

THE ACYCLIC GROUP DICHOTOMY

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To the memory of Karl Gruenberg – fine mathematician, exemplary colleague, and dear friend.

ABSTRACT. Two extremal classes of acyclic groups are discussed. For an arbitrary group G , there is always a homomorphism from an acyclic group of cohomological dimension 2 onto the maximum perfect subgroup of G , and there is always an embedding of G in a binate (hence acyclic) group. In the other direction, there are no nontrivial homomorphisms from binate groups to groups of finite cohomological dimension. Binate groups are shown to be of significance in relation to a number of important K -theoretic isomorphism conjectures.

0. INTRODUCTION

The theme of this note is that the world of acyclic groups appears to be dominated by two very distinct classes.

Recall that a discrete group G is called *acyclic* if all ordinary homology groups, with trivial integer coefficients, are zero in positive dimensions. In particular, considering dimension 1, all such groups are perfect. Among acyclic groups, we focus on classes that we may think of as the small and large acyclic groups, in the sense of cohomological dimension. In what follows, we prove that every group is sandwiched between these classes, in that it receives maps from small acyclic groups and embeds in a large acyclic, *binate* group. Schematically:

$$\begin{array}{ccccc}
 \text{acyclic group of cd 2} & & & & \\
 (0-1) \quad \exists \downarrow_{\text{onto}} & & & & \\
 \text{perfect radical} & \hookrightarrow & \text{arbitrary group} & \hookrightarrow & \text{binate group}
 \end{array}$$

We prove a rigidity result to the effect that the flow is irreversible, inasmuch as there are no nontrivial maps from large acyclic groups to small ones. Arising from this setup, large acyclic groups play a crucial role in certain long-standing conjectures. The note concludes with brief lists of examples of each class, for which the reader is referred to [9] for details.

1. THE SMALL: ACYCLIC GROUPS OF FINITE COHOMOLOGICAL DIMENSION

Arguably, the first (nontrivial) acyclic groups in the literature, those of Baumslag and Gruenberg [7], are of this kind. They consider a group G generated by two elements subject to a single relation, and show that if G_{ab} is infinite cyclic and G' is perfect, then G' is acyclic. Since two-generator one-relator groups have cohomological dimension 2, so does the acyclic group G' . Remarkably, a geometrically-defined example was provided almost simultaneously: in [21], Epstein performs a variant of the grope construction [33], [34], whereby a doubly-punctured torus is added at

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each stage. In [17], Epstein's group is shown to be acyclic of Baumslag-Gruenberg type: it is the commutator subgroup of the group

$$G = \langle x, y \mid x = [x, yx^{-1}y^{-1}] [x, y^{-1}xy] \rangle.$$

In general, one can describe a grope [19] as the direct limit L of a nested sequence of compact 2-dimensional polyhedra

$$L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \dots$$

obtained as follows. Take L_0 as some S_g , an oriented compact surface of positive genus g from which an open disk has been deleted. To form L_{n+1} from L_n , for each loop a in L_n that generates the group $H_1(L_n)$, attach to L_n some S_{g_a} by identifying the boundary of S_{g_a} with the loop a . Since the fundamental group of S_g punctured is a free group on $2g$ generators, this procedure embeds each $\pi_1(L_n)$, and thus each finitely generated subgroup of $\pi_1(L)$, as a subgroup of a free group, with each generator a of $\pi_1(L_n)$ becoming a product of g_a commutators in $\pi_1(L_{n+1})$. Hence $\pi_1(L)$ is a countable, perfect, locally free group. Since free groups have geometric dimension 1, and homology groups commute with direct limits, there is a key (well-known) observation that we record.

Lemma 1.1. *Perfect, locally free groups are acyclic, of geometric dimension 2. \square*

The converse to this lemma fails: [5, p.13] exhibits an acyclic group of geometric dimension 2 that is not locally free, indeed it contains the fundamental groups of all closed orientable surfaces of genus ≥ 2 . Moreover, there are acyclic groups of all geometric dimensions, as in the next result. For a stronger statement, we can speak of the *finite geometric dimension* of a group G , namely the minimum dimension of a finite CW-complex that is a $K(G, 1)$ (where such a complex exists).

