THE SAMELSON EX-PRODUCT

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[Received 29 May 1974: in revised form 22 November 1974]

0. Introduction

THE Samelson product [6] of two maps into an H-space corresponds broadly to their basic commutator in the loop of homotopy classes of maps into the H-space. In the burgeoning theory of ex-spaces and ex-maps (see, for example, [5]) it is of interest to determine under what conditions there is an analogue in this more general setting. This counterpart we derive in § 6 below, calling it the Samelson ex-product. En route we pass through some quite interesting ex-homotopy theory. In addition, our results extend the previously known range of definition of the 'ordinary' Samelson product. All definitions and notation may be found in § 1.

The purpose of this paper, then, is to establish Theorem (6.1) below, whose notation we now follow. Clearly, slightly different techniques are required to prove (6.1) under each of the seven sets of conditions. Nevertheless, a common thread runs through the proofs, which we now outline.

Since $\langle f_1, f_2\rangle_B^B \in [K_1 \wedge' K_2, E]_B^B$, it suffices to consider $[K_1 \times' K_2, E]_B^{K_1 \vee' K_2}$, where the map $K_1 \vee' K_2 \rightarrow E$ factors through the section $s\colon B \rightarrow E$. Indeed, we consider $[K_1 \times' K_2, E \times' E]_E^{K_1 \vee' K_2}$, where the maps $K_1 \vee' K_2 \rightarrow E \times' E \rightarrow E$ are induced from $K_1 \vee' K_2 \rightarrow E \rightarrow B$. In (2.3) we show that, if $K_1 \vee' K_2 \hookrightarrow K_1 \times' K_2$ is a closed cofibration (considered in § 3), then a f.h.e. (fibre homotopy equivalence) $g\colon E \times' E \rightarrow E \times' E$ induces a bijection from this set to a similar one. When $g = \pi_1 \times (\pi_1 . \pi_2)$ ($\pi_j\colon E \times' E \rightarrow E$ being projection to the j-th factor, $j = 1, 2$) corresponds to left-multiplication, then the fact that g_* is a bijection corresponds to the existence of a unique right-multiplicative inverse and so leads to the existence of a unique commutator, as desired. Thus it remains to show, as in (5.1) say, that g is truly a f.h.e.

The necessary steps in this argument are proved in rather greater generality than the above sketch indicates. Applications of this work are reserved for a subsequent paper. The author would like to thank the referee for his stimulating comments.

1. Notation and definitions

In general we follow [2]. A, B are arbitrary topological spaces.

1.1 (E, p, B, s) is an *ex-space* if it is an object of Top$_B^B$ [2] and if in

Quart. J. Math. Oxford (2), 27 (1976), 173–180

addition s is a *section* of $p\colon E \to B$, i.e. $s\colon B \to E$ with $p \circ s = id_B$. One sometimes refers to E itself as an ex-space (e.g. (1.3)). An *ex-map* is a morphism of Top^B_B (or, more generally, of Top^A_B); likewise, an *ex-homotopy* is required to be an ex-map at each stage in the homotopy.

1.2 $\times'$, $\vee'$, $\wedge'$ denote the fibre product, fibrewise wedge and smash product constructions respectively. If X, E are objects of Top^A_B, $[X, E]^A_B$ denotes the set of h-equivalence (in Top^A_B) classes of morphisms (of Top^A_B) $X \to E$. If $A = \phi$ or $B =$ point it is omitted from the notation.

1.3 An ex-space E is an *ex-H-space* if there exists a multiplication B-map $m\colon E \times' E \to E$ such that each of the B-maps

$$e \mapsto m(s \circ p(e), e), \qquad e \mapsto m(e, s \circ p(e)), \qquad e \in E$$

is B-homotopic to id_E. Thus E is fibrewise an H-space with the section determining the neutral elements. *Fibre homotopy (f.h.) inverses* and *associativity (f.h.a.)* are defined in the obvious way. We sometimes write $f_1 . f_2$ for the map $m \circ (f_1 \times f_2)\colon K_1 \times' K_2 \to E$, where $f_i\colon K_i \to E$; also $\pi_i\colon E \times' E \to E$ is the projection map to the i-th factor, $i = 1, 2$.

