

The minimal extension of P -localization on groups

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0 Introduction

Let P be a fixed set of primes, \mathcal{G} the category of all groups and group-homomorphisms, and \mathcal{N} the full subcategory of nilpotent groups. In [9], an idempotent functor $e: \mathcal{N} \rightarrow \mathcal{N}$, called P -localization, was defined so as to extend the \mathbf{Z} -module-theoretic localization of abelian groups. There are two well-known extensions of e to \mathcal{G} , namely, Bousfield's P -localization [2], [4], denoted by $E\mathbf{Z}_P$, and Ribenboim's P -localization [13], usually denoted by $(\)_P$. Ribenboim's P -localization is the maximal extension among localizations extending e to \mathcal{G} in that it maximizes the number of groups in its image [7]. The localized groups obtained after applying Ribenboim's P -localization are precisely the P -local groups, that is, the groups having unique n th-root for every n which is coprime to P , [13]. Being maximal is equivalent to this class of P -localized groups being the saturated class of groups generated by e -equivalences, that is, group homomorphisms between nilpotent groups which become isomorphisms after applying e .

This suggests the problem of whether e has a minimal extension to \mathcal{G} , a localization L_P , say, yielding the minimum number of L_P -local groups. A general attack on this problem is to be found in [6]. The solution below is arguably more concrete, concentrating attention on groups of a given cardinality, where L_P takes on a more recognizable form. The reason for this strategy is to overcome technical difficulties that arise because the collection of all (P -local nilpotent) groups fails to form a set.

In the final section, we discuss the effect of the localization L_P on groups whose lower central series stabilizes. We use our localization and Bousfield's P -

localization to characterize these groups. In an appendix, we apply Bousfield's technique to show that in some good categories like \mathcal{G} , there is a one-to-one correspondence between localizations and E -saturated factorization systems. This sheds further light on L_P . It also allows a characterization of those orthogonal pairs which are associated to localizations. We collect all necessary preliminaries in the next section before we embark on our proofs.

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1 Preliminaries

We need to explain terminology mentioned in the introduction and used in the sequel. We define localization and the related concept of an orthogonal pair [7]. Since we are discussing minimal extensions, we need to explain what we mean by extension and also the partial ordering between orthogonal pairs giving rise to the term 'minimal'. Finally, we explain the term factorization system [3]. We have also included some preliminary results.

For an object X and a morphism $f: A \rightarrow B$ in a category \mathcal{C} , we say X and f are orthogonal to each other, denoted as $X \perp f$ or $f \perp X$, if for each morphism $g: A \rightarrow X$, there exists a unique morphism $h: B \rightarrow X$ such that $h \circ f = g$, that is, $f^* : \mathcal{C}(B, X) \cong \mathcal{C}(A, X)$.

Let D be a collection of objects and S a collection of morphisms in \mathcal{C} . We shall write D^\perp and S^\perp for the orthogonal complements of D and S in \mathcal{C} respectively which are defined as follows :

$$D^\perp = \{f \in \text{Mor}(\mathcal{C}) \mid f \perp X \text{ for each } X \in D\}$$

$$S^\perp = \{X \in \text{Obj}(\mathcal{C}) \mid X \perp f \text{ for each } f \in S\}.$$

We say that D is saturated if $D = D^{\perp\perp}$ and S is saturated if $S = S^{\perp\perp}$.

An orthogonal pair (S, D) in \mathcal{C} comprises a collection S of morphisms in \mathcal{C} and a collection D of objects in \mathcal{C} satisfying $S = D^\perp$ and $D = S^\perp$. Thus, for an orthogonal pair (S, D) , S and D are orthogonal complements of each other and they are both saturated. The orthogonal pair generated by D (resp. S) is $(D^\perp, D^{\perp\perp})$ (resp. $(S^{\perp\perp}, S^\perp)$) which is the orthogonal pair whose object class (resp. morphism class) is the smallest containing D (resp. S).

A partial ordering between two orthogonal pairs, (S', D') and (S, D) , in \mathcal{C} can be defined via inclusion of their respective object classes, that is,

$$(S', D') \leq (S, D) \quad \text{if } D' \subseteq D, \text{ which is equivalent to } S \subseteq S',$$

that is, “the ‘smaller’ orthogonal pair is the one with the smaller collection of objects”.

Suppose \mathcal{C} is a subcategory of \mathcal{C}' and (S, D) and (S', D') are orthogonal pairs in \mathcal{C} and \mathcal{C}' respectively. We say that (S', D') is an extension of (S, D) in \mathcal{C}' if both $S \subseteq S'$ and $D \subseteq D'$. It follows from the property of orthogonality that any extension (S', D') of (S, D) satisfies

$$(D^\perp, D^{\perp\perp}) \leq (S', D') \leq (S^{\perp\perp}, S^\perp),$$

where the orthogonal complements are taken in \mathcal{C}' .

In view of this, we call $(D^\perp, D^{\perp\perp})$ and $(S^{\perp\perp}, S^\perp)$ the minimal extension and the maximal extension of (S, D) in \mathcal{C}' respectively. From the definition of orthogonality and inverse limits, we have the following result.