Theorem 1.2. [14] *For any integer $n > 1$, there is an acyclic group of finite geometric dimension n .*

The most economical example of L as above is the *minimal grope* M^* , for which one takes only one genus one surface at each step. Homotopically, each L_n is in this case just a bouquet of finitely many circles. So M^* is the classifying space of the group having: as generators, symbols x_w for each $w \in \Sigma$, the set of all (nonempty) words of finite length on the two symbols 0, 1; and as relations, $x_w = [x_{w0}, x_{w1}]$. Thus, the x_w with w of length n generate the free group $\pi_1(L_n)$, but each is a commutator when embedded in the larger free group $\pi_1(L_{n+1})$. For the 3-dimensional version of this construction, see [26, Theorem 2], where a nested sequence of handlebodies leads to a locally free fundamental group, and conversely.

To see the process more algebraically [10], one takes the free product of countable, perfect, locally free groups $F_{\mathbf{n}}$, one for each sequence $\mathbf{n} = (n_1, n_2, n_3, \dots)$ of positive integers. $F_{\mathbf{n}}$ is the direct limit of the following direct system $(F_{\mathbf{n},r}, \varphi_r)$. For $r = 0$, define $F_{\mathbf{n},0}$ to be infinite cyclic with a generator x_0 , and for each $r \geq 1$, let $F_{\mathbf{n},r}$ be the free group freely generated by the set of $2^r n_1 \cdots n_r$ symbols

$$\{x_r(\varepsilon_1, \dots, \varepsilon_r; i_1, \dots, i_r) \mid \varepsilon_k \in \{0, 1\}, 1 \leq i_k \leq n_k\}.$$

Then for $r \geq 0$, $\varphi_r: F_{\mathbf{n},r} \rightarrow F_{\mathbf{n},r+1}$ maps $x_r(\varepsilon_1, \dots, \varepsilon_r; i_1, \dots, i_r)$ to the product of commutators

$$\prod_{i_{r+1}=1}^{n_{r+1}} [x_{r+1}(\varepsilon_1, \dots, \varepsilon_r, 0; i_1, \dots, i_r, i_{r+1}), x_{r+1}(\varepsilon_1, \dots, \varepsilon_r, 1; i_1, \dots, i_r, i_{r+1})].$$

Now, for every perfect group P and each element $x \in P$, one can write x as a product of commutators. In consequence there is a sequence of integers n (which may be chosen to be increasing) and a homomorphism $\psi: F_{\mathbf{n}} \rightarrow P$ whose image

contains x [25]. A free product of such F_n then gives a perfect, locally free group with image containing any finitely generated subgroup of P . Since P is the direct limit of its finitely generated subgroups, by taking the direct limit of free products of groups F_n we obtain a perfect, locally free group that has P as a homomorphic image. Starting with an arbitrary group, recall that the group generated by all its perfect subgroups is also perfect, its *perfect radical*. By the above procedure, this perfect radical is the homomorphic image of a perfect, locally free group. The conclusion is thus the left hand side of the schematic flow of diagram (0-1), as follows.

Proposition 1.3. *For any group G , there is a homomorphism $\phi : A \rightarrow G$, where A is a locally free acyclic group, such that every perfect subgroup of G lies in the image of ϕ . \square*

2. THE LARGE: BINATE GROUPS

We begin with an elementary observation, that follows readily from the formula

$$[u, ab] = [u, a][u, b][[b, u], a],$$

where $[u, v] = uvu^{-1}v^{-1}$.

Lemma 2.1. *Let H be a subgroup of a group G , $u \in G$, and $\varphi : H \rightarrow G$ a set function such that for all $h \in H$,*

$$h = [u, \varphi(h)].$$

Then φ is a homomorphism if and only if both φ is injective and $[H, \varphi(H)] = 1$. \square

A nontrivial group G is called *binate* [8] if for any finitely generated subgroup H of G there exists a homomorphism (called a *structure map*) $\varphi = \varphi_H : H \rightarrow G$ and element (called a *structure element*) $u = u_H \in G$ such that for all $h \in H$,

$$h = [u, \varphi(h)].$$

Thus, the finitely generated subgroups of G are indeed arranged in commuting isomorphic pairs, as the name “binate” is intended to suggest.

Evidently, a binate group is perfect; indeed, every element is a commutator. Observe too that $u \in H$ would imply that $H = 1$. Therefore, binate groups cannot be finitely generated. Deeper facts are summarized as follows.