1.4 B is a *D-space* if for any two h-fibrations [2] $p'\colon E' \to B$, $p\colon E \to B$, and $f\colon E' \to E$ with $p \circ f = p'$, the restiction of f $f \mid p'^{-1}(b)\colon p'^{-1}(b) \to p^{-1}(b)$ being a h.e. (homotopy equivalence) for all $b \in B$ implies that f is a f.h.e. (fibre homotopy equivalence). (In (4.4) below we show that this definition is equivalent to that with "fibrations" in place of "h-fibrations".)

1.5 For ex-spaces (Z, q, B, t), (E, p, B, s), a B-map $f\colon Z \to E$ is *actionable* if there exists a B-map $\varphi\colon Z \times' E \to E$ such that

$$\varphi \circ (t \circ p, id_E) \underset{B}{\simeq} id_E\colon E \to E,$$

and

$$\varphi \circ (id_Z, s \circ q) \underset{B}{\simeq} f\colon Z \to E.$$

2. Bijections on sets of ex-homotopy classes

2.1 Lemma. *Let*

$$\begin{tikzcd} A \arrow[r, "f''"] \arrow[d, "i"'] & Y_1 \arrow[d, "g"] \\ X \arrow[r, "f'"'] & Y_2 \end{tikzcd}$$

be a commutative diagram, with i a closed cofibration, g a h.e. *Then there exists $f\colon X \to Y_1$ such that $f \circ i = f''$, $g \circ f \overset{A}{\simeq} f'$.*

(2.2) is proved by applying **[10]** Theorem 9 to the mapping path fibration of g. We leave the details to the reader. This result corresponds to **[7]** (7.6.22) in precisely the same way as **[9]** Theorem 4 corresponds to **[7]** (7.8.9). Thus (2.1) and **[9]** Theorem 4 may be combined to yield the following analogue of **[7]** (7.8.11) in just the same way as **[7]** (7.8.11) itself follows from **[7]** (7.6.22), (7.8.9).

2.2 THEOREM. *Let*

$$\begin{tikzcd}
A \arrow[r, "f''"] \arrow[d, "i"'] & E_1 \arrow[r, "g''"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\
X \arrow[r, "f'"'] & B_1 \arrow[r, "g'"'] & B_2
\end{tikzcd}$$

be a commutative diagram with i a closed cofibration, p_1, p_2 fibrations and g', g'' h.e. *Then any map $\bar{h}$: $X \to E_2$ in* $\mathrm{Top}^A_{B_2}$ *has a lifting $\bar{f}$: $X \to E_1$ in* $\mathrm{Top}^A_{B_1}$ *such that*

$$g'' \circ \bar{f} \overset{A}{\underset{B_2}{\simeq}} \bar{h}.$$

Now suppose in the commutative diagram of (2.2) that $B_1 = B_2 = B$ with $g' = id_B$; write g for g''.

2.3 THEOREM. (a) *Let i: $A \to X$ be a closed cofibration, p_j: $E_j \to B$ fibrations, $j = 1, 2$, with g: $E_1 \to E_2$ a* h.e. *Alternatively,* (b) *let (X, A) be a relative CW-complex (inclusion map i), p_j a weak fibration* [7], *$j = 1, 2$, and g a weak equivalence.*

Then composition with g induces a bijection

$$g_*\colon\ [X, E_1]^A_B \xrightarrow{\approx} [X, E_2]^A_B.$$

Proof. For (a), (2.2) immediately establishes surjectivity. Since the inclusion $\dot{I} \hookrightarrow I$ is a closed cofibration, so also, by **[10]** Theorem 6, is the induced map $A \times I \cup X \times \dot{I} \to X \times I$, which, when used in (2.2), shows that g_* is injective as well. The result under condition (b) is just **[7]** (7.8.12).

In passing we note that two-fold application of (2.3) (a) yields the following (see (1.4)).

2.4 THEOREM. *Suppose given closed cofibrations s_j: $A \to E_j$ and fibrations p_j: $E_j \to B$, $j = 1, 2$, where B is a D-space. If g: $E_1 \to E_2$ is a morphism of* Top^A_B *which restricts to a* h.e. *on each fibre of p_1, then g is an ex-homotopy equivalence.*

3. Fibrations and cofibrations

Let K_1, K_2 be spaces over B. Since B-maps from $K_1 \wedge' K_2$ correspond to B-maps from $K_1 \times' K_2$ with fibrewise constant restriction on $K_1 \vee' K_2$, for the present purposes we shall be interested in (2.3) when $A = K_1 \vee' K_2$, $X = K_1 \times' K_2$. Hence it is useful to know when the inclusion $K_1 \vee' K_2 \hookrightarrow K_1 \times' K_2$ is a closed cofibration. The appropriate result, below, follows from **[10]** Theorems 12 and 13.

