INTERTWINERS AND THE K-THEORY OF COMMUTATIVE RINGS

A. J. BERRICK

To the memory of Hamilton Berrick, in whose ebbing presence H was born.

Abstract. Since around 1970, the main approach to the K-theory of a ring A has been by means of the homotopy of the plus-construction applied to the classifying space of the general linear group of A. In the case of a commutative ring A, we show how to capture K_0A information that is neglected by this definition, while retaining the higher K-theory. To accomplish this, we expand the algebraic focus from invertible matrices to what we call intertwining matrices. S in M_nA is an *intertwining matrix* if it is not a zero divisor and satisfies $(M_nA)S = S(M_nA)$. We establish a number of properties of intertwiners in abstract monoids, and in particular of interwining matrices, so as to make the classifying space and its plus-construction more accessible. This ultimately leads to new insights on the action of K_0A on the higher K-groups, and traditional matters like the Rosenberg-Zelinsky theorem. The theory attains greatest power when A is a domain of dimension 1, where it provides a new description of torsion in the Picard group of A. Number fields are an abundant source of examples.

0. Introduction

The original, and most widely used, approach to the K-theory of a ring A is via the general linear group $\mathrm{GL}_n A$. The K-groups K_iA were defined by Quillen as $\operatorname{dirlim} \pi_i(B\,\mathrm{GL}_n A^+)$, where the limiting process, stabilization, occurs by taking the direct sum of $\mathrm{GL}_n A$ with identity matrices. (B refers to the classifying space of a group, and $^+$ to Quillen's plus-construction adjoining prescribed low-dimensional cells to a space [7].) The homotopy-theoretic nature of the definition greatly assists calculations, by the powerful machinery of algebraic topology (see [1], [28], [10] for surveys). It has, however, a significant drawback. Roughly half of classical K-theory, namely that dealing with the projective class group, is excluded. Subsequent alternative definitions of K-groups (see [30], [20], [19], [18]) have compensated for this defect by category-theoretic means, with accompanying loss of computational facility. The problem remains to formulate a definition that has the advantages of the $B\,\mathrm{GL}_n A^+$ route, and yet includes K_0A data as well.

This paper presents a solution to the problem when A is a commutative ring. Recall that $\mathrm{GL}_n A$ sits inside the monoid M_nA of all $n \times n$ matrices as its group of units. We focus instead on a larger, functorially defined, submonoid $\mathrm{Int}_n A$ of M_nA, comprising the *intertwining matrices.* Numerous equivalent definitions are given below. For example, they are the matrices that at each localization of A look like a scalar matrix times an invertible matrix. The characterization that is responsible for the name is that S is an intertwining matrix if it is regular and for each matrix M there is another matrix M^S such that $MS = SM^S$. The relationship between $\mathrm{GL}_n A$ and $\mathrm{Int}_n A$ is close and quite subtle, for $B\,\mathrm{GL}_n A^+$ turns out to be the covering of $B\,\mathrm{Int}_n A^+$ that kills the K_0A information contained in the fundamental

1991 *Mathematics Subject Classification. Primary:* 13D15, 15A09, 19D06. *Secondary:* 11C20, 11R29, 13C10, 13C20, 16S50, 20M25, 55R35.

group of $B\operatorname{Int}_n A^+$. There is a short-term price to be paid for acquiring this extra information, in that we must first provide the theoretical basis for the study of intertwiners in general monoids, in order to gain access to $B\operatorname{Int}_n A^+$. With this done, we then explore relations with the existing objects of K-theory. At the heart of these links is the function H that assigns to each intertwining matrix S the class of the A-ideal $A\langle S\rangle$ generated by its entries.

Here is a guide to the organization of the paper.

Table of Contents.

I Basic constructions and properties
1. Some intertwining definitions
2. Intertwiners for rings
3. Commutative rings
4. Intertwiners for projective modules
II Groups of intertwining matrices
5. Definitions of associated groups
6. Relation to Picard groups
7. A variant of Bass' Rosenberg-Zelinsky sequence
8. Relation to K-groups
9. Surjectivity of H
10. Quadratic field examples
III Stabilization
11. Direct sum stabilization
12. Tensor product stabilization
13. Automorphisms of Azumaya algebras
14. Action of $\tilde{K}_0(A)$ on K-theory

The first part of the paper lays the foundations of the subject. We begin with an axiomatic discussion of intertwiners, placed in an abstract categorical setting. The case of interest here is where the category has a single object, so that the setting is that of a monoid. Submonoids of intertwiners are defined, with properties that mimic those of normal subgroups and allow the formation of a closely related enveloping group. A sufficient condition is given to make the inclusion map from the monoid to its enveloping group induce a homotopy equivalence on classifying spaces. The second section provides examples of these phenomena among the multiplicative monoids of rings. It is followed by important fundamental results identifying the intertwiners in the multiplicative monoid M_nA of a commutative ring A. These lead to the key definition, of $\operatorname{Int}_n A$ as the submonoid of M_nA consisting of its intertwiners. The arguments extend to the monoid of endomorphisms of any finitely generated projective A-module.

The second part establishes the relations between the newly defined monoid $\operatorname{Int}_n A$ and its 'classical' group relatives, such as $\mathrm{GL}_n A$, $\mathrm{SL}_n A$ and E_nA. The vital tool here is the family of homomorphisms H, whose domains are quotient groups of $\operatorname{Int}_n A$. The first of these, sending an intertwining matrix S to the class of its first elementary ideal $A\langle S\rangle$, takes values in the n-torsion ${}_n\operatorname{Pic} A$ of the Picard group of A. We show that its image measures the extent to which the natural inclusion $\mathrm{GL}_n A \hookrightarrow \operatorname{Int}_n A$ fails to induce a homotopy equivalence of plus-constructed classifying spaces. Thus $B\operatorname{Int}_n A^+$ faithfully mimics the higher homotopy of $B\,\mathrm{GL}_n A^+$; however, its fundamental group also captures information about n-torsion in $\operatorname{Pic} A$.

So its K-groups are thereby richer than the traditional ones. This shows up in a variant of Bass' Rosenberg-Zelinsky sequence. The endomorphism map applied to $\operatorname{Int}_n A$ leads to Picard group information, in contrast to $\mathrm{GL}_n A$, which carries none.

The following section lifts the homomorphisms H to group derivations mapping to K-groups modulo n. For such well-behaved rings A as domains of dimension 1, there is a remarkable construction of intertwining matrices that shows that all n-torsion in the Picard group, or alternatively all of $K_1(A\,;\mathbb{Z}/n)$, may be captured by $\mathrm{Int}_n\,A$. (Here, $K_1(A\,;\mathbb{Z}/n)$ denotes the K_0 group of the n th direct sum functor on the category of fg projective A-modules.) Part of the power of this approach is that it yields interesting results that make no reference to $\mathrm{Int}_n\,A$. (See Theorem 0.1 below.) Some number theory may now be recast in the setting of matrix algebra; we explore examples in quadratic number fields, where there is interaction with recent work of [22].