Theorem 2.2. (a) [8] *Binate groups are acyclic.*

(b) [1] *Binate groups have no finite-dimensional representations over any field.*

M. V. Sapir raised with me the interesting question of whether the pairing of H with $\varphi(H)$ above yields a copy of $H \times H$ in the binate group G . While a weakened version of this query is readily answered (in (a) below), in its original form the question is more subtle, because the centre $\mathcal{Z}(H)$ of a finitely generated group H need not be finitely generated.

Theorem 2.3. (a) *If H is a finitely generated subgroup of a binate group G , then there are structure maps $\varphi_1, \varphi_2, \dots$ for H such that, for any distinct i, j , we have $\varphi_i(H) \times \varphi_j(H) \leq G$. Thus, G contains an infinite product of copies of H .*

(b) *Let $H \leq H_1$ be finitely generated subgroups of a binate group G such that $H \cap \mathcal{Z}(H_1)$ is finitely generated. Then there exists a structure map $\psi = \psi_H$ for H such that $H \cap \psi(H) = 1$; in other words, $H \times \psi(H) < G$.*

Proof. (a) Write $H_0 = H$ and, for $i \in \mathbb{N}$,

$$H_i = \langle H_{i-1}, \varphi_{i-1}(H_{i-1}), u_{i-1} \rangle$$

where φ_{i-1} is a structure map for H_{i-1} , with associated structure element u_{i-1} . Thus $[\varphi_i(H), \varphi_j(H)] = 1$. If for some $i < j$ we have $a, b \in H$ with $\varphi_i(a) = \varphi_j(b)$, then

$$a = [u_i, \varphi_i(a)] = [u_i, \varphi_j(b)] = 1,$$

whence $\varphi_i(a) = 1$. For the final assertion, recall that each φ_i is a monomorphism, and iterate.

(b) Since $H \cap \mathcal{Z}(H_1)$ is a finitely generated abelian group (hence isomorphic to $\mathbb{Z}^{\text{rk}} \oplus \text{Tor}$ with Tor finite), we argue by induction on the pair

$$(\text{rk}_{\mathbb{Z}}(H \cap \mathcal{Z}(H_1)), |\text{Tor}(H \cap \mathcal{Z}(H_1))|),$$

ordered lexicographically. When this pair is $(0, 1)$, we already have that

$$H \cap \varphi_{H_1}(H) \leq H \cap \mathcal{Z}(H_1) = 1,$$

so that the result is immediate with $\psi_H = \varphi_{H_1}$. For the induction step, assuming the result for lesser values of the pair, suppose that $x \in H - \{1\}$ has $\varphi(x) \in H$ where $\varphi = \varphi_{H_1}$. Then $\varphi(x) \in H \cap \mathcal{Z}(H_1)$, say $\varphi(x) = t^n$ for some element t of a minimal generating set for $H \cap \mathcal{Z}(H_1)$. Since

$$1 \neq x = [u, t^n],$$

where $u = u_{H_1}$, and φ is a monomorphism, one of the following must occur.

- (i) t has infinite order. Then x also must have infinite order. Therefore $\langle t \rangle \cap \mathcal{Z}(\langle H_1, u \rangle) = 1$. This implies that

$$\text{rk}_{\mathbb{Z}}(H \cap \mathcal{Z}(\langle H_1, u \rangle)) < \text{rk}_{\mathbb{Z}}(H \cap \mathcal{Z}(H_1)).$$

- (ii) If

$$\text{rk}_{\mathbb{Z}}(H \cap \mathcal{Z}(\langle H_1, u \rangle)) = \text{rk}_{\mathbb{Z}}(H \cap \mathcal{Z}(H_1)),$$

then, since x is nontrivial, t must have finite order with $\langle t \rangle \not\subseteq \mathcal{Z}(\langle H_1, u \rangle)$. Therefore

$$|\text{Tor}(H \cap \mathcal{Z}(\langle H_1, u \rangle))| < |\text{Tor}(H \cap \mathcal{Z}(H_1))|.$$

In either event, we can apply the induction hypothesis to obtain $\psi_H = \psi_{\langle H_1, u \rangle}$ as the desired structure map. \square

Remark 2.4. This argument extends to include the case where $H \cap \mathcal{Z}(H_1)$ is the direct sum of a finitely generated group with a finite number of copies of the rationals and a finite number of quasicyclic groups.

