

Homological realization of groups, via K-theory

Jon Berrick & Michel Matthey

Neuchâtel, 26 September 2006

Trieste 2002

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Question: Example of non- K -theory problem solved by algebraic K -theory?

History: Realization problems

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H_* -realization for groups

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Michel's response !

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Is function : $\text{Set}_1 \longrightarrow \text{Set}_2$ surjective?

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$$(b) \text{ RING} \ni R \xrightarrow{?} \dots, K_{-1}(R), K_0(R), K_1(R), \dots$$

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So, consider G torsion generated.

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$$G \in \text{STG} \Rightarrow H_1(G) = 0$$

$$\begin{aligned} A \in \text{ABGP} &\Rightarrow A = K_0(D)/\mathbb{Z} \\ &= K_2(S^2D)/\mathbb{Z} \\ &= \mathcal{Z}(\text{St}(S^2D))/\mathbb{Z} \\ &= \mathcal{Z}(\text{St}(S^2D)/\mathbb{Z}) \end{aligned}$$

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Later, using combinatorial group theory:

$\text{STG} \ni G \xrightarrow{\checkmark} H_1(G) = 0, H_2(G), \dots$ AJB-Miller 1992

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$$K_n^{\operatorname{top}}(\mathcal{K}) \cong \begin{cases} \mathbb{Z} & n \text{ even,} \\ 0 & n \text{ odd.} \end{cases}$$

Given an abelian group M , choose a presentation for M of the form

$$0 \longrightarrow \mathbb{Z}[J] \xrightarrow{f} \mathbb{Z}[I] \longrightarrow M \longrightarrow 0,$$

with free abelian groups having basis sets I, J .

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Its mapping cone \mathcal{C}_φ gives an extension of C^* -algebras

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This extension gives a 6-term exact sequence in K -theory of C^* -algebras.

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Proposition

For any abelian group M , there exist nonunital C^* -algebras $\mathcal{E}_M := S_{\mathbb{C}}^{\text{top}}(\mathcal{C}_\varphi)$ and $\mathcal{F}_M := \mathcal{E}_M \widehat{\otimes} \mathcal{K}$, such that, for every $n \in \mathbb{Z}$,

$$K_{2n}^{\text{alg}}(\mathcal{F}_M) \cong K_{2n}^{\text{top}}(\mathcal{E}_M) \cong M$$

and

$$K_{2n+1}^{\text{alg}}(\mathcal{F}_M) \cong K_{2n+1}^{\text{top}}(\mathcal{E}_M) = 0.$$

Using unitalization $\widetilde{}$ and algebraic suspension S , for abelian groups A, A_2, A_3 , define a group

$$G := \mathrm{St}(S^4(\widetilde{\mathcal{F}}_A)) \times \mathrm{E}(S^4(\widetilde{\mathcal{F}}_{A_2})) \times \mathrm{St}(S^5(\mathcal{F}_{A_3})).$$

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Theorem

Let A, A_2, A_3 be any three abelian groups. Then, there exists $G \in \text{STG}$ such that:

- (i) $\mathcal{Z}(G) \cong A$;
- (ii) $H_1(G) = 0$ (\because stg)
- (iii) $H_2(G) \cong A_2$ ($\because H_2(\text{E}(R)) \cong K_2(R)$)
- (iv) $H_3(G) \cong A_3$ ($\because H_3(\text{St}(R)) \cong K_3(R)$)
- (v) For infinitely many n , $H_n(G) \cong \mathbb{Z} \oplus \diamond$ (\because Arlettaz : $K_n^{\text{alg}}(R) \longrightarrow H_n(\text{St}(R)) \longrightarrow K_n^{\text{alg}}(R)$ is mult'n by $k > 0$).

KO-theory of real C^* -algebras is more subtle.

Proposition

Let M be an abelian group. Then, there exists a real C^* -algebra \mathcal{E}_M with the following topological *K*-theory:

$n \pmod{8}$	0	1	2	3	4	5	6	7
$KO_n^{\text{top}}(\mathcal{E}_M)$	M	0	0	0	M	$M/2$	$\Omega_2(M)$	${}_2M$

where $\Omega_2(M)$ is a suitable abelian group (may depend on presentation for M) sitting in a short exact sequence

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So can strengthen to $H_4(G) \cong A_4$ and $H_5(G) \cong A_5$.

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For real Banach algebras (if Karoubi conjecture holds for nine consecutive non-positive dimensions), best possible is

n	2	3	4	5	6	7	8
$H_n(G)$	A_2	A_3	A_4	A_5	$A \oplus A_6$	A_7	A_8

$$H_9(G) \cong {}_2A \oplus (A \otimes A_3) \oplus A_9 \oplus \text{Tor}(A, A_2)$$

$$H_{10}(G) \cong H \oplus (A \otimes A_4) \oplus A_{10} \oplus \text{Tor}(A, A_3)$$

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The extra terms involve Baues-Goerss' analysis of the Whitehead exact sequence.

Further strongly torsion generated groups

Theorem

$G_1, G_2 \in \text{STG} \implies G_1 \times G_2 \in \text{STG}.$

More generally, a (possibly infinite) cartesian product of stg groups is stg.

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$$\text{EU}(R, \Lambda) = [\text{U}(R, \Lambda), \text{U}(R, \Lambda)] \in \text{STG},$$

e.g. $\text{Sp}(\mathbb{Z}), [\text{O}(\mathbb{Z}), \text{O}(\mathbb{Z})] \in \text{STG}$.

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$$\begin{array}{ccccccc} \text{Aut}^+(F_n) & \longrightarrow & \text{Out}^+(F_n) & \longrightarrow & \text{SL}_n(\mathbb{Z}) & & \\ \downarrow & & \lrcorner & & \downarrow & & \\ \text{Aut}(F_n) & \longrightarrow & \text{Out}(F_n) & \longrightarrow & \text{GL}_n(\mathbb{Z}) & & \end{array}$$

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$$\text{Aut}^+(F_\infty), \text{Out}^+(F_\infty) \in \text{STG}.$$

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For odd n , $B_n(S^2)$ is strongly $2(n - 1)$ -torsion generated.