Proposition 1.1 *Let D be a collection of objects in \mathcal{C} . If D is saturated, then D is closed under inverse limits. \square*

Lemma 1.2 *Let D be a collection of objects in \mathcal{C} . If $X \in D^{\perp\perp}$ and $f: X \rightarrow Y$ is an epimorphism in D^\perp , then f is an isomorphism.*

PROOF. Since $D^{\perp\perp\perp} = D^\perp$, we have a morphism $g: Y \rightarrow X$ such that $g \circ f = I$ where I is the identity morphism on X . Thus, f is an epimorphism which is also a section, so that f is an isomorphism ([10], Proposition 6.15). \square

Let (L, η, χ) be a monad on a category \mathcal{C} , meaning that $L: \mathcal{C} \rightarrow \mathcal{C}$ is a functor and $\eta: I \rightarrow L$ and $\chi: L^2 \rightarrow L$ are natural transformations satisfying the following commutative diagrams for each object X in \mathcal{C} :

$$\begin{array}{ccc} L^3X & \xrightarrow{\chi_L} & L^2X \\ L\chi \downarrow & & \downarrow \chi \\ L^2X & \xrightarrow{\chi} & LX \end{array} \quad \text{and} \quad \begin{array}{ccccc} LX & \xrightarrow{L\eta} & L^2X & \xleftarrow{\eta^L} & LX \\ & \searrow I & \chi \downarrow & \swarrow I & \\ & & LX & & \end{array}$$

If (L, η, χ) is idempotent, that is, $L\eta_X = \eta_{LX} = 1_{LX}$ for each object X , then we say that (L, η, χ) is a localization. Since χ will be an equivalence, we may suppress χ and simply write (L, η) .

For a localization (L, η) on \mathcal{C} , the class D of objects which are LX for some X , forms the class of objects of a reflective full subcategory \mathcal{D} with L the left adjoint of the inclusion functor from \mathcal{D} into \mathcal{C} . Conversely, a reflective subcategory determines a localization so that there is no distinction between reflective subcategories and localizations.

The following illustrates the close relation between the concepts of localization and orthogonal pair.

Proposition 1.3 [7] *Let \mathcal{C} be a category.*

- (a) *Each localization (L, η) on \mathcal{C} determines a unique orthogonal pair (S, D) in \mathcal{C} where $S = \{f \in \text{Mor}(\mathcal{C}) \mid Lf \text{ is an isomorphism}\}$ and $D = \{X \in \text{Obj}(\mathcal{C}) \mid \eta_X: X \cong LX\}$.*
 - (b) *Let (S, D) be an orthogonal pair in \mathcal{C} . Suppose for each X of \mathcal{C} there is a morphism $\eta_X: X \rightarrow LX$ with $LX \in D$ and $\eta_X \in S$. Then (L, η) is the unique localization on \mathcal{C} of which (S, D) is the associated orthogonal pair.*
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In (a) of the above proposition, we say (S, D) is the associated orthogonal pair of (L, η) . Objects in D are known as L -local objects and morphisms in S are known as L -equivalences. We say $\eta_X: X \rightarrow LX$ L -localizes X . In particular, LX is L -local and η_X is an L -equivalence. A useful characterization of an L -localization of an object is as follows :

Proposition 1.4 $f: X \rightarrow Y$ L -localizes X if and only if f is an L -equivalence and Y is L -local. \square

In view of Proposition 1.2, we may define a partial ordering between localizations via their respective orthogonal pairs.

Consider localizations (L', η') and (L, η) on categories \mathcal{C}' and \mathcal{C} respectively where \mathcal{C} is a subcategory of \mathcal{C}' . Let $i: \mathcal{C} \rightarrow \mathcal{C}'$ be the inclusion functor. We say that (L', η') extends (L, η) if there exists a natural equivalence $\gamma: Li \rightarrow iL': \mathcal{C} \rightarrow \mathcal{C}'$ such that $\gamma_X \circ \eta_X = \eta'_X$, for each object X of \mathcal{C} (that is, the restriction of (L', η') to \mathcal{C} is just (L, η)). This is equivalent to the orthogonal pair of (L', η') extending that of (L, η) .

The last term related to localization and orthogonal pair to be discussed is factorization system [8]. For two morphisms f and g in \mathcal{C} , we write $f \downarrow g$ or $g \uparrow f$ if for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ a \downarrow & & \downarrow b \\ C & \xrightarrow{g} & D, \end{array}$$

there exists a unique morphism $h: B \rightarrow C$ such that $g \circ h = b$ and $h \circ f = a$.

Let S be a collection of morphisms of \mathcal{C} . We take S^\uparrow and S^\downarrow to be the following collections of morphisms in \mathcal{C} :

$$S^\uparrow = \{f \in \text{Mor}(\mathcal{C}) \mid f \downarrow s \text{ for each } s \in S\},$$

$$S^\downarrow = \{f \in \text{Mor}(\mathcal{C}) \mid f \uparrow s \text{ for each } s \in S\}.$$

A factorization system, denoted by (E, M) , in \mathcal{C} consists of two collections E and M of morphisms of \mathcal{C} , such that

- (i) $E = M^\uparrow$ and $M = E^\downarrow$, and
- (ii) each morphism f in \mathcal{C} can be factored as $f = m \circ e$ where $e \in E$ and $m \in M$.

We say that a factorization system (E, M) is E -saturated if E is saturated. The following two results relate factorization system, localization and orthogonal pair.

Theorem 1.5 ([3], [8]) *Let \mathcal{C} have a terminal object T and (E, M) be a factorization system in \mathcal{C} . Then, (E, M) determines a reflective category M/T of \mathcal{C} . Objects of M/T are objects X in \mathcal{C} whose unique morphism $x: X \rightarrow T$ is in M . The corresponding localization (\mathcal{L}, η) is such that $X \xrightarrow{\eta_X} \mathcal{L}X \xrightarrow{t} T$ is the unique (E, M) -factorization of x . \square*

The localization given in the above result we call (E, M) -localization.

Proposition 1.6 *Let \mathcal{C} be a category having a terminal object T and (E, M) be a factorization system in \mathcal{C} . Then $M/T = E^\perp$, and the orthogonal pair associated with (E, M) -localization is $(E^{\perp\perp}, E^\perp)$, the orthogonal pair generated by E .*

PROOF. Let $X \in M/T$, that is, the unique morphism $x: X \rightarrow T$ is in M . Let $s: A \rightarrow B \in E$ and $g: A \rightarrow X$ be any morphism in \mathcal{C} . Then we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ g \downarrow & \swarrow h & \downarrow b \\ X & \xrightarrow{x} & T, \end{array}$$

where b is the unique morphism from B to T and $h: B \rightarrow X$ is the unique morphism such that $h \circ s = g$ and $x \circ h = b$, since $x \uparrow s$. The uniqueness of h such that $h \circ s = g$ follows from that of h making both triangles commutative and that T is terminal. Hence, X is in E^\perp .