An easy consequence of this result (to be strengthened below) is that our classes of small and large acyclic groups are indeed disjoint.

Corollary 2.5. *Every binate group has infinite cohomological dimension.*

Proof. By the theorem applied initially to a cyclic subgroup, a binate group contains either free abelian groups of arbitrarily high rank, or elementary abelian groups of arbitrarily high rank. However, neither of these possibilities can occur in a group of finite cohomological dimension. \square

Here is another result, due to I. Agol (private communication), that shows the aversion of low-dimensional geometry for binate groups.

Theorem 2.6. *3-manifold groups cannot be binate.*

Proof. The important property is that for a binate group G , and any finitely generated subgroup H , there is an isomorphic subgroup H' such that $[H, H'] = 1$. First, we observe that G cannot be a free product, since if $G = G_1 * G_2$, then taking a finitely generated subgroup of the form $H = \langle h_1, h_2 \rangle$, with h_1 in $G_1 - \{1\}$, h_2 in $G_2 - \{1\}$, it's clear that there can be no subgroup $H' < G$ commuting with H .

This can be seen geometrically, e.g. take a $K(G_1, 1)$ and $K(G_2, 1)$, and link them together with an interval to get a $K(G, 1)$. If we have a commutator $[h_1, h'] = 1$, then we get a map of a torus into the $K(G, 1)$. This torus map can be deformed disjoint from the interval, otherwise some loop in the torus would be trivial, in which case some product $h_1^a h'^b$ would be trivial, which would imply that h' is in G_1 , and couldn't commute with h_2 . Thus, if we deformed the torus into $K(G_1, 1)$, we would get a conjugacy of h' into G_1 . Similarly, if h' commutes with h_2 , then it is conjugate into H_2 , which would be a contradiction. So we may assume that G is not a free product. Thus, if $G = \pi_1(M^3)$, then we may assume that M^3 is irreducible and noncompact (since G is not finitely generated), and therefore a $K(G, 1)$. However, by Corollary 2.5, G cannot have finite geometric dimension.

Alternatively, arguing geometrically, we can apply the Scott core theorem to deduce that any finitely generated subgroup of G is the fundamental group of a compact irreducible 3-manifold with nontrivial boundary. So, if we take finitely generated non-virtually-cyclic subgroups H and H' such that $[H, H'] = 1$, then we get a finitely generated subgroup $\langle H, H' \rangle$ which is the fundamental group of a compact manifold N with boundary. Each pair of commuting elements in H and H' gives a π_1 -injective map of a torus into N . By the characteristic submanifold theorem, there is a canonical decomposition of N along embedded incompressible tori, such that the complementary pieces are simple or Seifert fibered, and such that any map of a torus into N may be homotoped into one of the Seifert fibered pieces. Non-closed Seifert fibered spaces have a finite-sheeted covering which is of the form surface $\times S^1$, so it has fundamental group of form free group $\times \mathbb{Z}$. It's clear that the only pairs of commuting groups in here are of the form $\mathbb{Z} \times \mathbb{Z}$. Thus, H must be a virtually cyclic group, a contradiction. \square

This result should be seen alongside the result of [14] that nontrivial acyclic 3-manifold groups cannot be finitely generated. For examples of such groups, see Appendix A below.

We turn now to the right-hand half of the flow diagram (0-1). A universal method for embedding any group H in a binate group is the *universal binate tower* on H , constructed in [8] by means of HNN-extensions, as follows. Let $H_0 = H$ and for each $i \geq 0$

$$H_{i+1} = \text{gp} \langle H_i \times H_i, u_i \mid (g, g) = u_i(1, g)u_i^{-1} \text{ for each } g \in H_i \rangle,$$

with H_i embedded in H_{i+1} as $H_i \times 1$ and $\varphi_i(g) = (1, g)$. The direct limit of the H_i is a binate group (even though the homology of the groups H_i grows hyperexponentially with i). This tower is the initial object in a category of binate towers with base group H [16].

Theorem 2.7. [16] (a) *Every group H embeds in a universal binate group.*

(b) *Every binate group that contains H contains infinitely many images of the universal binate group containing H .*

Observe that, because homology cycles are always supported on finitely generated subgroups, and because homology commutes with direct limits, in an acyclic group G every countable subgroup lies in a countable acyclic subgroup of G . A similar result holds for binate groups G , by (b) above.

