

## ABSTRACTS OF RECENT PAPERS

A J BERRICK

### 1. $K$ -THEORY AND THE PLUS-CONSTRUCTION

1.1. [15] **A. J. Berrick & Carles Casacuberta: A universal space for plus-constructions.** We exhibit a 2-dimensional, acyclic, Eilenberg–Mac Lane space  $W$  such that, for every space  $X$ , the plus-construction  $X^+$  with respect to the largest perfect subgroup of  $\pi_1(X)$  coincides, up to homotopy, with the  $W$ -nullification of  $X$ ; that is, the natural map  $X \rightarrow X^+$  is homotopy initial among maps  $X \rightarrow Y$  where the based mapping space  $\text{map}_*(W, Y)$  is weakly contractible. Furthermore, we describe the effect of  $W$ -nullification for any acyclic  $W$ , and show that some of its properties imply, in their turn, the acyclicity of  $W$ .

1.2. [14] **A J Berrick: The plus-construction as a localization.** The paper begins with a brief survey of two different approaches to localization: first, by inversion of elements in a ring or morphisms in a category; the second considers localizations as idempotent monads. Using the second approach, the plus-construction is viewed here as a localization or nullification functor with respect to an acyclic space  $W$ . Such a functor applied to  $X$  gives rise to a family of perfect normal subgroups of  $\pi_1(X)$ . The case  $X = BGLR$  is examined. The perfect normal subgroups of  $GLR$  are described. There is also a study of those spaces  $W$  for which  $W$ -nullification produces algebraic  $K$ -theory of rings.

1.3. [13] **A J Berrick & W G Dwyer: The spaces that define algebraic  $K$ -theory.** We characterize spaces  $W$  such that the  $W$ -nullification functor  $P_W$ , applied to any  $BGL(R)$ , gives  $BGL(R)^+$ . The paper uses deep results of Hesselholt and Madsen on topological cyclic homology.

1.4. [7] **A J Berrick: Algebraic  $K$ -theory and algebraic topology.** These are lecture notes from an invited course of seven lectures at the Algebraic  $K$ -Theory and its Applications Summer School at ICTP in July 2002.

In the first three lectures, we see that the three main definitions of higher  $K$ -theory – via the plus-construction, symmetric monoidal categories and the  $Q$ -construction – all have at their core a ‘machine lemma’ (respectively, the fibration preservation theorem, the group completion

theorem, and Quillen's Theorem B) that ensures an adequate supply of fibrations, and thence long exact homotopy sequences upon which calculations may be based.

The next pair of lectures is devoted to the contribution of topological Hochschild and cyclic homology. We then give a topologically-oriented review of the current state of knowledge of  $K$ -theory of integers. The final lecture discusses recent work on classifying spaces for mapping class and related groups.

1.5. [6] **A J Berrick: Intertwiners and the  $K$ -theory of commutative rings.** This paper presents a solution to the problem, for a commutative ring  $A$ , of formulating a definition that has the advantages of the  $BGL_n A^+$  route, and yet includes  $K_0 A$  data as well. Recall that  $GL_n A$  sits inside the monoid  $M_n A$  of all  $n \times n$  matrices as its group of units. We focus instead on a larger, functorially defined, submonoid  $\text{Int}_n A$  of  $M_n A$ , comprising the *intertwining matrices*, namely  $S$  that is not a zero divisor and satisfies  $(M_n A)S = S(M_n A)$ .

We establish a number of properties of intertwiners in abstract monoids, and in particular of interwining matrices, so as to make the classifying space and its plus-construction more accessible. This ultimately leads to new insights on the action of  $K_0 A$  on the higher  $K$ -groups, and traditional matters like the Rosenberg-Zelinsky theorem. The theory attains greatest power when  $A$  is a domain of dimension 1, where it provides a new description of torsion in the Picard group of  $A$ . Number fields are an abundant source of examples.

1.6. [3] **A J Berrick & M Karoubi: Hermitian  $K$ -theory of the integers.** Rognes and Weibel used Voevodsky's work on the Milnor conjecture to deduce the strong Dwyer-Friedlander form of the Lichtenbaum-Quillen conjecture at the prime 2. In consequence (the 2-completion of) the classifying space for algebraic  $K$ -theory of the integers  $\mathbb{Z}[1/2]$  can be expressed as a fiber product of well-understood spaces  $BO$  and  $BGL(\mathbb{F}_3)^+$  over  $BU$ .