Let $X \in E^\perp$ and the following be a commutative square,

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ g \downarrow & & \downarrow b \\ X & \xrightarrow{x} & T, \end{array}$$

where $s: A \rightarrow B \in E$. Then there exists a unique morphism $h: B \rightarrow X$ such that $h \circ s = g$, since $s \perp X$. It follows from T being the terminal object that $x \circ h = b$. Thus, $x: X \rightarrow T$ is in M , and $X \in M/T$. \square

We shall apply the following existence theorem in the next section.

Theorem 1.7 ([3], Theorem 3.4) *Let \mathcal{C} be a cocomplete category having all set-indexed products and satisfying*

() for each X in \mathcal{C} , there is a set $\{i_\alpha: L_\alpha \rightarrow X\}$ such that each $g: K \rightarrow X$ can be factored as $g = i_\alpha \circ s$, for some α and epi s .*

Let D be a set of objects in \mathcal{C} and $E = D^\perp$. Then (E, E^\perp) is a factorization system in \mathcal{C} . \square

2 The Main Theorem

The category of groups \mathcal{G} satisfies the hypothesis in Theorem 1.7 [3]. However, we cannot apply this theorem directly to give our main result as the class of P -local nilpotent groups, which is denoted by D , is too big to be a set. Here, P is a fixed set of primes. Instead, we shall consider subcollections D_γ of D , where γ is an infinite cardinal and D_γ is the **set** of all P -local nilpotent groups of cardinality $\leq \gamma$ so that Theorem 1.7 works. We obtain the following result which applies to D when it is not a class with some additional hypothesis.

Theorem 2.1 *Let D be a class of groups and for each infinite cardinal λ , let D_λ be the subclass of D containing groups in D which have cardinality not greater than λ . Suppose there is a natural transformation $\alpha_G: G \rightarrow \mathcal{F}G$ where $\mathcal{F}G \in D_{\gamma(G)}^{\perp\perp}$ for some infinite cardinal $\gamma(G)$. If, for every $H \in D$, every morphism $\phi: G \rightarrow H$ factors through α_G (not necessarily uniquely), then the orthogonal pair $(D^\perp, D^{\perp\perp})$ generated by D admits a localization in \mathcal{G} .*

PROOF. By Theorem 1.7, $(D_\lambda^\perp, (D_\lambda^\perp)^\perp)$ is a factorization system in \mathcal{G} . Because \mathcal{G} has a terminal object, Theorem 1.5 tells us that $(D_\lambda^\perp, (D_\lambda^\perp)^\perp)$ defines a localization $(\mathcal{L}_\lambda, \eta_\lambda)$ on \mathcal{G} whose associated orthogonal pair is $(D_\lambda^\perp, D_\lambda^{\perp\perp})$, by Proposition 1.6.

For each G , α_G factors through $(\eta_\lambda)_G$ for every $\lambda \geq \gamma(G)$, say $\alpha_G = \beta \circ (\eta_\lambda)_G$. Suppose $\beta = m \circ e$ is the epi-mono factorization of β . To show that $e \in D^\perp$, let $H \in D_\lambda$ and $h: \mathcal{L}_\lambda G \rightarrow H$. By the assumption on α , there is a map $\delta: \mathcal{F}G \rightarrow H$ such that $h \circ (\eta_\lambda)_G = \delta \circ \alpha_G = \delta \circ (\beta \circ (\eta_\lambda)_G)$. However, $(\eta_\lambda)_G \in D_\lambda^\perp$ so that $\delta \circ \beta = h$. The uniqueness of $\delta \circ m$ such that $(\delta \circ m) \circ e = h$ follows

from e being epi. Thus $e \in D_\lambda^\perp$ and hence, by Lemma 1.2, e is an isomorphism. Therefore, $\mathcal{L}_\lambda G \twoheadrightarrow \mathcal{F}G$ for every $\lambda \geq \gamma(G)$. So there exists $\mu \geq \gamma(G)$ such that for all $\lambda \geq \mu$, $\mathcal{L}_\lambda G = \mathcal{L}_\mu G$ and $(\eta_\lambda)_G = (\eta_\mu)_G$. Hence, $(\eta_\mu)_G \in D_\lambda^\perp$ for every $\lambda \geq \mu$. On the other hand, for $\lambda \leq \mu$, $D_\mu^\perp \subseteq D_\lambda^\perp$; so $(\eta_\mu)_G \in D_\lambda^\perp$ for every infinite cardinal λ . Thus, $(\eta_\mu)_G \in D^\perp$. Since $D_\mu^{\perp\perp} \subseteq D^{\perp\perp}$, $\mathcal{L}_\mu G \in D^{\perp\perp}$. Defining $\mathcal{L}G$ as $\mathcal{L}_\mu G$ and η_G as $(\eta_\mu)_G$, we have a morphism $\eta_G: G \rightarrow \mathcal{L}G$ with $\mathcal{L}G \in D^{\perp\perp}$ and $\eta_G \in D^\perp$. This gives the desired localization, by Proposition 1.3(b). \square

Corollary 2.2 *With the above notation, for each group G , $\mathcal{L}G$ embeds in $\mathcal{F}G$ and hence $\mathcal{F}G$ embeds in $\mathcal{F}(\mathcal{F}G)$.*

PROOF. Note that $\eta: G \rightarrow \mathcal{L}G$ is just $\eta_\lambda: G \rightarrow \mathcal{L}_\lambda G$ for some $\lambda \geq \text{card}(G)$ and hence $\mathcal{L}G$ embeds in $\mathcal{F}G$, from the proof of Theorem 2.1. In particular, $\mathcal{L}(\mathcal{F}G)$ embeds in $\mathcal{F}(\mathcal{F}G)$. But $\mathcal{L}(\mathcal{F}G) = \mathcal{F}G$ and hence the result. \square