Corollary 2.8. *Every countable subgroup of a binate group G lies in a countable binate subgroup of G .* \square

An alternative way of embedding a given group in a binate group is provided by the Kan-Thurston construction of the cone of a group [27], discussed in Appendix B below.

We are now in a position to prove the rigidity result stated in the Introduction.

Theorem 2.9. *Every homomorphism from a binate group to a group that is residually of finite virtual cohomological dimension is trivial.*

Proof. First, if every map from a group G to a member of a class of groups \mathcal{X} is trivial, then every map from G to a group that is residually in \mathcal{X} must also be trivial, because any nontrivial element in the image of G would be mapped nontrivially to a member of \mathcal{X} . Next, by Theorem 2.2 (b) binate groups have no nontrivial finite quotients; thus, we may ignore the word “virtual”.

Since subgroups of groups of finite cohomological dimension also have finite cohomological dimension and so are torsionfree, it suffices to show that any torsionfree quotient Q of a binate group G contains a free abelian subgroup of infinite rank. To do this, let $\pi : G \twoheadrightarrow Q$ be an epimorphism and $x \in G - \text{Ker}\pi$. Define $H_0 = \langle x \rangle$ and, for $i \in \mathbb{N}$,

$$H_i = \langle H_{i-1}, \varphi_{H_{i-1}}(H_{i-1}), u_{i-1} \rangle.$$

Then the subgroup of Q generated by all $\pi\varphi_{H_i}(H_0)$ is abelian. Its rank must be infinite because if for some $i < j$ and m, n we have $\pi\varphi_{H_i}(x)^m = \pi\varphi_{H_j}(x)^n$, then

$$\pi(x)^m = [\pi(u_i), \pi\varphi_{H_i}(x^m)] = \pi([u_i, \varphi_{H_j}(x^n)]) = 1,$$

leaving $m = n = 0$ as the only possibility. \square

3. BINATE GROUPS AS CONJECTURE-TESTERS

The first hint that binate groups form an important class of groups for testing some famous conjectures comes from [16], where links to the Kervaire conjecture are discussed. Recall that this conjecture asserts that for any group G either G is trivial or the free product $G * \mathbb{Z}$ with an infinite cyclic group cannot be normally generated by a single element. By considering abelianizations one immediately sees that any counterexample to the conjecture must be perfect.

Theorem 3.1. [16] *The Kervaire conjecture holds for all groups if and only if it holds for all binate groups.*

Remarkably, [16] also shows that in any binate group the Kervaire conjecture is locally true, in that every finitely generated subgroup is contained in one with the property of the conjecture.

Next, we consider the Bass trace conjecture [3], as follows. With $\mathbb{Z}G$ the integral group ring of a group G , the *augmentation trace* is the \mathbb{Z} -linear map

$$\epsilon : \mathbb{Z}G \rightarrow \mathbb{Z}, \quad g \mapsto 1$$

induced by the trivial group homomorphism on G . The Hochschild homology group $HH_0(\mathbb{Z}G) = \mathbb{Z}G/[\mathbb{Z}G, \mathbb{Z}G]$, with $[\mathbb{Z}G, \mathbb{Z}G]$ the additive subgroup of $\mathbb{Z}G$ generated by the elements $gh - hg$ ($g, h \in G$), identifies with $\bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$, where $[G]$ is the set of conjugacy classes $[s]$ of elements s of G . The *Hattori-Stallings trace* of $M = \sum_{g \in G} m_g g \in \mathbb{Z}G$ is then defined by

$$\begin{aligned} \text{HS}(M) &= M + [\mathbb{Z}G, \mathbb{Z}G] \\ &= \sum_{[s] \in [G]} \epsilon_s(M)[s] \in \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s], \end{aligned}$$

where for $[s] \in [G]$, $\epsilon_s(M) = \sum_{g \in [s]} m_g$ is a partial augmentation. Now, an element of $K_0(\mathbb{Z}G)$ is represented by a difference of finitely generated projective $\mathbb{Z}G$ -modules, each of which is determined by an idempotent matrix having entries in $\mathbb{Z}G$. Combining the usual trace map to $\mathbb{Z}G$ of such a matrix with the above trace on $\mathbb{Z}G$ induces the Hattori-Stallings trace homomorphism on $K_0(\mathbb{Z}G)$, which is natural with respect to group homomorphisms.