The third part of the paper analyses the limiting process from particular n. It is first shown that the usual stabilization of $\mathrm{GL}_n\,A$, by taking the direct sum of a matrix with an identity matrix, is inappropriate here. Instead, it is necessary to consider the tensor product with an identity matrix. This leads to the monoid $\mathrm{Int}_\otimes\,A$. A Whitehead Lemma is established, identifying $E_\otimes A$ as the maximal perfect subgroup of the enveloping group. Stabilization of the Rosenberg-Zelinsky sequence combines with work of Weibel on Azumaya algebras, allowing $B\,\mathrm{Int}_\otimes\,A^+$ to be revealed as an infinite loop space, whose homotopy groups are described. As well as giving rational K-theory in higher dimensions, they again yield extra information about torsion in $\mathrm{Pic}\,A$. The natural action of an intertwining matrix on other square matrices may thereby be interpreted as an action of torsion elements in $\tilde{K}_0(A)$ on higher K-groups.

To indicate the kind of results that we obtain, here is a summary of some of our findings for domains of dimension 1.

Theorem 0.1. *Let A be a domain of dimension* 1, *with field of fractions K, and let $n \geq 2$.*
(a) (6.7), (12.7) *Inclusion induces a natural map*

$$\pi_i(B\,\mathrm{GL}_n\,A^+) \to \pi_i(B\,\mathrm{Int}_n\,A^+)$$

with the following properties.

(i) *For $i \geq 2$, it is an isomorphism.*
(ii) *For $i = 1$, it is a monomorphism, which splits when A is the ring of integers in a number field such that $U(A)$ is infinite and the fraction field K is a global field.*
(iii) *For $i = 1$, its abelian cokernel is given by a group extension*

$$U(K)/U(A) \rightarrowtail \mathrm{Cokernel} \twoheadrightarrow {}_n\,\mathrm{Pic}\,A.$$

(b) (9.3) *Write* $\mathrm{Pic}(n, A)$ *for the K_0 group of the n th power functor on the category of invertible A-modules. Then*

$$\det{}_1^{(n)} : K_1(A;\, \mathbb{Z}/n) \longrightarrow \mathrm{Pic}(n, A)$$

is an epimorphism, and

$$\det{}_0 : \mathrm{Tor}\,\tilde{K}_0(A) \longrightarrow \mathrm{Tor}\,\mathrm{Pic}\,A$$

is a split epimorphism.
(c) (6.8) *Let A be a Dedekind domain. With μ_n as the group of n th roots of unity in A, and* $\mathrm{Cart}A$ *as the group of Cartier divisors, there is an exact sequence*

$$\mu_n \hookrightarrow U(A) \overset{\otimes I_n}{\to} GL_nA/\,\mathrm{SL}_n\,A \to \mathrm{Int}_n\,A/(A^\times \cdot \mathrm{SL}_n\,A) \overset{\det}{\to}$$
$$\overset{\det}{\to} U(K)/(U(A)(U(K))^n) \to (\mathrm{Cart}A)/n \twoheadrightarrow (\mathrm{Pic}\,A)/n$$

whose middle term is isomorphic to $\mathrm{Pic}(n, A)$.

(d) (13.2) *There is a natural isomorphism*

$$\mathrm{End} : \mathrm{PInt}_{\otimes} A \to \mathrm{Aut}(\mathrm{Az}\, A),$$

from the stabilization of the projective group of intertwining matrices to the group of automorphisms of all Azumaya A-algebras.

.

I Basic constructions and properties

1. Some intertwining definitions

Intertwining operators form a vital tool in harmonic analysis and mathematical physics, yet they do not appear to have been studied previously from an algebraic or categorical viewpoint such as we now adopt.[1]

In general, an intertwining operator is just a natural transformation from a monoid. To expand on this, recall that for each object A of a category $\mathcal{C}$, composition of morphisms makes the endomorphism set $\mathrm{End}(A)$ a monoid. So monoids can be identified with categories having only one object. Then in $\mathcal{C}$ an intertwining operator comprises a monoid $S = \mathrm{End}(X)$, (where X is the sole object of some category), a $\mathcal{C}$-morphism $f : A \to B$ (the value in $\mathcal{C}$ of the natural transformation at X), together with monoid homomorphisms $\alpha : S \to \mathrm{End}(A),\ \beta : S \to \mathrm{End}(B)$, such that for all x in S the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow^{\alpha(x)} & & \downarrow^{\beta(x)} \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Although f can also be called an S-equivariant morphism, that description can be confusing in the situation of most interest to us, where $A = B$ but $\alpha \neq \beta$. In most applications, the monoids possess further structure. The 'classical' example is where S is a topological group and the morphisms in $\mathcal{C}$ are Hilbert space operators. For the applications in this paper, they are modules. The functions α, β as functors are then required to be structure-preserving. Five special cases of intertwining operators are especially noteworthy here.

Case 1. Our main interest is in the situation $S = \mathrm{End}(B)$ and $\beta = \mathrm{id}_{\mathrm{End}(B)}$, when we shall call f a *right intertwiner* if for each $g : B \to B$ there exists a unique h (namely $\alpha(g)$) in $\mathrm{End}(A)$ with $fh = gf$. There is then a monoid homomorphism $f^{\#} = \alpha : \mathrm{End}(B) \to \mathrm{End}(A)$. As the notation suggests, this association sets up a contravariant functor, the *endomorphism functor* End . Its domain is the category whose objects are those of $\mathcal{C}$ and whose morphisms are right intertwiners, while its codomain lies in the category of monoids. We also write $g^f = f^{\#}g$, expressing the fact that right intertwiners act on the right.

Case 2. Dually, a *left intertwiner* $f : A \to B$ is obtained by putting $\alpha = \mathrm{id}_{\mathrm{End}(A)}$ and requiring uniqueness in $\mathrm{End}(B)$. It defines a monoid homomorphism $f_{\#} : \mathrm{End}(A) \to \mathrm{End}(B)$, assigning to each $h : A \to A$ the unique $g : B \to B$ such that $fh = gf$.

Case 3. When $A = B$ and $\alpha = \beta$, the defining property $f\alpha(x) = \alpha(x)f$ makes f a *commuting* morphism (with respect to $\alpha : S \to A$).

Case 4. The intersection of the above cases is a morphism $f : A \to A$ which commutes with all endomorphisms of A. Such an f is *central.*

[1] R Street has pointed out that a special case of the definition appears in [24] p.41.

Case 5. Any isomorphism is both a right and left intertwiner. For then the equation $fh = gf$ corresponds to $h = g^f = f^{-1}gf$, and to $g = fhf^{-1}$. In the other direction, any left and right intertwiner $f : A \to B$ is easily seen (by the uniqueness condition) to induce inverse isomorphisms $f^\#$ and $f_\#$ between $\mathrm{End}(A)$ and $\mathrm{End}(B)$.

The uniqueness condition in the definition of a right intertwiner f can be strengthened by requiring that f be *right regular,* meaning that for all h_1, h_2 in $\mathrm{End}(A)$, $fh_1 = fh_2$ implies $h_1 = h_2$. From the dual definition of left regular, one sees that the right intertwiner f has $f^\#$ a monomorphism precisely when f is also left regular. Note that when, as in our applications, the categories are pre-additive and α is an additive functor, then every right intertwiner is a right regular intertwiner. For, additivity makes $f^\# : \mathrm{End}(B) \to \mathrm{End}(A)$ a ring homomorphism. So from $f(h_1 - h_2) = 0 = 0f$, we have that $h_1 - h_2 = f^\# 0 = 0$.

