

REMARKS ON THE STRUCTURE OF ACYCLIC GROUPS

A. J. BERRICK

1. The Theorem

THEOREM. *Any finite-dimensional complex representation of an acyclic group A restricts trivially to any finite subgroup of A .*

There are two reasons why this is a remarkable result. First, the definition of an acyclic group involves no representation theory but is strictly homological. (Recall that A is acyclic if $H_i(A; \mathbb{Z}) = 0$ for all $i > 0$.) Second, previous evidence had suggested an abundance of normal subgroups of acyclic groups. For example:

(1.1) [8] *Any group G may be normally embedded in a group which is itself a normal subgroup of an acyclic group.*

(1.2) [11] *Any perfect group is a homomorphic image of a (torsion-free) acyclic group.*

In recent years acyclic groups have received increasing attention as a result of their importance in algebraic K-theory, foliation theory, cohomology of groups and elsewhere. See [8, 10] for collections of examples.

2. Proof of the Theorem

Let $\mathcal{P}G$ denote the maximum perfect subgroup (perfect radical) of a group G .

(2.1) *If X is an acyclic space and $\mathcal{P}\pi_1(Y) = 1$ then $[X, Y]$ is trivial.*

Proof. We use Quillen's plus-construction [4, Chapter 5]. We have, for any map $f: X \rightarrow Y$,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X^+ & \xrightarrow{f} & Y^+ \end{array};$$

however, because $\mathcal{P}\pi_1(Y) = 1$, the map $Y \rightarrow Y^+$ is just $\text{id}: Y \rightarrow Y$. On the other hand $\pi_1(X)$ is perfect, so X^+ is both acyclic and simply-connected, hence contractible. Since f factors through a contractible space, it is nullhomotopic.

Now to the proof of the main result. In fact we prove the more general assertion that any homomorphism $\rho: A \rightarrow \text{GL}_\infty(\mathbb{C})$, where $\text{GL}_\infty(\mathbb{C}) = \text{dir lim } \text{GL}_n(\mathbb{C})$, has trivial

Received 25 February 1989; revised 28 July 1989.

1980 *Mathematics Subject Classification* 20C32.

Bull. London Math. Soc. 22 (1990) 227–232

restriction to any finite subgroup of A . Since any element of A of finite order is a product of elements of prime power order, it suffices to establish the triviality of ρ on any cyclic subgroup C of prime power order. We therefore show that the restriction $I(A) \rightarrow I(C)$ of augmentation ideals of complex representation rings is the zero map. In the commuting square

$$\begin{array}{ccc} I(A) & \longrightarrow & [BA, BGL_{\infty}(\mathbb{C})] \\ \downarrow & & \downarrow \\ I(C) & \twoheadrightarrow & [BA, BGL_{\infty}(\mathbb{C})], \end{array}$$

the lower horizontal map corresponds to the injective map [1, (6.11), (7.2)]

$$I(C) \twoheadrightarrow I(C)^{\wedge} \xrightarrow{\cong} \tilde{K}(BC),$$

while by (2.1) $[BA, BGL_{\infty}(\mathbb{C})] = 0$ since $\pi_1(BGL_{\infty}(\mathbb{C})) \cong \tilde{K}(S^1) = 0$.

Although I now have other, more recent proofs (also topological), the one presented here bears most directly on acyclic groups.

3. Consequences and remarks

Much of this section and the next is a group-theoretic exercise on the normal subgroup structure of a group A having the property of the theorem, under the assumption that A is perfect (occasionally, superperfect).

Thus for example a residually finite acyclic group must be torsion-free.

One immediate consequence of the theorem is (a) of the following; in view of (1.2), (b) is just a reformulation of the Feit–Thompson odd-order theorem.

- (3.1) (a) *A torsion-generated acyclic group has no subgroup of finite index.*
 (b) *A torsion-free acyclic group has no normal subgroup of odd index.*

(3.2) *If $N \trianglelefteq A$ is finitely generated and residually finite, then its torsion elements are central in N ; in particular, if N is generated by torsion elements, then N is abelian, hence finite abelian.*

Proof. Since the hypotheses imply that $\text{Aut}(N)$ is residually finite, apply the theorem to the map from the acyclic group to $\text{Aut}(N)$ induced by conjugation.