3.1 LEMMA. *Let* q_i: $K_i \to B$ *be a fibration with section* t_i: $B \to K_i$ *which is a closed cofibration,* $i = 1, 2$. *Then the inclusion* $K_1 \vee' K_2 \hookrightarrow K_1 \times' K_2$ *is a closed cofibration.*

[1] contains a general example illustrating the conditions of (3.1).

4. D-spaces

We introduce a class of spaces well suited to our discussion in §5 of maps into ex-H-spaces, namely D-spaces (D for Dold). See (1.4) for the definition. By **[3]** Theorem 6.3:—

4.1 THEOREM. *A space B which admits a numerable covering by sets whose inclusion in B is nulhomotopic, is a D-space.*

We now show that the class of D-spaces has the desirable quality of being an invariant of homotopy type.

4.2 PROPOSITION. *If B is dominated by a D-space A (i.e. there exist maps* α: $A \to B$, β: $B \to A$ *with* $\alpha \circ \beta \simeq id_B$*), then B is a D-space.*

Proof. Let f: $E' \to E$ be a B-map where p': $E' \to B$, p: $E \to B$ are h-fibrations; suppose f restricts to a h.e. over each point of B. Consider the pull-back of f by α, f_α: $E'_\alpha \to E_\alpha$: it is an A-map which restricts to a h.e. over each point of A. Since A is a D-space f_α is a f.h.e., inverse g say. But $\alpha \circ \beta$: $B \to B$ is a h.e.; so by **[2]** Satz (7.30) and **[3]** Theorem 6.1, $\tilde{\alpha} \circ \tilde{\beta}$: $E_{\alpha \circ \beta} \to E$ has f.h. inverse θ: $E \to E_{\alpha \circ \beta}$ say, while $(\tilde{\alpha} \circ \tilde{\beta})'$: $E'_{\alpha \circ \beta} \to E'$ is also a f.h.e. Then $(\tilde{\alpha} \circ \tilde{\beta})' \circ g_\beta \circ \theta$ is a f.h. inverse to f.

Another attractive feature of this class, given in (4.3) below, is also a property of the class of spaces of the homotopy type of a CW-complex (by **[8]** Proposition (0)) but not, it seems, of the class of spaces satisfying the condition of (4.1). Such being the case, the class of D-spaces is strictly larger than this last class.

By (4.2) it makes sense to speak of 'the' fibre of an h-fibration as a D-space.

4.3 PROPOSITION. *If the fibre and base space of an h-fibration* p: $E \to B$ *are D-spaces, then so is E.*

Proof. Let $f\colon \bar{E}' \to \bar{E}$ be an E-map between h-fibrations $q'\colon \bar{E}' \to E$, $q\colon \bar{E} \to E$; suppose that f restricts to a h.e. over each point of E. Let b be any point of B, whose inclusion in B induces an inclusion map $\alpha\colon p^{-1}(b) \hookrightarrow E$. Then, since f_α restricts over each point of $p^{-1}(b)$ to a h.e., and $p^{-1}(b)$ is a D-space, f_α is a f.h.e. That is, the restriction of f over each point b of B is a h.e. Since B is a D-space and $p \circ q'\colon \bar{E}' \to B$, $p \circ q\colon \bar{E} \to B$ are h-fibrations, f is a f.h.e. over B. In particular f is a h.e. So, by [3] Theorem 6.1, f is a f.h.e. over E.

Had D-spaces been defined in terms of fibrations rather than h-fibrations, it might appear at first glance that a larger class of spaces would have been obtained. (4.4) reveals this not to be the case, i.e. the two definitions are equivalent.

4.4 Proposition. *Let B be such that any B-map between fibrations which restricts over each point of B to a* h.e. *is a* f.h.e. *Then B is a D-space.*

The proof of (4.4) uses mapping path fibrations which in this case (see [3] Theorem 6.1) are f.h.e. to the h-fibrations. Details are left to the reader.

5. Actionable maps

In the notation of (1.5), $\varphi\colon Z \times' E \to E$ induces a Z-map

$$g\colon Z \times' E \to Z \times' E \quad (z, e) \mapsto (z, \varphi(z, e)) \quad z \in Z, \quad e \in E.$$

In particular, if E is an ex-H-space, multiplication $m\colon E \times' E \to E$, any B-map $f\colon Z \to E$ is actionable, by virtue of $\varphi = m \circ (f \times id_E)$; indeed, E is an ex-H-space precisely when $id_E\colon E \to E$ is actionable.

