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Torsion Generators for All Abelian Groups

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THEOREM A. *Let A be any abelian group. Then there exists a perfect group H such that*

(i) *A is isomorphic to the centre $\mathscr{Z}(H)$ of H, where*

(ii) *for any $n \geqslant 2$, H is normally generated by a single element of order n.*

THEOREM B. *Let A be any abelian group. Then for any $m \geqslant 2$ there exists a group G_m such that*

(i) *$A \cong H_m(G_m)$ and $H_i(G_m) = 0$ $(1 \leqslant i < m)$, where*

(ii) *for any $n \geqslant 2$, G_m is normally generated by a single element of order n.*

Throughout, homology is to be taken with trivial integer coefficients. Thus when $m = 2$, Theorem B asserts that any abelian group is the Schur multiplier of a torsion-generated perfect group. (Of course, an $m = 1$ version of Theorem B is not possible because any quotient of a torsion-generated group G_1, such as $H_1(G_1)$, is also torsion-generated.)

Proofs lean heavily on algebraic K-theory. The starting-point is the construction [6, 10] of a Dedekind domain whose class group is a given abelian group. Because this construction is non-functorial (it depends on a chosen free presentation for the abelian group), so too will ours be.

We recall the following constructions for a ring R (assumed associative, with 1). ER (resp. $E_k R$) is the subgroup of the general linear group over R generated by all (resp. all $k \times k$-) elementary matrices; we shall use the fact that the alternating group $\mathscr{U}_k$ is a subgroup of $E_k R$. Also ER is a quotient of the Steinberg group StR. The cone CR is the ring of all locally finite matrices over R, and the suspension SR is the quotient of CR modulo

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all finite matrices. (For further details, see [2], for example.) Torsion-generation stems from the following observation.

LEMMA 1. *For each $n \geqslant 2$, ER is normally generated by a single element of order n.*

Proof. The case $n=3$ is discussed in [2, (9.4)], where the normal generator may be taken to be a 3-cycle generating $\mathscr{U}_3 \leqslant E_3 R$. This effectively reduces the problem to one in $\mathscr{U}_k$. It is easy to check that any 3-cycle is a product of two n-cycles in $\mathscr{U}_n$ (n odd) and a product of two products $\sigma\tau \in \mathscr{U}_l$, where σ is an n-cycle and τ a disjoint transposition (n even, $l = \max(5, n+2)$). ▌

Proof of Theorem A. Let D be a Dedekind domain with class group A, so that $K_0(D) \cong A \oplus \mathbb{Z}$. Now also $K_0(D) \cong K_2(SSD) \cong \mathscr{Z}(St(SSD))$ [2, (3.3), (11.2), (3.7)]. We therefore have $A \cong \mathscr{Z}(St(SSD)/\mathbb{Z})$ (since $St(SSD)$ is perfect). Moreover, from the group extension $E(SD) \rightarrowtail E(CSD) \twoheadrightarrow St(SSD)$ [2, (11.12)] and Lemma 1, $H = St(SSD)/\mathbb{Z}$ is indeed normally generated by a single element of arbitrary finite order. ▌

It is possible to avoid the use of [6, 10] in the proof of Theorem A by means of the following technique kindly suggested to me by J. G. Thompson (in a different context). Let A have free abelian presentation

$$0 \to R \to F = \bigoplus_{\Lambda} \mathbb{Z} \to A \to 0,$$

where Λ is a suitable index set. Then for H we may choose the group $St(SS(\times_\Lambda \mathbb{Z}))$ factored out by an isomorphic copy of R inside its centre ($\cong F$).

For a more directly group-theoretic approach to Theorem A, at least when A is locally finite, see recent work on existentially closed groups by, for example, Hickin [8] and Phillips [9]. (O. Kegel tells me that he is making more general progress along these lines.) These methods also lead to the group H/A being simple. I do not know for which rings R the quotient of $St(SSR)$ by its centre is simple. However, by [5] we do know that this quotient admits no non-trivial finite-dimensional complex representation; for, by the above, it is the quotient of a torsion-generated acyclic group.

In the case of bounded abelian groups A, a functorial variant of Theorem A is available. For, such an A is the additive group of a functorially constructed torsion ring R [7, (120.8)] (namely, the trivial pseudoring on A with a multiplicative identity adjoined). We then have A isomorphic to the centre of the torsion-generated, perfect group

$M(A, (R, A))$ of [4] whenever A is a dense order with both a first and a last element.

Our proof of Theorem B makes use of the following.

LEMMA 2. *Suppose that* $K_i(R) = 0$ *for* $2 - m \leqslant i < 0$. *Then*

(a) *for* $3 \leqslant q \leqslant m$, $ES^qR = StS^qR$;

(b) *for* $0 \leqslant j < q \leqslant m$, $\tilde{H}_j(ES^qR) = 0$;

(c) *for* $3 \leqslant q \leqslant m$, $H_q(ES^qR) \cong H_{q-1}(ES^{q-1}R) \cong K_0(R)$.