Corollary 2.3 $D^{\perp\perp} = \cup_\lambda D_\lambda^{\perp\perp}$, where λ runs through all infinite cardinals. \square

To apply the above theorem to the case where D is the class of all P -local nilpotent groups where P is a fixed set of primes, we need to have a natural transformation α satisfying the hypothesis. Denote \mathcal{N}_γ the category of all nilpotent groups of cardinality $\leq \gamma$ and D_γ the **set** of all P -local nilpotent groups of cardinality $\leq \gamma$. Then $D = \cup_\gamma D_\gamma$ and $\cap_\gamma D_\gamma^\perp = D^\perp$.

Recall that the lower central series of G is defined as follows :

$$G = \Gamma_1 G \geq \Gamma_2 G \geq \dots \geq \Gamma_i G \geq \Gamma_{i+1} G = [G, \Gamma_i G] \geq \dots$$

so that for a nilpotent group G , its nilpotency class $\text{nil}(G)$ is the least c with $\Gamma_{c+1} G = 1$. We use (e, μ) to denote the P -localization of nilpotent groups [9], and write $G_P = e(G)$. From the observation that e increases neither cardinality nor nilpotency class, we have the following.

Proposition 2.4 *If $N \in \mathcal{N}_\gamma$, then $e(N) \in D_\gamma$. Thus (e, μ) restricts to a localization on \mathcal{N}_γ denoted by (e_γ, μ_γ) . \square*

Lemma 2.5 *Let i be a finite ordinal. For each group G , the natural transformation $\alpha^i: G \rightarrow (G/\Gamma_i G)_P$ defines a localization whose local objects are P -local nilpotent groups of nilpotency classes less than i . \square*

Consider the \mathbf{Z}_P -completion on \mathcal{G} [4], denoted by $(\widehat{})_P$, where \widehat{G}_P is defined to be $\lim_{\leftarrow} (G/\Gamma_i G)_P$. There is a natural transformation $\alpha: \text{Id} \rightarrow (\widehat{})_P$ giving rise to a monad. However, $((\widehat{})_P, \alpha)$ is not idempotent [5].

Now, let \mathcal{G}' be the full subcategory of \mathcal{G} whose objects are groups G such that $\alpha_{\widehat{G}_P}: \widehat{G}_P \cong ((\widehat{G}_P))_P$. Then, $((\widehat{})_P, \alpha)$ is a localization on \mathcal{G}' . Let (S', D') be its associated orthogonal pair in \mathcal{G}' . Note that $\widehat{N}_P = eN$ and $\alpha_N = \mu_N$ if N is a nilpotent group and $\widehat{G}_P = 1$ if G is a perfect group. Therefore, \mathcal{G}' contains all nilpotent groups and all perfect groups and $((\widehat{})_P, \alpha)$ extends (e, μ) in \mathcal{G}' . In fact, (S', D') is the minimal extension of (S, D) in \mathcal{G}' , that is, $D' = D^{\perp \mathcal{G}' \perp \mathcal{G}'}$ and $S' = D^{\perp \mathcal{G}'}$, where $\perp \mathcal{G}'$ means the orthogonal complement in \mathcal{G}' . Hence, for a group G in \mathcal{G}' , we have $\alpha_G: G \rightarrow \widehat{G}_P$ is in D^\perp and $\widehat{G}_P \in D^{\perp \perp}$.

From [4] (13.3), \mathcal{G}' contains all groups such that $H_1(G, \mathbf{Z}_P)$ is a finitely generated \mathbf{Z}_P -module. In particular, \mathcal{G}' contains all finitely generated groups.

Lemma 2.6 *Let G be a group and X be a P -local nilpotent group. Then each $\phi: G \rightarrow X$ factors through $\alpha: G \rightarrow \widehat{G}_P$.*

PROOF. Let $\text{nil}(X) < c$. By Lemma 2.5, ϕ factors through $\alpha^c: G \rightarrow G/\Gamma_c G \rightarrow (G/\Gamma_c G)_P$. However, α^c itself factors through α . \square

Lemma 2.7 *Let G be a group and λ an infinite cardinal such that $\text{card}(G) \leq \lambda$. Then $\widehat{G}_P \in D_\lambda^{\perp \perp}$.*

PROOF. Note that $\text{card}(G) \leq \lambda$ implies that $\text{card}(G/\Gamma_i G) \leq \lambda$, for each finite i . Hence, by Proposition 2.4, $e(G/\Gamma_i G) \in D_\lambda$ and thus $\widehat{G}_P \in D_\lambda^{\perp \perp}$, by Proposition 1.1. \square

The above lemmas demonstrate that the natural transformation $\alpha: G \rightarrow \widehat{G}_P$ satisfies the hypothesis of Theorem 2.1, and we obtain the following

Theorem 2.8 *Let P be a set of primes and D the class of P -local nilpotent groups. Then the orthogonal pair $(D^\perp, D^{\perp\perp})$ admits a localization L_P on the category of groups \mathcal{G} . \square*

Proposition 2.9 *L_P coincides with the P -completion $((\widehat{})_P, \eta)$ on the subcategory of groups \mathcal{G}' consisting of groups for which $((\widehat{})_P, \eta)$ is idempotent, and in particular on the subcategory of finitely generated groups.*

PROOF. This follows from Proposition 1.3(b) since for a group G in \mathcal{G}' , we have $\alpha_G: G \rightarrow \widehat{G}_P$ is in D^\perp and $\widehat{G}_P \in D^{\perp\perp}$. As noted, \mathcal{G}' contains all groups which are finitely generated. \square

Corollary 2.10 *For each group G , $L_P G$ embeds in \widehat{G}_P and hence \widehat{G}_P embeds in $(\widehat{G}_P)_P$. \square*

A different proof of the above theorem using Kan extensions can be found in [6](Theorem 3.2).