5.1 Proposition. *If $f\colon Z \to E$ is an actionable B-map where $p\colon E \to B$ is an h-fibration, Z is a D-space and $q^{-1}(b)$ is path-connected for all $b \in B$, then an induced Z-map $g\colon Z \times' E \to Z \times' E$ is a* f.h.e. *over Z.*

Proof. Since Z is a D-space and $p_q\colon Z \times' E \to Z$ is an h-fibration with $p_q^{-1}(z) = \{(z, e) \mid e \in p^{-1}(q(z))\}$, it suffices to show that for any $z \in Z$ the map

$$\varphi_z\colon p^{-1}(q(z)) \to p^{-1}(q(z)), \qquad e \mapsto \varphi(z, e)$$

is a h.e. Let $q(z) = b \in B$ and let $\lambda\colon I \to q^{-1}(b)$ be a path from z to $t(b)$ in $q^{-1}(b)$. Then there is a homotopy

$$\Phi\colon p^{-1}(b) \times I \to p^{-1}(b)\colon \varphi_z \simeq \varphi_{t(b)}, \quad (e, u) \mapsto \varphi(\lambda(u), e) \quad e \in E, \quad u \in I.$$

By definition, $\varphi_{t(b)} \simeq id_{p^{-1}(b)}$, so that φ_z is indeed a h.e.

Now suppose that E is an ex-H-space, this being where our applications lie. By tightening the other conditions, we may relax the assumption in (5.1) that $\pi_0(q^{-1}(b)) = 1$.

5.2 PROPOSITION. (a) *Let E be an ex-H-space where $p\colon E \to B$ is a weak fibration and $\pi_0(p^{-1}(b))$ is a loop for all $b \in B$. Then g, defined as above, is a weak equivalence.*

If in addition $p\colon E \to B$ is an h-fibration and either (b) *$Z \times' E$ has the homotopy type of a CW-complex, or* (c) *Z is a D-space and the fibres of p have the homotopy type of a CW-complex*†, *then g is a* h.e. *over Z.*

Proof. By the argument of Lemma 6 of [**11**], g induces a weak equivalence on the fibres of $p_q\colon Z \times' E \to Z$. (a) now follows by the Five Lemma; (b) follows in turn. When the fibres of p and so those of p_q have the homotopy type of a CW-complex, g is a h.e. on each fibre; so (c) follows.

5.3 We remark that if $Z = E$ and B has the homotopy type of a CW-complex, then Propositions (0), (12) and Corollary (13) of [**8**] show conditions (b) and (c) of (5.2) to be equivalent.

5.4 Finally, note that when E is a f.h.a. ex-H-space, g is a h.e. over Z for all Z if and only if E has a right f.h. inverse as an ex-H-space. This is the case when, for example, E is a fibrewise loop space.

6. Defining the product

We now assemble the jigsaw.

6.1 THEOREM. *Let (E, p, B, s) be an ex-H-space, (K_i, q_i, B, t_i) an ex-space, $f_i\colon K_i \to E$ an ex-map, $i = 1, 2$, with projection ex-map $k\colon K_1 \times' K_2 \to K_1 \wedge' K_2$. Then there exists a unique ex-homotopy class of ex-maps $\omega\colon K_1 \wedge' K_2 \to E$—the Samelson ex-product $\langle f_1, f_2\rangle_B^B$ of f_1, f_2—such that*

$$(f_2 \cdot f_1) \cdot (\omega \circ k) \underset{B}{\overset{B}{\simeq}} f_1 \cdot f_2 ,$$

provided any of the following conditions holds.

(i) *$p\colon E \to B$ is a weak fibration, $\pi_0(p^{-1}(b))$ is a loop for all $b \in B$, and $(K_1 \times' K_2, K_1 \vee' K_2)$ is a relative CW-complex.*

(ii)–(vii) *$p\colon E \to B$ is a fibration, $q_i\colon K_i \to B$ is a fibration, and $t_i\colon B \to K_i$ is a closed cofibration, $i = 1, 2$. Further:—*

(ii) *E is a D-space and p has path-connected fibre.*

(iii) *$K_1 \times' K_2$ is a D-space and q_i has path-connected fibre, $i = 1, 2$.*

(iv)–(vi) *$\pi_0(p^{-1}(b))$ is a loop for all $b \in B$. Further:—*

(iv) *$E \times' E$ has the homotopy type of a CW-complex.*

(v), (vi) *The fibre of p has the homotopy type of a CW-complex. Further:—*

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† Since the proof for (c) does not use the Five Lemma, the assumption that p is a weak fibration is unnecessary in that case.

