Theorem 1.2.

Unsurprisingly, there is also a biskew-map version. (In this we employ the join identifications $[w_1, 0, w_2] = w_1$, $[w_1, 1, w_2] = w_2$.)

(3.10) PROPOSITION: Let $f': S^p \times S^p \rightarrow S^p$, $q': S^q \times S^q \rightarrow S^q$ be symmetric biskew maps which extend respectively to biskew

$$f: S^{p+q+1} \times S^p \rightarrow S^{p+r}$$

($r \geq 0$) and to

$$g: S^q \times (S^q * S^{r-1}) \rightarrow S^q$$

with

$$g(-x, [-z_1, u, z_2]) = g(x, [z_1, u, z_2]) = -g(x, [-z_1, u, -z_2]) \quad x, z_1 \in S^q, \\ z_2 \in S^{r-1}, u \in I.$$

Then there is a symmetric biskew map

$$h: S^{p+q+1} \times S^{p+q+1} \rightarrow S^{p+q+1}.$$

Proof. Given $x = [x_1, s, x_2]$, $y = [y_1, t, y_2] \in S^p * S^q$ with $s \geq t$,
if $t < 1$ let

$$f\left([x_1, \frac{s-t}{1-t}, x_2], y_1\right) = [w, u, z] \in S^p * S^{r-1},$$

so that $u = 0$ if $s = t$ or $r = 0$.

Then define

$$h(x, y) = \begin{cases} [w, \max(2t-1, u), g(x_2, [y_2, \max(1-2t, 0), z])] & t < 1, \\ g(x_2, y_2) & t = 1, \end{cases}$$

and $h(y, x) = h(x, y)$.

4. Summary of immersion and embedding results

The number t listed below for a given n is the largest known value of t for which P^n immerses (\subseteq) or embeds (\subset) in \mathbb{R}^{2n-t} . An asterisk indicates that this value is also known to be best possible, a cross that if $8 \leq n \pmod{16} \leq 11$ then this value can be improved to $t+1$. Where the preceding bound or reference continues to apply the relevant square is left blank.

$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17
$n \equiv 0 \pmod{8}$														
\subseteq	1*	9					11 ⁺		13		α			
Refs	[44]	[36]					[14]		[15]		[26]			
\subset	0*	6						$\alpha-1$						
Refs	[43]	[42]						(3.1)						
$n \equiv 1 \pmod{8}$														
\subseteq	0*	3*	4*	8		11	13	13 ⁺	15			α		
Refs		[38]	[2]*	[36]		[14]			[15]			[33]		
\subset	0*	1*	3	6				$\alpha-1$						
Refs		[28]*	[29]	[42]				[41]						
$n \equiv 2 \pmod{8}$														
\subseteq	1*	4*	6*	7	8	13	15 ⁺						α	
Refs	[44]	[38]		[14]	[13]	[14]							[33]	
		[7]*	[2]*											
\subset	0*	3*	4	6				$\alpha-1$						
Refs	[23]	[37]		[42]				[41]-[8]						
		[30]*	-[28]*											
$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17

$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17
$n \equiv 3 \pmod{8}$														
\subseteq		2*	6*	8*	9	10	15	17	17 ⁺					$\alpha+1$
Refs		[34]	[38]-[2]*	[14]	[13]	[14]								[33]
\subset		1*	3	6	7		α							
Refs		[23]	[37]	[42]	[10]		[41]-[8]							
$n \equiv 4 \pmod{8}$														
\subseteq	1*	6*	7	9	10		11	13			α			
Refs		[35]	[36]	[10]		[14]		[15]			[33]			
			[17]*											
\subset	0*	3	6				$\alpha-1$							
Refs		[29]	[42]				[41]-[8]							
$n \equiv 5 \pmod{8}$														
\subseteq		3*	4*	8		12		13	15		$\alpha+1$			
Refs		[22]	[38]	[36]		[10]	[14]		[15]		[33]			
			[2]*											
\subset		1*	3*	6			7	α						
Refs		[23]	[29]	[42]		[37]		[41]-[8]						
		[32]*	[4]*											
$n \equiv 6 \pmod{8}$														
\subseteq		5*	6*	8	9	14		15	16		$\alpha+3$			
Refs		[22]	[38]	[10]	[14]	[10]	[14]				[33]			
			[2]*											
\subset		2	5*	7		$\alpha+2$								
Refs		[37]	[4]*	[10]		[41]-[8]								
$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17

$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17
$n \equiv 7 \pmod{8}, n \neq 63$														
\subseteq			6*	8*	9	14		16	17	18	$\alpha+4$			
Refs			[22]	[39]	[14]						[33]			
				[2]*										
\subset			4	7*	8	$\alpha+3$								
Refs			[37]	[4]*		[41]-[8]								
$n=2^\alpha-1 \geq 15$														
\subseteq														
$\alpha \equiv 0 \pmod{4}$				$2\alpha^*$										
$\alpha \equiv 1, 2 \pmod{4}$				$2\alpha-1^*$										
$\alpha \equiv 3 \pmod{4}$				$2\alpha+1^*$										
Refs				[24]-[20]*										
$\alpha(n)$	1	2	3	4	5	6	7	8	9	10-13	14	15	16	≥ 17

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