As left adjoints preserve colimits [12] and every group is a direct limit (which is a special type of colimit) of its finitely generated subgroups, the next result expresses $L_P G$ in a somewhat more recognizable form.

Proposition 2.11 *Let $G = \varinjlim H$ where H runs through all finitely generated subgroups of G . Then $L_P G = \varinjlim_D \widehat{H}_P$, since $L_P H = H_{\widehat{P}}$. Here, $\varinjlim_D \widehat{H}_P$ means the direct limit of \widehat{H}_P in the full subcategory \mathcal{D} whose object class is $D^{\perp\perp}$. \square*

In the case where D is the collection of all finite groups, we have the profinite completion as $\mathcal{F}G$ which is in $D_\gamma^{\perp\perp}$ for any infinite cardinal. However, since D is a set, Theorem 1.7 is already applicable to give a factorization system which defines a localization $\eta: G \rightarrow \mathcal{L}G$. Moreover, $\mathcal{L}G \twoheadrightarrow \widehat{G}$ where \widehat{G} is the profinite completion of G .

3 Groups with stable lower central series

If G has a stable lower central series, (that is, there is a finite ordinal c , the minimum such being called its stability, with $\Gamma_d G = \Gamma_c G$ for each $d \geq c$), then Bousfield's p -localization of G , $E\mathbf{Z}_p G$, is $(G/\Gamma_c G)_p$ at each single prime p [4]. It is plain that the nilpotency class of $E\mathbf{Z}_p G$ is less than c for each p . In this section, we show that the converse is true and that this is also equivalent to $L_p G$ being nilpotent of nilpotency class not more than some finite ordinal for all p .

We shall show that for a group G whose abelianization is finitely generated, if $L_p G$ is nilpotent at each single prime p , then G has a stable lower central series. (Here, we do not impose a uniform bound on the nilpotency class as p varies.) The following simple yet useful result follows from the universal property of localization.

Lemma 3.1 *Let (L_1, κ_1) and (L_2, κ_2) be localizations on a category \mathcal{C} such that $(L_1, \kappa_1) \geq (L_2, \kappa_2)$. If G is such that $L_1 G$ is L_2 -local, that is, $L_2 L_1 G \cong L_1 G$, then $\kappa_1: G \rightarrow L_1 G$ is equal to $\kappa_2: G \rightarrow L_2 G$. \square*

Theorem 3.2 *Let G be a group. The following statements are equivalent :*

- (a) G has a stable lower central series with stability at most c ;
- (b) $E\mathbf{Z}_p G$ is nilpotent of nilpotency class less than c at each prime p ;
- (c) $L_p G$ is nilpotent of nilpotency class less than c at each prime p .

PROOF. For (a) implying (b), see [4], while (b) implying (c) follows from Lemma 3.1. It remains to prove that (c) implies (a).

Suppose that at each prime p , $L_p G$ is nilpotent of nilpotency class less than c . By Lemma 2.5 and Lemma 3.1, $L_p G = (G/\Gamma_c G)_p$ and hence by Lemma 3.1 again, $L_p G = (G/\Gamma_j G)_p = (G/\Gamma_c G)_p$, for all $j \geq c$.

Since p -localization preserves short exact sequences in \mathcal{N} [9], we obtain the short exact sequence :

$$(\Gamma_j G/\Gamma_{j+1} G)_p \twoheadrightarrow (G/\Gamma_{j+1} G)_p \rightarrow (G/\Gamma_j G)_p.$$

Thus, for $j \geq c$, we have $(\Gamma_j G / \Gamma_{j+1} G)_p = 0$ for each prime p . Since $\Gamma_j G / \Gamma_{j+1} G$ is abelian, we must have $\Gamma_j G / \Gamma_{j+1} G$ a trivial group. Hence, $\Gamma_j G = \Gamma_c G$ for all $j \geq c$, that is, G has a stable lower central series of stability at most c . \square

Every finite group has a stable lower central series with stability c for some finite ordinal c , so that its $E\mathbf{Z}_p$ -localization and L_p -localization coincide. In fact, for a finite group G its Ribenboim p -localization G_p is a finite p -group at each single prime p . Thus, G_p is a finite p -group and hence G_p is p -local nilpotent with nilpotency class less than c . Thus, at each single p , all localizations in \mathcal{G} which extend e coincide on finite groups. Hence, by Theorem 3.2, we get the result of [1] that finite perfect groups are generically trivial.

There exist groups G with $E\mathbf{Z}_p G$ nilpotent at each prime p , but with no finite ordinal c , such that $\text{nil}(E\mathbf{Z}_p G) \leq c$ for each p . An example is the infinite direct product G of G_i , where G_i is a finite p_i -group whose nilpotency class is more than i , and p_i runs through infinitely many distinct primes. Here, $E\mathbf{Z}_p G = G_i$ if $p_i = p$.

However, if we restrict to finitely generated groups, we have a desirable result, namely Theorem 3.4. We first record the following key to Theorem 3.4.

Lemma 3.3 *Let G be a group. The following statements are equivalent :*

- (a) $E\mathbf{Z}_p G = (G / \Gamma_c G)_p$;
- (b) $\Gamma_c G / \Gamma_{c+1} G$ is p' -torsion;
- (c) $E\mathbf{Z}_p G = (G / \Gamma_d G)_p$, for all $d \geq c$;
- (d) $\Gamma_c G / \Gamma_d G$ is p' -torsion, for all $d \geq c$.