A further strengthening is obtained by requiring that f be *monic,* that is, for h_1, h_2 in $\mathrm{Mor}(Z, A)$, $fh_1 = fh_2$ implies $h_1 = h_2$. Of course, this reduces to right regularity when A is the unique object of $\mathcal{C}$. Again this condition is natural to the setting of the paper. For, assume that A is a generator of $\mathcal{C}$, in other words, $\mathrm{Mor}(A, -)$ embeds $\mathcal{C}$ in the category of sets, and that f is right regular. Given h_1, h_2 in $\mathrm{Mor}(Z, A)$ with $fh_1 = fh_2$, suppose that $h_1 \neq h_2$. Then $h_{1*} \neq h_{2*} : \mathrm{Mor}(A, Z) \to \mathrm{Mor}(A, A)$. Thus there exists $k : P \to N$ with $h_1 k \neq h_2 k : P \to P$. Since f is right regular, this is in contradiction of the fact that $fh_1 k = fh_2 k$. The conclusion is that f must be monic.

We now specialize to the case where the category $\mathcal{C}$ has a unique object. Let M be an arbitrary monoid. Write M^* for the submonoid of right regular elements of M. Then $x \in M^*$ is a right intertwiner (in fact a right regular intertwiner) precisely when $Mx \subseteq xM$. Denote by $M^\times$ the monoid of right regular intertwiners of M. If $x \in M$ is right intertwining, then the monoid homomorphism $x^\#$ becomes an endomorphism of M, and $\theta(x) = x^\#$ defines a canonical monoid homomorphism $\theta : M^\times \to \mathrm{End}(M)$. When $M = M^\times$ and every $x^\#$ is an automorphism, M has been called a *rack*, or *quandle* (since also $x^x = x$).

More generally, for two submonoids N, N' of M, we call (N, N') an *ri pair* if for all $n' \in N'$ we have $Nn' \subseteq n'N$. Thus an element x of M^* is right intertwining just when the cyclic submonoid $\langle x \rangle$ has $(M, \langle x \rangle)$ an ri pair. We call a submonoid N of M^*

- *right intertwining* when (M, N) is an ri pair;
- *stable* when both (M, N) and (N, N) are ri pairs; and
- *normal* when both (M, N) and (N, M) are ri pairs.

Strictly speaking, the last two terms should be prefixed by 'right'. However, we have no need for the left-handed concepts. We record the following observations, the first three of which help to explain the terminology. For example, the condition of stability ensures that the sequence $N \supseteq N^\times \supseteq (N^\times)^\times \supseteq \cdots$ stabilizes.

Lemma 1.1. *Let N be a submonoid of M^*.*

(a) *N is right intertwining if and only if $N \subseteq M^\times$.*

(b) *Suppose that $N \subseteq M^\times$. Then N is stable if and only if $N = N^\times$.*

(c) *N is a normal submonoid of $U(M)$ if and only if N is a normal subgroup of $U(M)$.*

(d) *Suppose that $M = \mathrm{colim}\, M_i$ and $N = \mathrm{colim}\, N_i$ ($i \in \mathbb{N}$), with commuting maps*

$$\begin{array}{ccc} N_i & \hookrightarrow & M_i \\ \downarrow & & \downarrow \\ N_{i+1} & \hookrightarrow & M_{i+1} \end{array}$$

If for each i, N_i is stable (resp. normal) in M_i, then N is stable (resp. normal) in M.

Now let P be a submonoid of N.

(e) *If (M,N) is an ri pair then (M,P) is an ri pair.*

(f)

$$\text{normal} \Rightarrow \text{stable} \Rightarrow \text{right intertwining.}$$

(g) *If P is normal in M and N is stable, then P is normal in N.*

(h) *Suppose that M is a quandle. If P is characteristic in N ($\subseteq U(M)$), and N is normal in M, then P is normal in M.* □

We have already noted that both units and regular central elements are right regular intertwiners.

Definition 1.2. With $U(M)$ as the subgroup of units of M, and $\mathcal{Z}(M)$ as the centre, we define the *senate*[2] $\mathrm{Sen}(M)$ of M to be the submonoid of $M^\times$

$$\mathrm{Sen}(M) = (M^* \cap \mathcal{Z}(M)) \cdot U(M) = M^* \cap (\mathcal{Z}(M) \cdot U(M)).$$

(Members of $\mathrm{Sen}(M)$ are called senators.)

Lemma 1.3. (a) *Suppose that $h \in \mathrm{Sen}(M)$ and $x \in M$. If any of x, hx or xh lies in $M^\times$, then all three do.*

(b) *$M^* \cap \mathcal{Z}(M)$ and $\mathrm{Sen}(M)$ are normal in $M^\times$.*

Proof. (a) Since $\mathrm{Sen}(M) \subseteq M^\times$, this is obvious if $x \in M^\times$. Since θ above is clearly trivial on $M^* \cap \mathcal{Z}(M)$, it suffices to check the result for invertible h. Then, for hx a right intertwiner, $wx = x(hwh^{-1})^{hx}$; when xh is a right intertwiner, $wx = x(hw^{xh}h^{-1})$. Regularity is easily checked.

(b) Normality of $M^* \cap \mathcal{Z}(M)$ is obvious. $(M^\times, \mathrm{Sen}(M))$ is an ri pair since by (a) if $h \in U(M)$ and $x \in M^\times$, then $xh = h(h^{-1}xh)$ where by (a) $h^{-1}xh \in M^\times$. On the other hand, because $x^\# \in \mathrm{End}(M)$, it preserves $\mathrm{Sen}(M)$. So also $(\mathrm{Sen}(M), M^\times)$ is an ri pair. □

Given a monoid M, we shall be interested in the formation of the enveloping group of a right intertwining submonoid Σ. By definition, Σ is stable precisely when it admits a calculus of right fractions [17], or constitutes a right-denominator set relative to itself. The following is easily checked (cf. [33](3.1)). The last assertion follows because the enveloping group is a set of formal fractions st^{-1}; see [26] Example (i).

Lemma 1.4. *Let Σ be a stable submonoid of M.*

(a) *The enveloping group $G = \Sigma[\Sigma^{-1}]$of Σ may be constructed as $(\Sigma \times \Sigma)/\sim$ where $\sim$ is the symmetric and transitive closure of the relation*

$$(s,t) \sim (su, tu)$$

and

$$(s_1,t_1) \cdot (s_2,t_2) = (s_1(s_2)^{t_1}, t_2t_1).$$

The homomorphism $s \mapsto (s,1)$ embeds Σ in G, where $(s,t) = st^{-1}$.

(b) *G is naturally isomorphic to the subgroup generated by the image of Σ in the monoid $M[\Sigma^{-1}]$ constructed by right fractions. Here, $M[\Sigma^{-1}]$ is $(M \times \Sigma)/\sim$ where $\sim$ is the symmetric and transitive closure of the relation*

$$(r,s) \sim (ru, su) \qquad u \in \Sigma$$

and

$$(r_1,s_1) \cdot (r_2,s_2) = (r_1(r_2)^{s_1}, s_2s_1).$$

The homomorphism $r \mapsto (r,1)$ embeds Σ in $M[\Sigma^{-1}]$, where $(r,s) = rs^{-1}$.

[2]This is intended to sound like a mix of the words 'centre' and 'unit'.

(c) *The induced map on classifying spaces $B\Sigma \to BG$ is a homotopy equivalence.* □

In the sequel we find ourselves having to deal with derivations as well as the special case of homomorphisms. Thus the 'First Isomorphism Theorem' below is presented in the more general framework.