Similar results are now obtained for Hermitian  $K$ -theory and the classifying spaces of the integral symplectic and orthogonal groups. For the integers  $\mathbb{Z}[1/2]$ , this leads to computations of the 2-primary Hermitian  $K$ -groups and affirmation of the Lichtenbaum-Quillen conjecture in the framework of Hermitian  $K$ -theory.

1.7. [5] **A J Berrick, I Chatterji & G Mislin: From acyclic groups to the Bass conjecture for amenable groups.** We prove that the Bost Conjecture on the  $\ell^1$ -assembly map for countable discrete groups implies the Bass Conjecture on the  $K$ -theory of the integral group ring of discrete groups. It follows that a large class of groups, including all amenable groups, satisfies the Bass Conjecture. The proof is obtained via a tour from geometric functional analysis

(using Lafforgue's work on the Bost assembly map), through operator algebra  $K$ -theory, algebraic topology and combinatorial group theory (using Berrick's work on acyclic groups), and ultimately to general linear algebra.

1.8. [1] **A J Berrick, I Chatterji & G Mislin: Homotopy idempotents on manifolds and Bass' conjectures.** The Bass trace conjectures on the  $K$ -theory of the integral group ring of discrete groups are placed in the setting of homotopy idempotent selfmaps of manifolds. For the strong conjecture, this is achieved via a formulation of Geoghegan. It results in a theorem that the conjecture is equivalent to every homotopy idempotent selfmap of a manifold of dimension at least 4 being homotopic to a map with a unique fixed point. (The condition is proved in dimension 3, but is false in dimension 2.)

The weaker form of the conjecture is reformulated as a comparison of ordinary and  $L^2$ -Lefschetz numbers.

## 2. COHOMOLOGY OF GROUPS, ACYCLIC GROUPS

2.1. [12] **A J Berrick & P H Kropholler: Groups with infinite homology.** It is shown that non-locally-finite groups in a class including all soluble groups of finite rank cannot be acyclic, and that in fact their homology in all positive degrees, regarded as a single abelian group, is infinite. The proof is a surprising application of Miller's affirmation of the Sullivan Conjecture.

2.2. [10] **A J Berrick: A topologist's view of perfect and acyclic groups.** Motivated by Berrick's work on acyclic groups, this invited book chapter is a discussion, at the level of a graduate student, of results and open problems on perfect and acyclic groups (those with trivial homology in positive dimensions). The list of examples of acyclic groups is drawn from a very wide range of mathematics. There are some interesting, and unexpected, patterns.

## 3. ACYCLIC GROUPS AND LOW-DIMENSIONAL TOPOLOGY

3.1. [4] **A J Berrick & Y-L Wong: Acyclic groups and wild arcs.** This paper introduces unifying ideas to the study of acyclic groups of low dimension. Acyclic groups, those groups whose positive degree trivial-coefficients homology vanishes, occur in many areas of mathematics, so it is perhaps surprising that some systematization is possible.

We discuss two classes of acyclic groups that are commutator subgroups of finitely presented groups with infinite cyclic abelianization. The first is algebraic and includes groups first exhibited by Baumslag & Gruenberg, of which it is shown that Epstein's acyclic group is a special case. The second class is geometric, and is shown to include a number of wild arc groups in the literature, including those studied by

Fox and Artin more than half a century ago, and also those investigated recently by Freedman and Freedman.

3.2. [8] **A J Berrick & J A Hillman: Perfect and acyclic subgroups of finitely presentable groups.** We consider acyclic groups of low dimension. To indicate our results simply, let  $G'$  be the nontrivial perfect commutator subgroup of a finitely presentable group  $G$ . Then  $\text{def}(G) \leq 1$ . When  $\text{def}(G) = 1$ ,  $G'$  is acyclic provided that it has no integral homology in dimensions above 2 (a sufficient condition for this is that  $G'$  be finitely generated); moreover,  $G/G' \cong \mathbb{Z}$  or  $\mathbb{Z}^2$ . Natural examples are the groups of knots and links with Alexander polynomial 1. We give a further construction based on knots in  $S^2 \times S^1$ . In these geometric examples,  $G'$  cannot be finitely generated; in general, it cannot be finitely presentable. When  $G$  is a 3-manifold group it fails to be acyclic; on the other hand, if  $G'$  is finitely generated it has finite index in the group of a  $\mathbb{Q}$ -homology 3-sphere.