PROOF. To show (a) is equivalent to (b), we first note that $E\mathbf{Z}_p G = (G / \Gamma_c G)_p$ if and only if $G \rightarrow G / \Gamma_c G \rightarrow (G / \Gamma_c G)_p$ is an $H\mathbf{Z}_p$ -map. (We recall from [4] that a group homomorphism $f: A \rightarrow B$ is an $H\mathbf{Z}_p$ -map if f induces an isomorphism $f_1: H_1(A; \mathbf{Z}_p) \cong H_1(B; \mathbf{Z}_p)$ and an epimorphism $f_2: H_2(A; \mathbf{Z}_p) \rightarrow H_2(B; \mathbf{Z}_p)$.) $G / \Gamma_c G$ being nilpotent, we have for each i , $H_i(G / \Gamma_c G; \mathbf{Z}_p) \cong H_i((G / \Gamma_c G)_p; \mathbf{Z}_p)$,

by [9]. Therefore, $E\mathbf{Z}_p G = (G/\Gamma_c G)_p$ if and only if $G \rightarrow G/\Gamma_c G$ is an $H\mathbf{Z}_p$ -map. From the short exact sequence $\Gamma_c G \twoheadrightarrow G \twoheadrightarrow G/\Gamma_c G$, we have the exact sequence :

$$\begin{aligned} H_2(G; \mathbf{Z}_p) &\rightarrow H_2(G/\Gamma_c G; \mathbf{Z}_p) \rightarrow (\Gamma_c G/\Gamma_{c+1} G) \otimes \mathbf{Z}_p \rightarrow H_1(G; \mathbf{Z}_p) \rightarrow \\ &\rightarrow H_1(G/\Gamma_c G; \mathbf{Z}_p) \rightarrow 0. \end{aligned}$$

The above exact sequence tells us that $G \rightarrow G/\Gamma_c G$ is an $H\mathbf{Z}_p$ -map if and only if $(\Gamma_c G/\Gamma_{c+1} G) \otimes \mathbf{Z}_p$ is trivial. Thus, $E\mathbf{Z}_p G = (G/\Gamma_c G)_p$ if and only if $(\Gamma_c G/\Gamma_{c+1} G) \otimes \mathbf{Z}_p$ is trivial, which is equivalent to $\Gamma_c G/\Gamma_{c+1} G$ being p' -torsion.

To show (a) implies (c), we note that $E\mathbf{Z}_p G$ being nilpotent implies that $(G/\Gamma_d G)_p = E\mathbf{Z}_p G = (G/\Gamma_c G)_p$ for all $d \geq c$, by Lemma 3.1. To show (c) implies (d), we note that $(G/\Gamma_d G)_p \cong (G/\Gamma_c G)_p$ implies that the surjection $G/\Gamma_d G \rightarrow G/\Gamma_c G$ is a p -isomorphism. Thus, the kernel is p' -torsion; but the kernel is just $\Gamma_c G/\Gamma_d G$.

The implications (c) \implies (a) and (d) \implies (b) are trivial. This completes the proof. Incidentally, for an alternative, directly group-theoretic proof of (b) \implies (d), see [14] p.127. \square

With this result, we are now in a position to establish the following theorem which is closely related Theorem 3.2. We observe that both Theorem 3.2 and Theorem 3.4 require some kind of finiteness condition, namely, finite stability class and finite generation respectively.

Theorem 3.4 *Let G be a group whose abelianization $H_1(G; \mathbf{Z})$ is a finitely generated group. Then the following statements are equivalent :*

- (a) G has a stable lower central series;
- (b) $E\mathbf{Z}_p G$ is nilpotent at each prime p ;
- (c) $L_p G$, which coincides with \widehat{G}_P , is nilpotent at each prime p .

PROOF. (b) implies (c) follows from Lemma 3.1 and (a) implies (b) from [4]. To show (b) implies (a), note that if $E\mathbf{Z}_p G$ is nilpotent, then there exists an

integer c such that $E\mathbf{Z}_pG = (G/\Gamma_dG)_p$ for each $d \geq c$, where c depends on p . Let \bar{c} be the smallest integer such that $E\mathbf{Z}_2G = (G/\Gamma_{\bar{c}}G)_2$. By the above lemma, $\Gamma_{\bar{c}}G/\Gamma_{\bar{c}+1}G$ is $2'$ -torsion. However, $H_1(G; \mathbf{Z})$ being finitely generated implies that $\Gamma_{\bar{c}}G/\Gamma_{\bar{c}+1}G$ is also finitely generated, by [15] (Theorem 3.1) or [14] (p.127). Together with its being $2'$ -torsion, it now follows that $\Gamma_{\bar{c}}G/\Gamma_{\bar{c}+1}G$ is a finite abelian group.

Let n be the order of the group $\Gamma_{\bar{c}}G/\Gamma_{\bar{c}+1}G$ and P be the finite set of primes dividing n . For any prime q not in P , $\Gamma_{\bar{c}}G/\Gamma_{\bar{c}+1}G$ is q' -torsion. So, by the above lemma, for each $d \geq \bar{c}$, $E\mathbf{Z}_qG = (G/\Gamma_dG)_q$. However, for each $p \in P$, there exists a finite number c_p such that $E\mathbf{Z}_pG = (G/\Gamma_dG)_p$, for each $d \geq c_p$. Let c be the maximal number among \bar{c} and c_p where $p \in P$ (recall that P is finite). Then $E\mathbf{Z}_pG = (G/\Gamma_dG)_p$, for each $d \geq c$. By Theorem 3.2, G has a stable lower central series with stability at most c .