Conjecture 3.2. *For any group G , the induced map*

$$\mathrm{HS}_G : K_0(\mathbb{Z}G) \longrightarrow HH_0(\mathbb{Z}G) = \bigoplus_{[s] \in [G]} \mathbb{Z} \cdot [s]$$

has image in $\mathbb{Z} \cdot [e]$.

There is likewise a version of the conjecture for the complex group ring, as well as for intermediate rings. For a recent description of groups G satisfying the conjecture, see [11], where the following adaptation of the universal binate group is introduced. With the notation

$$\begin{aligned} \Delta_C &= \{(c, c) \in C \times C \mid c \in C\}, \\ \Delta'_F &= \{(f^{-1}, f) \in F \times F \mid f \in F\}, \end{aligned}$$

let G be a group with $\{F_i\}_{i \in I}$ as the set of all its finitely generated abelian subgroups. For $i \in I$, write C_i for the centralizer in G of F_i . Then in $G \times G$ the subgroups $(1 \times F_i) = \{(1, f) \mid f \in F_i\}$ and Δ_{C_i} commute, so that their product $(1 \times F_i) \cdot \Delta_{C_i}$ is also a subgroup of $G \times G$. Likewise, $\Delta'_{F_i} \cdot (1 \times C_i)$ is also a subgroup, and the obvious bijection

$$\begin{aligned} (1 \times F_i) \cdot \Delta_{C_i} &\longrightarrow \Delta'_{F_i} \cdot (1 \times C_i) \\ (k, fk) &\longmapsto (f^{-1}, fk) \end{aligned}$$

is a group isomorphism. Now define $A_1(G)$ to be the generalized HNN extension

$$A_1(G) = \mathrm{HNN}(G \times G; (1 \times F_i) \cdot \Delta_{C_i} \cong \Delta'_{F_i} \cdot (1 \times C_i), t_i)_{i \in I}$$

meaning that, whenever $f \in F_i$ and $k \in C_i$,

$$(k, fk) = t_i(f^{-1}, fk)t_i^{-1}.$$

For $n \geq 2$, inductively define $A_n(G) = A_1(A_{n-1}(G))$. Since $A_{n-1}(G) \leq A_n(G)$, we put $A(G) = \cup A_n(G)$.

Theorem 3.3. [11] *The homomorphism $G \rightarrow A(G)$ has the following properties.*

- (a) *It is a functorial Frattini embedding.*
- (b) *Every finitely generated abelian subgroup of $A(G)$ has its centralizer in $A(G)$ binate.*
- (c) *The prime powers that occur as orders of elements of $A(G)$ are precisely those that occur as orders of elements of G .*
- (d) *If G is infinite, then $A(G)$ has the same cardinality as G .*

Here, recall that a Frattini embedding is one in which nonconjugate elements remain nonconjugate. (Also, observe that by taking the trivial subgroup in (b), one sees that $A(G)$ itself is binate.) In [11], properties (a) and (b) are combined, with (b) used to show that the equivariant K -homology $K_*^A(\underline{EA})$ of the universal proper A -CW-complex \underline{EA} reduces to a direct sum of complex vector spaces afforded by the complex representation rings of the finite subgroups of A . Since the Hattori-Stallings trace HS_A is known to exclude nontrivial finite conjugacy classes from its image [13], [30], it follows that it yields only the trivial conjugacy class on elements of $K_0(\mathbb{C}A)$ that originate in $K_0^A(\underline{EA})$. When G satisfies the Bost conjecture [28], then it can be deduced that the image of the Hattori-Stallings trace HS_G in $HH_0(\mathbb{C}A)$ is just the trivial conjugacy class. Thereby, using (a), [11] shows that if G satisfies the Bost conjecture, then it also satisfies the Bass conjecture.

The point of (a) of the theorem is that for any ring R the induced map $HH_0(RG) \rightarrow HH_0(RA)$ is a monomorphism. Therefore, from the commuting square

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathbb{Z}A) \\ \downarrow \text{HS}_G & & \downarrow \text{HS}_A \\ HH_0(\mathbb{Z}G) & \twoheadrightarrow & HH_0(\mathbb{Z}A) \end{array}$$

we immediately obtain a noteworthy consequence.