Lemma 1.5. *Let Σ be a normal submonoid of a monoid M.* (a) *M/Σ inherits a natural monoid structure from M.*

(b) *If also $M[\Sigma^{-1}]$ is the enveloping group of M, then $\Sigma[\Sigma^{-1}]$ is a normal subgroup of $M[\Sigma^{-1}]$ and the inclusion of M in $M[\Sigma^{-1}]$ induces a natural group isomorphism from M/Σ to $M[\Sigma^{-1}]/\Sigma[\Sigma^{-1}]$.*

(c) *Suppose that $d : M \to G$ is a derivation compatible with an action of M on a group G given by*

$$d(xy) = d(x)(d(y))^x.$$

If $d(\Sigma) = 1$, then d factors uniquely through a derivation $d' : M/\Sigma \to G$. Further, if $\Sigma = d^{-1}(1)$ and $M[\Sigma^{-1}]$ is the enveloping group of M, then d' is a bijection onto $\operatorname{Im} d$.

Proof. (a) We form an equivalence relation $\equiv$ on M by $x \equiv y$ if $x\Sigma \cap y\Sigma \neq \emptyset$. Its transitivity follows easily because Σ is stable. Moreover, because Σ is normal, this relation is a congruence, in that multiples of equivalent elements remain equivalent. This induces a monoid multiplication on the equivalence classes, and M/Σ coincides with $M/\equiv$.

(b) Normality is readily checked. From the data we have a commuting diagram

$$\begin{array}{ccccc} \Sigma & \rightarrowtail & M & \twoheadrightarrow & M/\Sigma \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma[\Sigma^{-1}] & \rightarrowtail & M[\Sigma^{-1}] & \twoheadrightarrow & M[\Sigma^{-1}]/\Sigma[\Sigma^{-1}] \end{array}$$

in which the rows are exact and the two left vertical arrows are injective. The projection from M to M/Σ is easily seen to factor uniquely through a homomorphism $\phi : M[\Sigma^{-1}] \to M/\Sigma$. Because the kernel of ϕ contains $\Sigma[\Sigma^{-1}]$, ϕ in turn factors through $M[\Sigma^{-1}]/\Sigma[\Sigma^{-1}]$. That the maps so constructed between M/Σ and $M[\Sigma^{-1}]/\Sigma[\Sigma^{-1}]$ are mutually inverse follows from the universal properties of $M[\Sigma^{-1}]$.

(c) First observe that d respects the congruence relation of (a) above. Thus d' is defined. For the second claim we use the group structure given by (b) to show that if $x, x' \in M$ have $d(x) = d(x')$ then in M/Σ $[x] = [x']$. Let $y \in M$ be such that $yx \in \Sigma$, that is, $[y] = [x]^{-1}$. Then in G

$$d(y) = ((d(x))^y)^{-1} = ((d(x))^{-1})^y.$$

This implies that

$$\begin{aligned} d'([x]^{-1}[x']) &= d(y)(d(x'))^y \\ &= ((d(x))^{-1}d(x'))^y \\ &= 1^y = 1, \end{aligned}$$

whereupon in M/Σ, $[x] = [x']$. □

2. Intertwiners for rings

The above considerations apply in particular to the multiplicative monoid of any ring S. (All our rings contain a multiplicative identity, 1.) By additivity, every right intertwiner is right regular. S^* consists of those elements with no right zerodivisor. When $S = M_n(R)$, S^* is easily seen (by consideration of action on standard basis vectors) to consist of those matrices that act monomorphically on R^n as column space. Sen(S) comprises invertible matrices multiplied by regular central scalars.

Lemma 2.1. *Let R be a ring in which every element is a sum of senators. Let s be a right regular element of R. Then $Rs \subseteq sR$ if and only if $U(R)s \subseteq sU(R)$.*

Proof. Sufficiency is easily checked. Necessity holds for any ring (in fact, for any monoid), because if α is invertible, then so too must be its homomorphic image $\theta(s)(\alpha)$. □

Recall that any element of a local ring may be written as a sum of (at most two) units, since for any nonunit r the element $1 + r$ is a unit.

Corollary 2.2. *If R is a local ring, and $s \in R$ is right regular, then $Rs \subseteq sR$ if and only if $U(R)s \subseteq sU(R)$.* □

We show below that this result holds more generally for any matrix ring over a local ring.

Lemma 2.3. *$a \in R^{\times}$ if and only if $aI_n \in (M_nR)^{\times}$.*

Proof. 'only if': Obvious.

'if': First note that any right zerodivisor b of a yields bI_n as a right zerodivisor of aI_n. So a must be right regular. For $r \in R$, $(rI_n)(aI_n) = (aI_n)B$ for some unique matrix B. Since a is right regular, $B = r^aI_n$ for some $r^a \in R$. Then $ra = ar^a$ as required. □

Lemma 2.4. *Let R be a local ring.* (a) *The following are equivalent.*

(i) *$A \in (M_nR)^{\times}$;*
(ii) *$(\mathrm{GL}_n R)A \subseteq A\,\mathrm{GL}_n R$, and A is right regular;*
(iii) *$(\mathrm{GL}_n R)A(R^n) \subseteq A(R^n)$, and A is right regular;*
(iv) *$A \in R^{\times} \cdot \mathrm{GL}_n R$.*

(b) *If also R is commutative, then each of the above assertions is equivalent to its left-handed counterpart, and so to the following.*

(i) *$(M_nR)A = A(M_nR)$, and A is regular;*
(ii) *$(\mathrm{GL}_n R)A = A\,\mathrm{GL}_n R$, and A is regular;*
(iii) *for any $G \in \mathrm{GL}_n R$, $GA(R^n) = A(R^n)$, and A is regular;*
(iv) *$A \in R^{*} \cdot \mathrm{GL}_n R$.*

Proof. We first discuss (a).

(i) ⇒ (ii): This is immediate from (2.2) above.

(ii) ⇒ (iii): This is obvious.

(iii) ⇒ (ii): Let $G \in \mathrm{GL}_n R$. By assumption, GA may be regarded as an R-module homomorphism from R^n to $A(R^n)$. Since there is a surjection $A : R^n \to A(R^n)$ and R^n is projective, there is a linear map G' from R^n to itself with $GA = AG'$. As A is right regular, G' is unique. From this uniqueness, the same argument applied to G^{-1} reveals G' to be invertible.

(ii) ⇒ (iv): Let E_{pq} be the standard matrix unit having as its unique nonzero entry 1 in the (p, q) position. Then E_{pq} is the sum of two invertible matrices, namely, for $p \neq q$,

$$E_{pq} = E_{pq}(1) + (-I_n),$$
$$E_{pp} = E_{qp}(-1)E_{pq}(1) + E_{pq}(-1)E_{qp}(1)E_{pq}(-1),$$

where $E_{pq}(r) = I_n + rE_{pq}$. So for each p, q we have that

$$E_{pq}A = AF_{pq}$$

where F_{pq} is also a sum of two invertible matrices. Since A is right regular, the matrices F_{pq} also form a set of matrix units for M_nR. Then by an application of the Krull-Schmidt Theorem to semiperfect rings [33](2.9.19), there is an invertible matrix A' with $F_{pq} = (A')^{-1}E_{pq}A'$. It follows that $A(A')^{-1}$ commutes with each E_{pq} and so is a (right regular) scalar matrix aI_n.