(v) *E is a D-space.*

(vi) *$K_1\times' K_2$ is a D-space.*

(vii) *E has a right* f.h. *inverse and is* f.h.a. *as an ex-H-space.*

Proof. Since E is an ex-H-space, the existence of $g\colon Z\times' E\to Z\times' E$ (see §5) is guaranteed, where $Z=K_1\times' K_2$ or $Z=E$ as indicated below. The results of §2 and §5—we record below the specific results required for each case (i)–(vii)—now show the existence of a unique element

$$\bar\omega\in[K_1\times' K_2, Z\times' E]_Z^{K_1\vee' K_2}$$

such that

$$g_*(\bar\omega)=h\in[K_1\times' K_2, Z\times' E]_Z^{K_1\vee' K_2},$$

where

$$h(k_1,k_2)=\begin{cases}((k_1,k_2), m(f_1(k_1), f_2(k_2))) & Z=K_1\times' K_2,\\ (m(f_2(k_2), f_1(k_1)), m(f_1(k_1), f_2(k_2))) & Z=E,\ k_i\in K_i,\end{cases}$$

by choosing $f\colon Z\to E$ in § 5 to be $f_2\,.\,f_1$ or id_E as $Z=K_1\times' K_2$ or E. Note that, despite the notation, the domain and range of g_* are not identical, in that they refer to different sections $K_1\vee' K_2\to Z\times' E$. The former section $r\colon K_1\vee' K_2\to Z\times' E$ is given by

$$r=\begin{cases}(id_Z, s\circ(q_1\vee q_2)) & Z=K_1\times' K_2,\\ (f_1\vee f_2, s\circ(q_1\vee q_2)) & Z=E.\end{cases}$$

Let $\pi_2\colon Z\times' E\to E$ be projection to the second factor. Then $\pi_{2*}(\bar\omega)\in[K_1\times' K_2, E]_B^{K_1\vee' K_2}$ where the section $K_1\vee' K_2\to E$ is given by $s\circ(q_1\vee q_2)$. Thus $\pi_{2*}(\bar\omega)$ induces a unique class $\omega\in[K_1\wedge' K_2, E]_B^B$, with the requisite property.

We now detail the lemmas used to derive $\bar\omega$ in each of the seven cases, as well as the appropriate choice of Z where it matters.

(i) (2.3) (b), (5.2) (a);

(ii)–(vii) (2.3) (a), (3.1) and

(ii) (5.1), $Z=E$;

(iii) (5.1), $Z=K_1\times' K_2$;

(iv) (5.2) (b), $Z=E$;

(v) (5.2) (c), $Z=E$;

(vi) (5.2) (c), $Z=K_1\times' K_2$;

(vii) (5.4).

6.2 Of course the arguments of § 5 hold good when right multiplication rather than left multiplication is considered. If then we replace $K_1\times' K_2$ by an ex-space X in (i)–(vi) of (6.1) (so that, for example, $B\to X$ is a closed cofibration), this shows that *$[X, E]_B^B$ is a loop,* provided that $s\,.\,s=s\colon B\to E$. (Note that if $s\colon B\rightarrowtail E$ is a closed cofibration, $p\colon E\to B$

a fibration, **[10]** Theorem 12 implies that $s(B) \times' s(B) \hookrightarrow E \times' E$ is a cofibration, whence m is deformable to a multiplication satisfying this last condition.) Hence one obtains ex-homotopy analogues of (1.1), (1.2), (1.3) of **[4]**.

6.3 Observe that for consideration of f.h. classes § 2 is unnecessary. One can therefore relax conditions (ii)–(vii) of (6.1) as follows, in order to conclude that $[X, E]_B$ *is a loop.* X is merely assumed to be a space over B, i.e. in Top_B, while $p\colon E \to B$ need only be an h-fibration. The arguments of § 5 now suffice to yield that $[X, E]_B$ is a loop, and, in the case of (vii), a group. That is, they provide the f.h. extensions of the results of **[4]** § 1, whose restriction to the 'ordinary' homotopy case ($B = pt$) is a generalisation of those results.

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