(3.3) Two consequences of (3.2) are especially striking. The second is an observation of C. Soulé.

- (a) *A finite group is normal in an acyclic group if and only if it is abelian.*
 A stronger version of sufficiency appears as (3.6) below.
 (b) *$GL_n(\mathbb{Z})$ is normal in an acyclic group if and only if $n = 1$ or ∞ .*

For the case $n = \infty$, see [14]. Similar assertions hold for more general rings R with $K_1(R)$ finitely generated and satisfying a suitable stable range condition.

(3.4) *If a 1-relator group with torsion is normal in an acyclic group, then it is not residually finite.*

(3.5) (3.4) has an interesting relation to the following two areas of research.

(a) G. Baumslag's Conjecture: *All 1-relator groups with torsion are residually finite.*

(b) [3] *Any torsion-free 1-relator group which is perfect-by-infinite cyclic is acyclic-by-infinite cyclic. There exist such.*

(3.6) The following result, when taken together with the part of (3.3)(a) that follows from the theorem, says that among finite groups only the abelian groups can be normal in acyclic groups, but among the abelian groups all conceivable normalities may be realised.

Let G be abelian, $P \leq \text{Aut}(G)$. Then there exists an acyclic group A with G normal in A and $A/C_A(G) \cong P$ if and only if P is perfect.

Proof. 'Only if'. Obvious, since A is perfect.

'If'. As in (1.2) there exists an acyclic group B and epimorphism $B \rightarrow P$. With $A_G = M(\Lambda, (\mathbb{Z}, G))$ as in [7], observe that any $\alpha \in \text{Aut}(G)$ extends to $\bar{\alpha} \in \text{Aut}(A_G)$ via

$$\bar{\alpha}(x_{\lambda\mu}^r) = \begin{cases} x_{\lambda\mu}^{\alpha(r)} & \lambda = 0, \\ x_{\lambda\mu}^r & \lambda > 0. \end{cases}$$

Therefore set $A = A_G \rtimes B$ via the B -action on A_G given by $B \rightarrow P \rightarrow \text{Aut}(A_G)$. Then $C_A(G) = A_G \rtimes \text{Ker}(B \rightarrow P)$, so $A/C_A(G) \cong P$. Thus A is acyclic because A_G acyclic forces $\tilde{H}_*(A) \approx \tilde{H}_*(B) = 0$ from $A_G \rightarrow A \rightarrow B$.

(3.7) Because finitely generated subgroups of linear groups over fields are residually finite [12, (7.11)], it follows from the theorem that, at least for a finitely generated acyclic group, the conclusion of the theorem holds for finite-dimensional representations over arbitrary fields.

(3.8) It is immediate from the theorem that torsion-generated acyclic groups belong to the class of groups which admit no non-trivial finite-dimensional complex representation. By the result quoted in (3.7), this class also contains any finitely generated group which has no proper subgroups of finite index (such as Higman's torsion-free 4-generator acyclic group [2]). According to [9], it also contains the commutator subgroup of $\text{GL}_n D$ whenever D is a division algebra which is infinite-dimensional over its centre. A continuous analogue is the result [13] that a connected semisimple Lie group without compact simple constituents admits no non-trivial finite-dimensional unitary representation.

4. Torsion-generated acyclic groups

(4.1) Let A be a torsion-generated acyclic group, and write $A_1 = A/\mathcal{L}(A)$. Because A is perfect, A_1 is centreless.