Corollary 3.4. *The Bass conjecture holds for all groups if and only if it holds for all binate groups.* \square

As noted in [32], there are several related conjectures (mostly for torsionfree groups), such as the weak Bass conjecture, Atiyah conjecture, trace conjecture, strong trace conjecture, embedding conjecture and zerodivisor conjecture, as well as the fibered Farrell-Jones conjecture [2], that are subgroup-closed. It is therefore immediate from Theorem 3.3 (a) that they hold for all (respectively all torsionfree) groups if and only if they hold for all (resp. all torsionfree) binate groups. With regard to the weak Bass conjecture, there is a further simplification available. For, according to Bass (see [12]), this conjecture holds for all groups if and only if it is valid for all finitely presented groups. Now, from [15, Theorem 6], there is a finitely presented, strongly torsion generated, acyclic group, BM say, with the property that it contains (an isomorphic copy of) every finitely presented group. These two simplifications thereby combine to reveal that a single binate group suffices to test the validity of the conjecture.

Corollary 3.5. *The following assertions are equivalent:*

- (i) *the weak Bass conjecture holds for all groups;*
- (ii) *the weak Bass conjecture holds for the finitely presented acyclic group BM;*
- (iii) *the weak Bass conjecture holds for the universal binate group on BM;*
- (iv) *the weak Bass conjecture holds for the binate group $A(\text{BM})$.* \square

Property (c) of the theorem sheds light on two further conjectures. With the notation that i is the map induced on K -theory from the standard inclusion of the reduced C^* -algebra in the group von Neumann algebra, and $\text{tr}_{\mathcal{N}(G)}$ is the map induced by the standard complex-valued von Neumann trace on $\mathcal{N}(G)$, the modified trace conjecture of Baum-Connes [4], [31, Conjecture 0.2] asserts that, for any discrete group G , the image of the Kaplansky trace

$$\kappa : K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{C}$$

lies in the ring Λ_G of rational numbers whose denominators are products of the orders of finite subgroups of G . We note that, as in [11, p.616], [32, p.58], orders of finite subgroups do indeed occur as denominators.

Also, there is the Farkas conjecture [22], which asserts that a discrete group G has elements of all prime orders that occur as denominators of nonintegral rational numbers in the image of the composition

$$K_0(\mathbb{C}G) \longrightarrow K_0(C_r^*(G)) \xrightarrow{i} K_0(\mathcal{N}(G)) \xrightarrow{\text{tr}_{\mathcal{N}(G)}} \mathbb{C}.$$

Now, observe from (c) of Theorem 3.3 that $\Lambda_G = \Lambda_{A(G)}$, and that if $A(G)$ has an element of order p , then so does G . On the other hand, since G embeds in $A(G)$,

there is a commuting diagram

$$\begin{array}{ccccccc}
 K_0(\mathbb{C}G) & \longrightarrow & K_0(C_r^*(G)) & \xrightarrow{i} & K_0(\mathcal{N}(G)) & & \\
 \downarrow & & \downarrow & & \downarrow & \begin{array}{c} \text{tr}_{\mathcal{N}(G)} \\ \searrow \\ \mathbb{C} \\ \nearrow \\ \text{tr}_{\mathcal{N}(A(G))} \end{array} & \\
 K_0(\mathbb{C}A(G)) & \longrightarrow & K_0(C_r^*(A(G))) & \xrightarrow{i} & K_0(\mathcal{N}(A(G))) & &
 \end{array}$$

By combining these facts, we immediately obtain the following.

Corollary 3.6. (a) *The modified trace conjecture of Baum-Connes holds for all groups if and only if it holds for all binate groups.*

(b) *The Farkas conjecture holds for all groups if and only if it holds for all binate groups.* \square

In this connection, we note that for binate groups A there is a stronger possible result. For, if F_α and F_β are finite subgroups of a binate group A , then together they generate a finitely generated subgroup H of A . Thus, the structure map φ_H yields by Theorem 2.3 above a copy of $H \times H$, which contains $F_\alpha \times F_\beta$ as a subgroup. Thereby, A contains finite subgroups isomorphic to any direct product of the form $F_\alpha \times \cdots \times F_\alpha \times F_\beta \times \cdots \times F_\beta$. Hence, the whole ring Λ_A occurs in the image of κ .