To see that $a \in R^\times$, observe that for any $\alpha \in U(R)$,

$$\alpha a A' = aA'G$$

for some invertible matrix G. Since a is right regular, it follows from the equation

$$\alpha a I_n = a(A'G(A')^{-1})$$

that $(A'G(A')^{-1})$ is an (invertible) scalar matrix, βI_n say. On substitution in this equation, we find that $\alpha a = a\beta$. As R is local we now obtain the result from (2.2).

(iv) $\Rightarrow$ (i): Each factor $R^\times$, $\mathrm{GL}_n R$ lies in the monoid $(M_nR)^\times$.

This completes the proof of (a). Likewise there are left-handed versions of each of the above (in the case of (iii) one has to consider instead the free left R-module ${}^nR$, on which M_nR acts on the right), and virtually the same arguments show them to be equivalent. Now suppose R to be commutative. Then $R^\times = R^*$ and condition (iv) becomes left-right symmetric; so its equivalence to (b)(i) and (ii) follows. Finally, (b)(iii) is an immediate consequence of (ii). □

Note that in statement (iv) of the above result, the scalar need not be central. Thus the assertion is weaker than the claim that $\mathrm{Sen}(M_nR) = (M_nR)^\times$, which, as the example below shows, may be false.

Example 2.5. Let $\mathcal{E}$ denote the set of all finite binary strings of positive length, and let R be the polynomial ring on indeterminates x_ϵ ($\epsilon \in \mathcal{E}$) over the field of two elements, subject to the relations

$$x_\epsilon x_0 = x_0 x_{\epsilon 0} \ (\epsilon \neq 0), \qquad x_\epsilon x_1 = x_1 x_{\epsilon 1} \ (\epsilon \neq 1).$$

Then the only right intertwiners in $R = M_1(R)$ are the products of x_0, x_1. We therefore have $x_1x_0 = x_0x_{10}$ with x_{10} failing to be a right intertwiner. The senate of R is trivial.

When the monoid M in Lemma 1.4 is the multiplicative monoid of a ring R, other constructions of the ring of right fractions, or localization, $R[\Sigma^{-1}]$ are available, e.g. [12](6.1). In particular, when $R^\times$ is a stable monoid of right intertwiners, we say that the ring R has *stable right intertwiners.* This is therefore the desirable case, in which the enveloping group of $R^\times$ is readily described. Commutative rings clearly have this property. On the other hand, the above example exhibits a local ring that does not have stable right intertwiners.

Lemma 2.6. *If R is local and has stable right intertwiners, then M_nR has stable right intertwiners.*

Proof. After (2.4), we consider the equation

$$bB'aA' = aA'C$$

where $a, b \in R^\times$ and $A', B' \in \mathrm{GL}_n R$, and seek to prove that $C \in R^\times \cdot \mathrm{GL}_n R$. Since aI_n is a right intertwiner, $B'a = aB''$ for some B'' that is invertible by (2.4). Now $ba = ab^a$ for some $b^a \in R^\times$, since R is assumed to have stable right intertwiners. Therefore $(A')^{-1}b^a = b^aA''$ with $A'' \in \mathrm{GL}_n R$ by (2.4) again, and $C = b^aA''B''A'$, as required. □

3. Commutative rings

Henceforth, let A be a commutative ring. Then $A^\times = A^*$ is just the monoid (under multiplication) of regular elements (non-zerodivisors) of A.

By the principal result of [16] for example, the monoid $(M_nA)^*$ of right regular matrices consists of all $n \times n$ matrices over A of regular determinant. By consideration of the transpose matrix, it follows easily that right regularity and left regularity of matrices coincide.

Let $A \cdot \mathrm{GL}_n A$ denote the monoid (under matrix multiplication) of all $n \times n$ scalar-times-invertible matrices. Clearly, such a matrix is regular if and only if the scalar is regular. Therefore Definition 1.2 here reduces to:

$$\mathrm{Sen}(M_nA) = (M_nA)^* \cap A \cdot \mathrm{GL}_n A = A^\times \cdot \mathrm{GL}_n A.$$

Definition 3.1. For any matrix S in M_nA, let $A\langle S\rangle$ be the A-ideal generated by the n^2 entries of S.

When $S \in \mathrm{GL}_n A$, this ideal reduces to A itself. Moreover, $S \in \mathrm{GL}_n A$ precisely when $A\langle \operatorname{adj} S\rangle = A(\det S)$, where $\operatorname{adj} S$ denotes the classical adjoint (of cofactors) of S.

Theorem 3.2. *If $S \in (M_nA)^*$, then the following are equivalent.*

(i) $A\langle S\rangle A\langle \operatorname{adj} S\rangle = A(\det S)$ (the 'ideal equation');

(ii) *S is locally intertwining, that is, for each maximal ideal $\mathfrak{m}$ of A, S is scalar-times-invertible,*
$$S \in A_{\mathfrak{m}}^\times \cdot \mathrm{GL}_n A_{\mathfrak{m}};$$

(iii) $(A\langle S\rangle)^n = A(\det S)$;

(iv) $(\mathrm{GL}_n A)S(A^n) \subseteq S(A^n)$;

(v) *S is a right unit-intertwiner, that is,*
$$(\mathrm{GL}_n A)S \subseteq S(\mathrm{GL}_n A);$$

(vi) *S is a right intertwiner, that is,*
$$(M_nA)S \subseteq S(M_nA);$$

(vii) *for each $G \in \mathrm{GL}_n A$, $GS(A^n) = S(A^n)$;*

(viii) *S is a unit-intertwiner, that is,*
$$(\mathrm{GL}_n A)S = S(\mathrm{GL}_n A);$$

(ix) *S is an intertwiner, that is, $(M_nA)S = S(M_nA)$.*

Proof. We establish the equivalences in groups of three.

(i) $\Rightarrow$ (ii): Since regularity of elements is preserved by localization, we may assume that A has unique maximal ideal $\mathfrak{m}$. Let $S = (s_{ij})$ have determinant d and (i,j)-cofactor c_{ij}. So for fixed i we have that $d = \sum_j s_{ij}c_{ij}$. Now from the ideal equation we may write $s_{ij}c_{ij} = df_{ij}$ for some $f_{ij} \in A$. Since d is regular we obtain $\sum_j f_{ij} = 1$, from which it follows that some $j = J$ has f_{iJ} outside $\mathfrak{m}$ and thus a unit of A. Since $d = s_{iJ}c_{iJ}f_{iJ}^{-1}$ is regular, so are s_{iJ}, c_{iJ}. Now for each p, q we have an element e_{pq} of A with

$$s_{pq}c_{iJ} = de_{pq} = s_{iJ}c_{iJ}f_{iJ}^{-1}e_{pq},$$

from which $s_{pq} = s_{iJ}f_{iJ}^{-1}e_{pq} \in A(s_{iJ})$. This makes $A\langle S\rangle = A(s_{iJ})$ and $S = s_{iJ}S'$ for some matrix S' with $A\langle S'\rangle = A$ (because s_{iJ} is regular). The ideal equation now reduces to

$$A(s_{iJ})A\left\langle s_{iJ}^{n-1} \operatorname{adj} S'\right\rangle = A(s_{iJ}^n \det S'),$$

whence $A\langle \operatorname{adj} S'\rangle = A(\det S')$ and $S' \in \mathrm{GL}_n A$ as required.

(ii) ⇒ (iii): The equation $(A_{\mathfrak{m}} \langle S \rangle)^n = A_{\mathfrak{m}}(\det S)$ evidently holds for each maximal ideal $\mathfrak{m}$ of A. Hence the global equality is also true.