(4.2) Here it is useful to consider the class \mathcal{T} of groups which contain no non-trivial homomorphic image of a torsion-generated acyclic group. Evidently \mathcal{T} is closed under the formation of subgroups, cartesian products (hence residually-closed) and extensions. Clearly, \mathcal{T} contains all torsion-free and soluble groups, and, by the theorem, all groups admitting faithful complex representations. Since acyclic

groups A are superperfect, \mathcal{T} also contains all quotients of \mathcal{T} -groups G by subgroups N in the hypercentre of G or which have $\text{Out}(N)$ in \mathcal{T} ; for in either case the induced extension over the acyclic group is trivial (compare [6]), making the map $A \rightarrow G/N$ factor through G . Also, because A_1 is centreless, it can admit no normal subgroup whose automorphism group lies in \mathcal{T} , and so, in particular, no finitely generated residually finite normal subgroup.

(4.3) Since A has no finite non-trivial quotient while A_1 is centreless, A_1 has no proper subgroups of finite index; each element and each non-normal subgroup of A_1 has infinitely many conjugates. (Compare [8, (2.3)].)

(4.4) Any just-infinite normal subgroup of A_1 is an infinite simple-by-finite abelian group.

Proof. First, we check that the subgroup, M say, may not be residually finite. For by [15, Proposition 9] M has only finitely many subgroups of each finite index n ; their intersection $H(n)$ is therefore characteristic in M , hence normal in A_1 . Because $M/H(n)$ is finite the induced map $A_1 \rightarrow \text{Aut}(M/H(n))$ is trivial, making $[A_1, M] \leq H(n)$. Since $\bigcap_n H(n) = 1$ for M residually finite, this forces M to be central in A_1 , a contradiction of (4.1). Thus by [15, Proposition 1] M is monolithic with monolith a finite direct product of isomorphic simple groups. Then by (4.3) there is just one factor in the product and it cannot be finite. To check that the monolith of M is the commutator subgroup M' observe by [15, Proposition 9] again that any normal subgroup of M contains a characteristic subgroup K . Since $\text{Aut}(M/K)$ is finite, the A -action is trivial and $[A_1, M] \leq K$. The same conclusion may be deduced for any non-central just-infinite normal subgroup of A .

(4.5) For M normal in A_1 (or A) the following are equivalent.

- (i) M/M'' is f.g.
- (ii) There exists $n > 1$ such that, for the lower central series term $\gamma_n M$ and for any prime field ℓ , the abelianisations $M_{\text{ab}} \otimes \ell$ and $(\gamma_n M)_{\text{ab}} \otimes \ell$ are finite-dimensional over ℓ .
- (iii) M is f.g. abelian or infinite perfect-by-f.g. abelian.

Proof. Clearly (iii) implies the others. Note as before that the perfect subgroup of (iii) cannot be finite. We work first with M normal in A_1 . For (i) \Rightarrow (iii), observe that, being f.g. metabelian, M/M'' is f.g. residually finite; hence, as with (3.2), it is abelian.

To check (ii) \Rightarrow (iii), we note that the action of A on $M_{\text{ab}} \otimes \ell$ corresponds to a map through A to some $\text{GL}_n(\ell)$. Since $\text{GL}_n(\ell)$ admits a faithful finite-dimensional complex representation, the homomorphism must be trivial. It follows that A_1 acts trivially on $M_{\text{ab}} = M/M'$. Therefore $[A_1, M] = M'$. Since A_1 is perfect, application of the Three Subgroups Lemma to A_1, A_1, M and to A_1, M, M yields that

$$M' = [A_1, M] = [A_1, [A_1, M]] = [A_1, [M, M]] = [[A_1, M], M] = \gamma_3(M).$$

Thus in the second part of the hypothesis $\gamma_n(M) = M'$; repetition of the above argument now shows that $M' = M''$ as required.

Finally, we obtain the corresponding results for N normal in A . Since (i) and (ii)

are preserved by central quotients, we deduce from the above that $[A, N] = [A, N\mathcal{Z}(A)] \leq N'\mathcal{Z}(A) = N''\mathcal{Z}(A)$. It follows that $[A/N'', [A/N'', N/N'']] = 1$, whence N/N'' is central in perfect A/N'' . This leaves $N' = N''$.