It now remains to prove (c) implies (b). Note that \widehat{G}_P being nilpotent implies that there exists a finite number c such that $\widehat{G}_P = (G/\Gamma_dG)_p$, for each $d \geq c$. However, as in the proof of the above lemma, this implies that the surjection $G/\Gamma_dG \rightarrow G/\Gamma_cG$ is a p -isomorphism. Thus, Γ_cG/Γ_dG is p' -torsion. By the above lemma, $E\mathbf{Z}_pG = (G/\Gamma_cG)_p$, which is nilpotent. \square

Appendix

Factorization systems and localization in \mathcal{G}

In \mathcal{G} , $(D_\gamma^\perp, D_\gamma^{\perp\downarrow})$ is a factorization system, by Theorem 1.7. We shall now show that $(D^\perp, D^{\perp\downarrow})$ is also a factorization system. This follows from Theorem A.6 which is also a special case of Corollary 3.4 [8] and Proposition 10 [11], although the proof given here holds for a more general setting. Here, we have applied the technique of Bousfield [3] in proving this theorem. We begin with two lemmas whose routine proofs are left to the reader.

Lemma A.1 *Let (S, D) be an orthogonal pair. Then*

- (a) *S contains all isomorphisms;*

(b) if f and g are morphisms such that $g \circ f$ is defined, then any two of f , g and $g \circ f$ are in S implies the third is in S ;

(c) if

$$\begin{array}{ccc} A' & \rightarrow & A_1 \\ \alpha \downarrow & & \downarrow \beta \\ A_2 & \rightarrow & A \end{array}$$

is a push-out diagram and α is in S , then β is in S ;

(d) S is closed under small colimits. \square

Lemma A.2 Let \mathcal{C} be a category with a terminal object T and (S, D) be an orthogonal pair in \mathcal{C} . Let X be an object of \mathcal{C} . Then the unique morphism $X \xrightarrow{x} T$ is in S^\perp if and only if $X \in D$. \square

Proposition A.3 Let \mathcal{C} be a category with a terminal object T and (S, D) be an orthogonal pair in \mathcal{C} . Then $(S^\perp)^\uparrow = S$.

PROOF. It suffices to show that $(S^\perp)^\uparrow \subseteq S$. Let $f: G \rightarrow K$ be in $(S^\perp)^\uparrow$, $X \in D$ and $g: G \rightarrow X$. Then we have $k \circ f = x \circ g$ where $k: K \rightarrow T$ and $x: X \rightarrow T$ are unique morphisms to T . By Lemma A.2, $x \in S^\perp$ and hence there is a unique morphism $\phi: K \rightarrow X$ such that $\phi \circ f = g$ and $x \circ \phi = k$. Thus, $f \in S$. \square

The next two results can be easily checked.

Lemma A.4 Let \mathcal{C} be a category and D be a class of objects of \mathcal{C} . Let $f: A \rightarrow B$ be an epimorphism and $g: B \rightarrow C$ be a morphism in \mathcal{C} . Then $g \circ f \in D^\perp$ implies $f \in D^\perp$ and $g \in D^\perp$. \square

Lemma A.5 Let \mathcal{C} be a category and S be a class of morphisms of \mathcal{C} . Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms in \mathcal{C} . Then

(a) if f and g are in S^\perp , then $g \circ f$ is in S^\perp ;

(b) if $g \circ f$ and g are in S^\perp , then f is in S^\perp ;

(c) if g is a monomorphism and $g \circ f \in S^\perp$, then $f \in S^\perp$;

(d) if

$$\begin{array}{ccc} A' & \xrightarrow{u} & A_1 \\ \downarrow & & \downarrow \\ A_2 & \xrightarrow{v} & A \end{array}$$

is a pull-back diagram and v is in S^\perp , then u is in S^\perp . \square

Now we shall prove our main result in this section, namely,

Theorem A.6 *Let (L, η) be a localization on the category of groups and (S, D) be its associated orthogonal pair. Then (S, S^\perp) is a factorization system.*

PROOF. In view of Proposition A.3, it suffices to show that every group homomorphism has a (S, S^\perp) -factorization. Let $f: G \rightarrow K$ be a group homomorphism, and η the (L, η) -localization of G . Suppose $W = LG \times K$ and $\pi_1: W \rightarrow LG$ and $\pi_2: W \rightarrow K$ are projections. Then there is a unique group homomorphism a such that $\pi_1 \circ a = \eta$ and $\pi_2 \circ a = f$. Let $G \xrightarrow{\bar{a}} W_a \xrightarrow{i} W$ be the epi-mono factorization of a . By Lemma A.4, \bar{a} and $\pi_1 \circ i$ are in S , since $\eta = (\pi_1 \circ i) \circ \bar{a} \in S$.

Now we define a new category \mathcal{F}_f as in [3], where objects of \mathcal{F}_f are factorizations $G \xrightarrow{s} Z \xrightarrow{t} W$ of a with $s \in S$ and morphisms of \mathcal{F}_f are commuting diagrams :

$$\begin{array}{ccccc} G & \xrightarrow{s} & Z & \xrightarrow{t} & W \\ \parallel & & \downarrow & & \parallel \\ G & \xrightarrow{s'} & Z' & \xrightarrow{t'} & W . \end{array}$$

By Lemma A.1, S is closed under small colimits so that \mathcal{F}_f is cocomplete. Let \mathcal{R}_f be a set of objects of \mathcal{F}_f containing precisely objects $G \xrightarrow{s} Z \xrightarrow{t} W$, where t is a monomorphism. Then \mathcal{R}_f is non-empty as it contains $G \xrightarrow{\bar{a}} W_a \xrightarrow{i} W$. We first show that \mathcal{R}_f is a weakly terminal set in \mathcal{F}_f .