(iii) ⇒ (i): Since for any matrix S

$$A(\det S) \subseteq A \langle S \rangle \, A \langle \operatorname{adj} S \rangle \subseteq (A \langle S \rangle)^n,$$

we obtain the ideal equation for S from the given equation.

Hence the first three assertions are equivalent. We now show their equivalence to the second triad of assertions.

(ii) ⇒ (iv): By (2.4)(a) for any invertible G, the module inclusion

$$S(A^n) \hookrightarrow S(A^n) + GS(A^n)$$

is locally an isomorphism. This makes it an isomorphism, leaving $GS(A^n) \subseteq S(A^n)$.

(iv) ⇒ (v): This is just the implication (ii) ⇒ (iii) of (2.4), which does not require the ring to be local.

(v) ⇒ (vi): We exploit the fact that any matrix in M_nA can be written as an A-linear combination of invertible matrices (already exhibited for the standard matrix units in (2.4)). Since A is commutative, scalars are central in M_nA, and the result follows.

(vi) ⇒ (i): For each standard matrix unit E_{pq} we have

$$(\operatorname{adj} S)E_{pq}S = (\det S) \cdot F_{pq}$$

for some $F_{pq} \in M_nA$. As the (i,j)-th entry of $(\operatorname{adj} S)E_{pq}S$ is $c_{pi}s_{qj}$, we deduce that $c_{pi}s_{qj} \in A(\det S)$ for all p, i, q, j, as required.

Thus each of the second triad of assertions is equivalent to the first three. This leaves the strengthened versions (vii) - (ix) of (iv) - (vi). For (vii), the inclusion of $GS(A^n)$ in $S(A^n)$ is locally an isomorphism by (2.4), and so an isomorphism, whence equality results. Now the first three assertions are left-right symmetric, so the left-handed counterparts of (v) and (vi) are also equivalent to these. Combination with (v) and (vi) clinches the equivalence of (viii) and (ix) to what we already have. □

In this theorem, the ideal equation (i) suggests a generalization to other elementary ideals. Let $I_k(S)$ be the ideal of A generated by the $k \times k$ minors of the matrix S. Thus, for $S \in M_nA$,

$$I_0(S) = A, \qquad I_1(S) = A \langle S \rangle,$$
$$I_{n-1}(S) = A \langle \operatorname{adj} S \rangle, \qquad I_n(S) = A(\det S).$$

Then (i) above is just the case $i = 1$ of the equation $I_i(S)I_{n-i}(S) = I_n(S)$. For other values of i, we apply (ii) to see that the equation holds at each localization of the ring A, and hence globally.

Corollary 3.3. *A matrix $S \in (M_nA)^*$ is an intertwiner if and only if*

$$I_i(S)I_{n-i}(S) = I_n(S)$$

whenever $0 \leq i \leq n$. □

It seems to be a delicate question as to which of the other cases imply the case $i = 1$.

With the above theorem as the key tool, we now look at the monoid structure of M_nA and its intertwiners.

Corollary 3.4. *The monoid M_nA has stable intertwiners.*

Proof. Suppose that $S, T \in (M_nA)^\times$ with $TS = ST^S$. Then since each localization of A has stable right intertwiners, it follows from (2.6) that T^S is locally scalar-times-invertible and hence a right intertwiner. □

We now introduce a highly useful item of notation.

Definition 3.5.

$$\operatorname{Int}_n A = (M_n A)^{\times},$$

the monoid of intertwining matrices in $M_n A$.

The first observation about this object can be read off immediately from the ideal equation in the theorem.

Corollary 3.6. $\operatorname{Int}_n A$ *is functorial in* A. □

Corollary 3.7. $\operatorname{Int}_n A$ *is*
(a) *a unitary submonoid of* $M_n A$, *and*
(b) *a quandle.*

Proof. (a) This means that a matrix is intertwining whenever its product with an intertwiner is an intertwiner. The theorem permits the claim to be tested locally. But then $\operatorname{Int}_n A = \operatorname{Sen}(M_n A)$ by (2.4), and the result follows from (1.3).

(b) Since left and right intertwiners coincide, combination of (3.4) with (3.2)(ix) shows that for any $S \in \operatorname{Int}_n A$, we have $(\operatorname{Int}_n A)S = S(\operatorname{Int}_n A)$; moreover, S is both left and right regular. Hence $S_{\#}$ and $S^{\#}$ are mutually inverse automorphisms of $\operatorname{Int}_n A$. □

Corollary 3.8. (a) *For* $n \geq 2$, $A^{\times}$, $A^{\times} \cdot \mathrm{SL}_n A$ *and* $A^{\times} \cdot \mathrm{GL}_n A$ *are normal submonoids of* $\operatorname{Int}_n A$.

(b) $A^{\times} \cdot E_n A$ *is a normal submonoid of* $\operatorname{Int}_n A$ *whenever* $E_n A$ *is a characteristic subgroup of* $\mathrm{SL}_n A$ *or* $\mathrm{GL}_n A$. *In particular, this holds for* $n \geq 3$.

Proof. (a) Normality of $A^{\times}$ and $A^{\times} \cdot \mathrm{GL}_n A$ follows from (1.3). For $A^{\times} \cdot \mathrm{SL}_n A$, observe that if $S \in \operatorname{Int}_n A$, then the automorphism $S^{\#}$ preserves determinants.

(b) In view of the preceding corollary, this is just an application of (1.3)(h). When $n \geq 3$, $E_n A$ is a characteristic subgroup of $\mathrm{GL}_n A$ by [34]. □

Another straightforward local argument gives the following.

Corollary 3.9. $\operatorname{Int}_n A$ *is invariant under passage to adjoints.* □

We note that there is a more concise description of the generators of $A\langle S\rangle$. For in (vii) above, we may take G to be any permutation matrix. It follows that for any $h, i \in \{1, 2, ..., n\}$ the ideals $A(s_{h1}, ..., s_{hn})$ and $A(s_{i1}, ..., s_{in})$ coincide. Since also the map from A^n to $S(A^n)$ is a monomorphism when S is regular, we obtain the following.

Corollary 3.10. *If* $S \in \operatorname{Int}_n A$, *then*
(i) *for any row* $(s_{i1} \cdots s_{in})$ *of* S,

$$A\langle S\rangle = A(s_{i1}, ..., s_{in}),$$

and
(ii) $\bigoplus_n A\langle S\rangle \cong A^n$. □

Conditions (i) and (iii) of the above theorem suggest the condition $A\langle \operatorname{adj} S\rangle = (A\langle S\rangle)^{n-1}$. Indeed its necessity is easily checked locally from (ii) above, so that it holds for any intertwining matrix S. However, the condition is not sufficient, since when $n = 2$ it holds for any matrix S.

Condition (ix) suggests the condition $(E_n A)S = S(E_n A)$. In [11] this is taken to be the definition of *bireducibility* of S, and the following is proven in [11](3.1),(3.3).

Theorem 3.11. *Every regular bireducible matrix is an intertwiner. The converse holds under the following conditions.*
(a) $n \geq 3$ *or*
(b) $n = 2$ *and at least one of the following holds:*

(i) $E_2(A) = \mathrm{SL}_2(A)$ *or a characteristic subgroup of* $\mathrm{SL}_2(A)$ *(such as its commutator subgroup);*
(ii) A *is an* $\mathbb{R}$*-algebra such that for all* $a \in A$*, there exists* $\epsilon > 0$ *in* $\mathbb{R}$ *for which* $1 + xa$ *is a unit in* A *whenever* $|x| < \epsilon$*;*
(iii) $E_2(A) = \mathrm{SL}_2(A) \cap E_4(A)$.