(4.6) Note that the hypothesis that A be torsion-generated may be weakened in (4.5), provided that the subgroup $T(A)$ generated by all elements of A of finite order is large in relation to the normal subgroup N . Specifically, similar arguments show that (i) \Rightarrow (iii) still when $N \leq T(A)N'C_N(N/N'')$, and (ii) \Rightarrow (iii) provided $N \leq \mathcal{P}(T(A)N)$. Thus, for example, an f.g. soluble normal subgroup of an acyclic group must be abelian.

(4.7) Of course, in this context there is always a trivial method for obtaining perfect normal subgroups, namely by taking for A a direct product of acyclic groups. For a partial converse to this, suppose that a perfect normal subgroup P of A has $\text{Out}(P) \in \mathcal{T}$. Then because $\text{Hom}(A, \text{Out}(P))$ is trivial, the obstruction to non-triviality of the extension $P \rightarrow A \rightarrow A/P$ lies in $H^2(A/P; \mathcal{Z}(P))$ [6]. Now the exact sequence

$$H_2(A) \longrightarrow H_2(A/P) \longrightarrow H_1(P)_A \longrightarrow H_1(A) \longrightarrow H_1(A/P)$$

and the universal coefficient theorem together make the obstruction group vanish. Hence $A \cong P \times A/P$, so that P is after all a torsion-generated acyclic group.

(4.8) Similarly to the previous arguments, if N normal (but non-central) in A has the map $\text{Aut}(N) \rightarrow \text{Out}(N/\mathcal{P}N)$ factoring through a group in \mathcal{T} , then, using [5, (2.5)], one has that N is infinite perfect-by-abelian.

It is a pleasure to acknowledge the stimulus of conversations with D. Benson and P. Kropholler to this work, and to thank the referee for very helpful comments on its presentation.

References

1. M. F. ATIYAH, 'Characters and cohomology of finite groups', *Math. Publ. IHES* 9 (1961) 23–64.
2. G. BAUMSLAG, E. DYER and A. HELLER, 'The topology of discrete groups', *J. Pure Appl. Algebra* 16 (1980) 1–47.
3. G. BAUMSLAG and K. W. GRUENBERG, 'Some reflections on cohomological dimension and freeness', *J. Algebra* 6 (1967) 394–409.
4. A. J. BERRICK, *An approach to algebraic K-theory*, Research Notes in Math. 56 (Pitman, London, 1982).
5. A. J. BERRICK, 'Group epimorphisms preserving perfect radicals, and the plus-construction', *Algebraic K-theory, number theory, geometry and analysis (Proc. Bielefeld, 1982)*, Lecture Notes in Math. 1046 (Springer, Berlin, 1984), pp. 1–12.
6. A. J. BERRICK, 'Group extensions and their trivialisation', *Enseign. Math.* 31 (1985) 151–172.
7. A. J. BERRICK, 'Two functors from abelian groups to perfect groups', *J. Pure Appl. Algebra* 44 (1987) 35–43.
8. A. J. BERRICK, 'Universal groups, binate groups and acyclicity', *Proc. Singapore Group Theory Conference 1987* (Walter de Gruyter, Berlin, 1989), pp. 253–266.
9. Y. CHEN, 'On the representations of skew linear groups', *Bull. London Math. Soc.* 21 (1989) 267–269.
10. P. DE LA HARPE and D. MACDUFF, 'Acyclic groups of automorphisms', *Comment. Math. Helv.* 58 (1983) 48–71.
11. A. HELLER, 'On the homotopy theory of topogenic groups and groupoids', *Illinois J. Math.* 24 (1980) 576–605.
12. R. C. LYNDON and P. E. SCHUPP, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete 89 (Springer, Berlin, 1977).

13. I. E. SEGAL and J. VON NEUMANN, 'A theorem on unitary representations of semisimple Lie groups', *Ann. of Math* (2) 52 (1950) 509-517.
14. J. B. WAGONER, 'Delooping classifying spaces in algebraic K-theory', *Topology* 11 (1972) 349-370.
15. J. S. WILSON, 'Groups with every proper quotient finite', *Proc. Cambridge Philos. Soc.* 69 (1973) 373-391.

Department of Mathematics
National University of Singapore
Kent Ridge 0511
Singapore