Proof. (a) Recall that $\mathrm{Ker}[StS^qR \twoheadrightarrow ES^qR]$ is $K_2(S^qR)$, which is isomorphic to $K_{2-q}(R)$, here assumed trivial.

(b) We argue by induction on q, the cases $q = 1$, 2 being trivial. For the inductive step from $q - 1$ to q, on combining (a) with [2, (11.2), (10.1)], we have a group extension $ES^{q-1} \rightarrowtail ECS^{q-1}R \twoheadrightarrow ES^qR$ whose classifying space fibration $BES^{q-1}R \to BECS^{q-1}R \to BES^qR$ is quasi-nilpotent (and hence orientable because ES^qR is perfect—the "joke" of [3, (4.4)]). Since $H_i(ES^{q-1}R) = 0$ for $0 < i < q - 1$ by inductive hypothesis, there is an exact Serre homology sequence

$$H_q(ES^{q-1}R) \to \cdots \to H_j(ECS^{q-1}R) \to H_j(ES^qR) \to H_{j-1}(ES^{q-1}R) \to \cdots.$$

Now cone rings are homologically trivial, so the induction goes through.

(c) The results of (b) imply that the above exact sequence may be extended further to the left (to dimension $2q - 2$). From the triviality of $\tilde{H}_*(ECS^{q-1}R)$ we obtain, after iteration,

$$H_q(ES^qR) \cong H_{q-1}(ES^{q-1}R) \cong \cdots \cong H_2(ES^2R) \cong K_2(S^2R) \cong K_0(R). \quad ▮$$

Proof of Theorem B. Let F be a field inside a Dedekind domain D whose class group is isomorphic to A (according to [10] any infinite perfect F is possible). The inclusions $F \subsetneq D$ and $S^mD \subsetneq CS^mD$ induce $ES^mF \subsetneq ES^mD$. So we may define $G_m = ES^mD *_{ES^mF} ECS^mF$, the generalised free product of ES^mD and ECS^mF with amalgamated subgroup ES^mF. As in the proof of Lemma 1, for each $n \geqslant 2$, G_m is normally generated by an element of order n in a copy of $\mathcal{U}_\infty$ inside the subgroup ES^mF. The homology properties of G_m may be checked (by Lemma 2) from the Mayer-Vietoris sequence

$$\begin{aligned} \cdots &\to H_j(ES^mF) \to H_j(ES^mD) \oplus H_j(ECS^mF) \\ &\to H_j(G_m) \to H_{j-1}(ES^mF) \to \cdots, \end{aligned}$$

using the triviality of $K_i(F)$, $K_i(D)$ for negative i (since both rings are regular) and the fact that the ring inclusion induces $K_0(F) \to K_0(D)$ with cokernel isomorphic to A. ▮

One may ask whether it is possible to sharpen Theorem B by obtaining for G_m a torsion-generated group with $\tilde{H}_j(G_m) = A$ when $j = m$, zero otherwise (as is done for finitely-generated, but possibly torsion-free, G_m in [1]). This suggests the following question.

QUESTION 1. Let G be a torsion-generated group with only finitely many non-zero homology groups. Is G acyclic? (That is, is $\tilde{H}_*(G) = 0$?)

This question has an affirmative answer when G is locally finite (see [11, Theorem A]). A related question also suggested by the above is as follows.

QUESTION 2. Let R be a ring with $K_i(R) = 0$ for $i \geqslant 2$. Is $K_1(R) = 0$?

In principle, the hypothesis of Question 2 may arise in the following way. Suppose that the outer automorphism group Out(ER) of ER lies in the class $\mathscr{J}$ of groups which receive no non-trivial homomorphism from a torsion-generated acyclic group. ($\mathscr{J}$ evidently includes all torsion-free and soluble groups, and, according to [5], all residually finite groups.) Then by [5], ER is acyclic and so $K_i(R) = 0$ whenever $i \geqslant 2$. A consequence of this remark is the following amusing observation. Let R be a finite field $\mathbb{F}_q$. Then for each finite n, the group Out($E_n\mathbb{F}_q$) is both soluble and finite. However, from the above we deduce, since $K_3(\mathbb{F}_q) \neq 0$ [12], that the direct limit $E\mathbb{F}_q$ (a simple group) has its outer automorphism group neither soluble nor even residually finite. This is an indication of the sharpness of the Schreier Hypothesis on the solubility of the outer automorphism groups of finite simple groups.

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Note added in proof. For a subsequent treatment of this topic that uses combinatorial group theory instead of algebraic K-theory, see [13].

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