As an illustration of the subtleties for the case $n = 2$, here is one of the examples constructed in [11], of intertwining matrices which are not bireducible.

Example 3.12. Let F be a field and $A = F[x, y]$. Then

$$\begin{bmatrix} 1 - xy & -x^2 \\ y^2 & 1 + xy \end{bmatrix}$$

is intertwining (indeed invertible), but not bireducible.

4. Intertwiners for projective modules

In this section, P will be a finitely generated projective module over a commutative ring A. We claim that the Characterization Theorem 3.2 generalizes to this situation, although the meaning of some of the terms there is not obvious.

Our first task is to show that Lemma 2.1 applies to the ring $\mathrm{End}_A(P)$ of A-module endomorphisms of P.

Lemma 4.1. *Let A be a commutative ring and P be a finitely generated projective A-module. Then $\mathrm{End}_A(P)$ is generated as an A-module by A-automorphisms of P.*

Proof. Let $A\,\mathrm{Aut}(P)$ be the A-submodule, indeed subring, of $\mathrm{End}_A(P)$ generated by all the A-automorphisms of P. To show that the module inclusion of $A\,\mathrm{Aut}(P)$ in $\mathrm{End}_A(P)$ is an isomorphism, we prove that for each maximal ideal $\mathfrak{m}$ of A, with complement the multiplicative set Σ,

$$A_{\mathfrak{m}}\,\mathrm{Aut}(P) = \Sigma^{-1}\,\mathrm{End}_A(P) \cong \mathrm{End}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}}) \cong M_k(A_{\mathfrak{m}})$$

for some k. These isomorphisms exploit the natural isomorphism of [3](III.4.5) (which applies because P is finitely presented), and the fact that P is locally free. The following argument was kindly suggested by W. van der Kallen.

With $k = 1$, both sides reduce to $A_{\mathfrak{m}}$. For higher k, as shown in the proof of (2.4), $M_k(A_{\mathfrak{m}})$ is equal to the subring generated by its nilpotents. Now if $r^{-1}\alpha \in \Sigma^{-1}\,\mathrm{End}_A(P)$ is nilpotent, then, for some $s \in \Sigma$ and $n \in \mathbb{N}$, $\alpha^n s = 0$. Since Σ is central in $\mathrm{End}_A(P)$, αs is nilpotent in $\mathrm{End}_A(P)$ and we can write $r^{-1}\alpha = (rs)^{-1}(\alpha s)$. Then the rewriting $\alpha s = 1 + (\alpha s - 1) \in A\,\mathrm{Aut}(P)$ gives the result. □

Of course an endomorphism of P in general fails to give rise to a matrix in the way that one may associate a matrix to an endomorphism of a finitely generated free module (for a given choice of basis). If a basis for a free module is not selected, then the matrix S of an endomorphism is determined only up to pre- and post-multiplication by invertible matrices. Since evidently $A\langle SB\rangle \subseteq A\langle S\rangle \cap A\langle B\rangle$, such multiplication does not alter the ideal $A\langle S\rangle$ of (3.2). (We may similarly speak of $A\langle \mathrm{adj}\, S\rangle$ and $A(\det S)$ for an endomorphism S of a free module.) This suggests that, even though the matrix of an endomorphism S of P need not be well-defined (even after choice of a minimal generating set for P), it may be possible to define the corresponding ideal $A\langle S\rangle$. The next lemma accomplishes this.

Lemma 4.2. *Let P be a finitely generated projective module with embedding $\iota : P \to A^n$ having left inverse $\pi : A^n \to P$. Then for any A-endomorphism S of P the ideal $A\langle S\rangle$ defined by*

$$A\langle S\rangle = A\langle \iota S\pi\rangle$$

is independent of the choice of n, ι *and* π, *and in particular coincides with the earlier definition of* $A\langle S\rangle$ *when* P *is a finitely generated free module. Similarly,* $A\langle \operatorname{adj} S\rangle$ *is well-defined as* $A\langle \iota \circ \operatorname{adj} S \circ \pi\rangle$, *while the principal ideal* $A(\det S)$ *is well-defined as* $A(\det(\iota S\pi \oplus \mathrm{id}_{\operatorname{Ker}\pi}))$.

Proof. Here is the argument for $A\langle S\rangle$. First we consider the case when P is free. Exploiting the invariance of the ideal $A\langle S\rangle$ with respect to composition with isomorphisms, we can reduce to the case where the given basis for P has its image as the first k elements of the standard basis for A^n. Then the matrix corresponding to $\iota S\pi$ is just the direct sum of the matrix for S with the zero matrix. This gives the result.

Now for the general case, consider any two choices n, ι, π and n', ι', π'. Then upon localization at any maximal ideal $\mathfrak{m}$ the module $P_{\mathfrak{m}}$ is free, so the previous case shows that the localized endomorphisms satisfy

$$A_{\mathfrak{m}}\langle \iota S\pi\rangle = A_{\mathfrak{m}}\langle S\rangle = A_{\mathfrak{m}}\langle \iota' S\pi'\rangle .$$

Thus $A\langle \iota S\pi\rangle = A\langle \iota' S\pi'\rangle$ too.

Clearly the above argument also works for $A\langle \operatorname{adj} S\rangle$. For $A(\det S)$, the treatment is similar, relying on the fact that the determinant of an endomorphism of a free module is unchanged by taking its direct sum with an identity map (*cf.* [27](3.2)). □

Theorem 4.3. *Let* A *be a commutative ring and* P *a finitely generated projective* A*-module.*

(a) *The following are equivalent assertions about* $S \in \operatorname{End}_A(P)$.

(i) $S \in \operatorname{Int}(P)$, *defined as* $(\operatorname{End}_A(P))^{\times}$;
(ii) *for each maximal ideal* $\mathfrak{m}$ *of* A, $S \in \operatorname{Int}(P_{\mathfrak{m}}) \cong \operatorname{Int}_k(A_{\mathfrak{m}})$ *for some* k *(equivalently,* S *is regular scalar-times-invertible);*
(iii) $\operatorname{Aut}_A(P)S(P) \subseteq S(P)$, *and* S *is right regular;*
(iv) $\operatorname{Aut}_A(P)S \subseteq S\operatorname{Aut}_A(P)$, *and* S *is right regular.*

(b) *Each of the above is equivalent to its left-handed counterpart, and to the following.*

(i) $A\langle S\rangle A\langle \operatorname{adj} S\rangle = A(\det S)$;
(ii) $(A\langle S\rangle)^n = A(\det S)$.

Proof. We start with (a). It is easy to see that regularity is a local property.

(i) $\Longrightarrow$(ii): Exploiting the isomorphism between $\operatorname{End}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}})$ and $\operatorname{End}_A(P)_{\mathfrak{m}}$, we can find, for each $T \in \operatorname{End}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}})$, an element $s \in A - \mathfrak{m}$ such that $sT \in \operatorname{End}_A(P)$. Then by (i), $sTS = S(sT)^S$, where $(sT)^S \in \operatorname{End}_A(P)$. It follows that $T^S = s^{-1}(sT)^S \in \operatorname{End}_{A_{\mathfrak{m}}}(P_{\mathfrak{m}})$.