#### 4. HOMOTOPY THEORY

4.1. [11] **A J Berrick & A A Davydov: Splitting of Gysin extensions.** Let  $X \rightarrow B$  be an orientable sphere bundle. Its Gysin sequence exhibits  $H^*(X)$  as an extension of  $H^*(B)$ -modules. We prove that the class of this extension is the image of a canonical class that we define in the Hochschild 3-cohomology of  $H^*(B)$ , corresponding to a component of its  $A_\infty$ -structure, and generalizing the Massey triple product. We identify two cases where this class vanishes, so that the Gysin extension is split. The first, with rational coefficients, is that where  $B$  is a formal space; the second, with integer coefficients, is where  $B$  is a torus.

4.2. [9] **A J Berrick & E Dror Farjoun: Fibrations and nullifications.** The paper investigates preservation of a fibration  $F \rightarrow E \rightarrow B$  by a localization functor  $L$ , meaning that  $LF \rightarrow LE \rightarrow LB$  is also a fibration. We obtain necessary and sufficient conditions for this, in case  $L$  is equivalent to a nullification functor  $P_W$ . Applications include the fact that if such a fibration is preserved by a nullification, then so too are all of its pullbacks. Another application is a homological criterion for a fibration to be preserved.

#### 5. HOMOTOPY THEORY AND LOW-DIMENSIONAL TOPOLOGY

5.1. [2] **A J Berrick, F R Cohen, Y-L Wong & J Wu: Configurations, braids and homotopy groups.** This paper introduces connections between the topology of configuration spaces and certain objects of a simplicial nature described below. These connections ultimately lead to geometric descriptions of elements of the homotopy groups of spheres in terms of special kinds of braids.

Simplicial and  $\Delta$ -structures of configuration spaces are investigated. New connections between the homotopy groups of the 2-sphere and the

braid groups are given. The higher homotopy groups of the 2- sphere are shown to be derived groups of the braid groups over the 2-sphere. Moreover the higher homotopy groups of the 2-sphere are shown to be isomorphic to the Brunnian braids over the 2-sphere modulo the Brunnian braids over the disk. We also exhibit a presentation of the homotopy groups of the 2-sphere solely in terms of Brunnian braids over the disk.

It was observed that the Artin representation is simplicial and so the direction was moved to investigate the simplicial structure on the sequence of the braid groups. The simplicial structure on braids was introduced in [?] in the canonical way by using the method of doubling/deleting strands. According to the terminology in low dimensional topology, a braid is called *Brunnian* if it becomes a trivial braid after removing any one of its strands. (For instance, the Borromean Rings is a link by closing up a 3-strand Brunnian braid.) By using terminology of simplicial groups, the Brunnian braids are the *Moore cycles* after establishing simplicial or  $\Delta$ -structure on braids. Let  $\text{Brun}_n(M)$  denote the group of  $n$ -strand Brunnian braids over the manifold  $M$ .

**Theorem 5.1.** [2, Theorem 1.2] *There is an exact sequence of groups*

$$1 \rightarrow \text{Brun}_{n+1}(S^2) \rightarrow \text{Brun}_n(D^2) \xrightarrow{f_*} \text{Brun}_n(S^2) \rightarrow \pi_{n-1}(S^2) \rightarrow 1$$

for  $n \geq 5$ , where  $f_*$  is induced from the canonical embedding  $f: D^2 \rightarrow S^2$ .

Thus the torsion homotopy groups of  $S^2$  (or  $S^3$ ) are the invariants for measuring the difference of the Brunnian braids between  $S^2$  and  $D^2$ . Moreover there is a differential on the sequence of the classical Brunnian braids  $\{\text{Brun}_n(D^2)\}$ , which is essentially induced from complex-conjugation operation on configuration spaces. This makes  $\{\text{Brun}_n(D^2)\}$  is a chain complex of non-commutative groups.

**Theorem 5.2.** [2, Theorem 1.3] *For all  $n$  there is an isomorphism of groups*

$$H_n(\text{Brun}(D^2)) \cong \pi_n(S^2).$$

This result describes the homotopy groups of the sphere as derived groups of the classical Brunnian braids.

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