Let $G \xrightarrow{s} Z \xrightarrow{t} W$ be an object of \mathcal{F}_f and $Z \xrightarrow{\bar{t}} Imt \xrightarrow{j} W$ the epi-mono factorization of t . Since $s \in S$, there exists ζ such that in the following diagram

the triangle commutes

$$\begin{array}{ccccc}
G & \xrightarrow{s} & Z & \xrightarrow{\bar{t}} & Imt \\
& \searrow \eta & \zeta \downarrow & & \downarrow j \\
& & LG & \xleftarrow{\pi_1} & W
\end{array}$$

and $\zeta \circ s = \eta = \pi_1 \circ a = \pi_1 \circ (t \circ s) = \pi_1 \circ (j \circ \bar{t} \circ s) = (\pi_1 \circ j \circ \bar{t}) \circ s$. Hence, $\zeta = \pi_1 \circ j \circ \bar{t}$, since $s \in S$ and $LG \in D$. But $\zeta \in S$, by Lemma A.1(b), since both $\eta = \zeta \circ s$ and s are in S . Therefore, $\bar{t} \in S$, by Lemma A.4 since \bar{t} is epi, and $\bar{t} \circ s \in S$ by Lemma A.1(b) so that

$$\begin{array}{ccccc}
G & \xrightarrow{s} & Z & \xrightarrow{t} & W \\
\parallel & & \bar{t} \downarrow & & \parallel \\
G & \xrightarrow{\bar{t} \circ s} & Imt & \xrightarrow{j} & W
\end{array}$$

is a morphism in \mathcal{F}_f showing that \mathcal{R}_f is a weakly terminal set.

By the Existence Theorem [12] for terminal objects, \mathcal{F}_f has a terminal object we denote by $G \xrightarrow{v} V \xrightarrow{w} W$. We show that w and π_2 are in S^\downarrow . Given $g \in S$ such that $\beta \circ g = w \circ \alpha$, let U be the push-out of g and α , so that $h: V \rightarrow U$ is in S and we have the commutative diagram :

$$\begin{array}{ccccc}
& & A & & \\
& & \alpha \swarrow & & \searrow g \\
G & \xrightarrow{v} & V & \xrightarrow{h} & U & \xleftarrow{k} & B \\
& & \searrow w & & \swarrow \beta \\
& & & u \downarrow & & \\
& & & W & & .
\end{array}$$

Thus, $G \xrightarrow{h \circ v} U \xrightarrow{u} W$ is an object in \mathcal{F}_f and there is a morphism $m: U \rightarrow V$ such that $m \circ h \circ v = v$, $w \circ m = u$ and $m \circ h = 1$. Hence, letting $\rho = m \circ k: B \rightarrow V$, we have $\rho \circ g = \alpha$ and $w \circ \rho = \beta$. To show uniqueness, let $\rho': B \rightarrow V$ satisfy $\rho' \circ g = \alpha$ and $w \circ \rho' = \beta$. Replacing w by the identity map I on V and using the push-out argument, there is a morphism $m': U \rightarrow V$ satisfying $m' \circ h = I$, $m' \circ k = \rho'$ and $w \circ m' = u$, and we have the following morphism in \mathcal{F}_f :-

$$\begin{array}{ccccc}
G & \xrightarrow{h \circ v} & U & \xrightarrow{u} & W \\
\parallel & & m' \downarrow & & \parallel \\
G & \xrightarrow{v} & V & \xrightarrow{w} & W .
\end{array}$$

By the universal property of the terminal object, $m' = m$ and hence $\rho' = \rho$. Thus, $w \in S^\perp$. By Lemma A.2, the unique map $\sigma: LG \rightarrow \langle 1 \rangle$ is in S^\perp where $\langle 1 \rangle$ is the trivial group. Noting that π_2 is the pull-back of σ along the morphism $K \rightarrow \langle 1 \rangle$, it is in S^\perp by Lemma A.5(a). Consequently, by Lemma A.5(a), $\pi_2 \circ w \in S^\perp$, and $f = (\pi_2 \circ w) \circ v$ is the required (S, S^\perp) -factorization. \square

Corollary A.7 $(D^\perp, D^{\perp\perp})$ is a factorization system in \mathcal{G} . \square

We use the partial ordering on factorization systems defined in [8], that is, $(E, M) \leq (E', M')$ whenever E' is contained in E . Let Φ denote the order-preserving map sending a localization (L, η) with associated orthogonal pair (S, D) to the factorization system (S, S^\perp) , and Ψ the order-preserving map sending a factorization system (E, M) to the (E, M) -localization, whose associated orthogonal pair is $(E^{\perp\perp}, E^\perp)$. Then $\Psi \circ \Phi = 1$ and $\Phi \circ \Psi(E, M) = (E^{\perp\perp}, E^{\perp\perp\perp})$. Thus, we get a bijection between localizations and E -saturated factorization systems.

Corollary A.8

- (a) *The pair (Φ, Ψ) forms an adjunction between factorization systems and localizations in \mathcal{G} .*
- (b) *In \mathcal{G} , there is a one-to-one order-preserving correspondence between localizations and E -saturated factorization systems.* \square

Because E -saturated factorization systems are thus equivalent to reflective subcategories, they are called reflective factorization systems in [8]. The conclusions of Theorem A.6 and Corollary A.8 hold for a cocomplete category \mathcal{C} having a terminal object and products and also having the class E of epimorphisms and the class M of monomorphisms forming a factorization system on \mathcal{C} . In fact, a similar result has been obtained in [8] (Theorem 3.3) for categories which are finitely well-complete.

We may now deduce a characterization of orthogonal pairs admitting localizations in suitable categories.

Corollary A.9 *Given an orthogonal pair (S, D) in \mathcal{G} , the following statements are equivalent :*

- (a) (S, D) admits a localization;
- (b) (S, S^\perp) is a factorization system;
- (c) every group homomorphism has an (S, S^\perp) -factorization;
- (d) every trivial group homomorphism $f : G \rightarrow 1$ has an (S, S^\perp) -factorization;
- (e) to each group G , there is a group homomorphism $G \rightarrow H$ in S with H in D .

PROOF. The equivalence of (a) and (b) follows from Corollary A.8(b), whereas that of (b) and (c) follows from Proposition A.3. The implication (c) \implies (d) is clear whereas the implication (d) \implies (a) follows from [8](Theorem 2.3), Proposition A.1(b) and Proposition A.3. Moreover, the equivalence of (d) and (e) is immediate from Lemma A.2 (compare Proposition 1.3(b)). \square

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