Proposition 3.7. *If the modified trace conjecture of Baum-Connes holds for a binate group A , then the image of its Kaplansky trace is precisely Λ_A .* \square

Finally, we remark that the proof of Atiyah’s L^2 -index theorem in [18] consists in first proving it for countable acyclic groups, and then embedding an arbitrary countable group G in a countable acyclic group, such as $A(G)$ (using (d) of Theorem 3.3 above).

APPENDIX A. EXAMPLES OF LOW-DIMENSIONAL ACYCLIC GROUPS

Examples described in [9] that are of low cohomological dimension are as follows. First, Higman’s four-generator, four-relator group ($k = 4$)

$$\langle x_i \mid x_{i+1} = [x_i, x_{i+1}] \rangle_{i \in \mathbb{Z}/k}$$

(with $k \geq 0$, but $k = 1, 2, 3$ trivial) is a candidate for being the “oldest” acyclic group in the literature, although its acyclicity was not proved until much later in [20], where it is shown to be the fundamental group of both an acyclic, aspherical finite 2-complex and an homology 4-sphere. In [17] it is observed that this group is the commutator subgroup of

$$\langle x, y \mid x[x, yxy^{-1}], [x, y^4] \rangle.$$

For a discussion of how the Higman group relates torsionfree acyclic groups and groups G with $\underline{E}G/G$ contractible, see [29].

Stitch groups [17] are generalizations of the wild arc groups of [23], corresponding to iteration (infinitely, towards a cluster point, in both directions) of an underlying 2-tangle called a *stitch*. They are also acyclic of Baumslag-Gruenberg type. In fact, with the *deficiency* $\text{def}(G)$ as the maximum excess of the number of generators over the number of relators in a finite presentation of G , the following hold.

Theorem A.1. [14], [17] *Let λ be a smooth knot in $S^2 \times S^1$ such that $[\lambda]$ generates $H_1(S^2 \times S^1)$, and let M be the closed complement of a tubular neighbourhood of λ in $S^2 \times S^1$. Then*

- (a) *M is an aspherical Haken manifold, and is an integral homology circle;*

- (b) $\pi = \pi_1(M)$ is finitely presentable, $\text{def}(\pi) = 1$ and $\text{gd}(\pi) \leq 2$;
- (c) π is locally indicable and in particular contains no nontrivial finitely generated perfect subgroups;
- (d) π is residually finite;
- (e) the commutator subgroup π' is acyclic.

Here is an algebraic counterpart of this situation.

Theorem A.2. [14] *Let G be a finitely presentable group with $\text{def}(G) > 0$, and such that G' is finitely generated and perfect, nontrivial. Then the following hold.*

- (a) $\text{def}(G) = 1$, G requires at least three generators, and $\text{gd}(G) \leq 2$.
- (b) G_{ab} is \mathbb{Z} or \mathbb{Z}^2 .
- (c) G' is acyclic, but not FP_2 .

APPENDIX B. EXAMPLES OF BINATE GROUPS

Examples discussed in [9] include:

- algebraically closed groups (which are relevant to the study of the Kervaire conjecture cited above [16]);
- Philip Hall's countable universal locally finite group;
- groups of self-homeomorphisms with support symmetries;
- the general linear group of the cone on a ring;
- the cone on a group G , as constructed in [27] as the semidirect product of $G^{\mathbb{Q}}$ by $\text{Aut}(\mathbb{Q})$. Here, $G^{\mathbb{Q}}$ is the set of functions from the rationals \mathbb{Q} to G which map all numbers outside some finite interval to the identity; the group structure on G determines that on $G^{\mathbb{Q}}$. Likewise, $\text{Aut}(\mathbb{Q})$ denotes the restricted symmetric group on \mathbb{Q} comprising those permutations with compact support. Here, G is embedded as a two-step subnormal subgroup of a binate group.
- mitotic groups of [5] providing combinatorial, finitely generated and, later [6], finitely presented results analogous to those of [27]. They have the remarkable property that all quotients are also binate.
- automorphism groups of large structures elaborated in [24]. Examples include the group of all continuous linear automorphisms, or of invertible isometries, of an infinite-dimensional Hilbert space, the group of invertible or of unitary elements in a properly infinite von Neumann algebra, the group of measure-preserving automorphisms of a Lebesgue measure space, and the group of permutations on an arbitrary infinite set.

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