(ii) $\Longrightarrow$ (iii): If G is an automorphism of P, then it induces an automorphism of $P_{\mathfrak{m}}$. So by (ii) and (2.4) $GS(P_{\mathfrak{m}}) \subseteq S(P_{\mathfrak{m}})$. Hence $GS(P) \subseteq S(P)$.

(iii) $\Longrightarrow$ (iv): This is the argument of (2.4)(a)((iii)$\Rightarrow$(ii)) with A^n replaced by P.

(iv) $\Longrightarrow$ (i): This follows by combination of (2.1) and the above lemma, since the elements of $A\operatorname{Aut}(P)$ are precisely the sums of senators of $\operatorname{End}_A(P)$.

Likewise, the left-handed counterparts of (i) to (iv) are equivalent to each other. However, by (2.4), the above assertion (ii) is equivalent to its left-handed counterpart. So the initial assertion of (b) follows. Finally, (b)(i) and (ii) can be verified locally. □

II Groups of intertwining matrices

5. Definitions of associated groups

Since by (3.4) $\operatorname{Int}_n A$ consists of stable right intertwiners, (1.4) applies. Thus the enveloping group of $\operatorname{Int}_n A$ is formed by constructing fractions. Now for any submonoid of $(M_nA)^*$ containing $A^\times$ and invariant under the adjoint operation, inversion of elements is equivalent to inversion of scalar matrices, since then $S^{-1} = (\operatorname{adj} S)(\det S)^{-1}$. In particular, by (3.9), this applies to $\operatorname{Int}_n A$. We therefore draw the following conclusion.

Proposition 5.1. *The enveloping group of* $\operatorname{Int}_n A$ *is the group* $(A^\times)^{-1}\operatorname{Int}_n A$ *obtained by constructing fractions as in (1.4). The embedding* $\operatorname{Int}_n A \to (A^\times)^{-1}\operatorname{Int}_n A$ *induces a homotopy equivalence of classifying spaces.* □

We may also remark that, by [11] (2.5), a similar assertion holds for bireducible matrices.

Now observe that the determinant (monoid) homomorphism from $\operatorname{Int}_n A$ to $A^\times$ extends to a group homomorphism

$$\det : (A^\times)^{-1}\operatorname{Int}_n A \to (A^\times)^{-1}A^\times$$

with kernel $\operatorname{SL}_n A$. Thus the group $((A^\times)^{-1}\operatorname{Int}_n A)/\operatorname{SL}_n A$ is abelian, as are its quotients. In particular,

$$((A^\times)^{-1}\operatorname{Int}_n A)/((A^\times)^{-1}A^\times \cdot \operatorname{SL}_n A) \cong \operatorname{Int}_n A/(A^\times \cdot \operatorname{SL}_n A)$$

is abelian (the isomorphism is from (1.5), (3.8)). (Alternatively, one can prove directly from (3.2) that $ST(\operatorname{adj} S)(\operatorname{adj} T) \in A^\times{\cdot}\operatorname{SL}_n A$ whenever S, T are intertwiners.) Further, $((A^\times)^{-1}\operatorname{Int}_n A)/\operatorname{GL}_n A$ is abelian. For the next result we write μ_n for the group of nth roots of unity in A; also, $\otimes I_n$ denotes the map which assigns to an element of A its corresponding scalar matrix in M_nA.

Proposition 5.2. *There is an exact sequence of abelian groups*

$$\mu_n \hookrightarrow U(A) \overset{\otimes I_n}{\to} GL_nA/\operatorname{SL}_n A \to \operatorname{Int}_n A/(A^\times \cdot \operatorname{SL}_n A) \twoheadrightarrow (\operatorname{Int}_n A)/(A^\times \cdot \operatorname{GL}_n A).$$

Proof. Only the kernel of the epimorphism is less than obvious. From the above isomorphism, the kernel is $((A^\times)^{-1}A^\times \cdot \operatorname{GL}_n A)/((A^\times)^{-1}A^\times \cdot \operatorname{SL}_n A)$, which is in turn isomorphic to $\operatorname{GL}_n A/(((A^\times)^{-1}A^\times \cap \operatorname{GL}_n A)\cdot \operatorname{SL}_n A)$. Since $(A^\times)^{-1}A^\times \cap \operatorname{GL}_n A$ comprises the invertible scalar matrices, this quotient is just the cokernel of $\otimes I_n$, as required. □

A similar argument proves the following. Its hypotheses are known to hold whenever $n \geq 3$ as well as in certain cases with $n = 2$ [34]. They guarantee that E_nA is normal in $\operatorname{Int}_n A$. We also use the fact that $\operatorname{SL}_n A \cap A^\times = \mu_n \leq E_nA$.

Proposition 5.3. *Suppose that* E_nA *is a characteristic subgroup of* $\operatorname{SL}_n A$ *or* $\operatorname{GL}_n A$*. Then there are short exact sequences of groups*

$$\operatorname{SL}_n A/E_nA \rightarrowtail \operatorname{Int}_n A/(A^\times \cdot E_nA) \twoheadrightarrow \operatorname{Int}_n A/(A^\times \cdot \operatorname{SL}_n A),$$
$$\operatorname{GL}_n A/(E_nA \cdot U(A)) \rightarrowtail \operatorname{Int}_n A/(A^\times \cdot E_nA) \twoheadrightarrow \operatorname{Int}_n A/(A^\times \cdot \operatorname{GL}_n A).$$

□

Proposition 5.4. *Suppose that* $n > \max(2, \operatorname{sr} A)$*. Then the maximum perfect subgroup* $\mathcal{P}((A^\times)^{-1}\operatorname{Int}_n A)$ *of the group* $(A^\times)^{-1}\operatorname{Int}_n A$ *is its second derived subgroup* $E_nA = ((A^\times)^{-1}\operatorname{Int}_n A)^{(2)}$.

Proof. The fact that $n > 2$ ensures that E_nA is characteristic in $\mathrm{GL}_n A$. and so normal in $(A^\times)^{-1}\,\mathrm{Int}_n A$, and also that E_nA is perfect, hence contained in $\mathcal{P}((A^\times)^{-1}\,\mathrm{Int}_n A)$. In the other direction, because $n > \mathrm{sr}\,A$, the group $\mathrm{SL}_n A/E_nA$ is abelian, making $(A^\times)^{-1}\,\mathrm{Int}_n A/E_nA$ metabelian. Thus

$$\mathcal{P}((A^\times)^{-1}\,\mathrm{Int}_n A) \le ((A^\times)^{-1}\,\mathrm{Int}_n A)^{(2)} \le E_nA.$$

The result follows. □

For further information about the groups involved here, see Corollary 12.6 below.

6. Relation to Picard groups

To relate intertwining matrices to Picard groups, we adopt some language and results of Bass' book [3]. As there, let $\mathbb{P}\mathrm{ic}\,A$ denote the category of all invertible A-modules and their A-homomorphisms, equipped with the product $\otimes_A$. For any cofinal, product-preserving functor $F : \mathbb{P}\mathrm{ic}\,A \to \mathbb{P}\mathrm{ic}\,A$, from [3] VII (Theorem 5.3) there is an exact sequence of abelian groups

$$K_1(F) \to K_1(\mathbb{P}\mathrm{ic}\,A) \to K_1(\mathbb{P}\mathrm{ic}\,A) \to K_0(F) \to K_0(\mathbb{P}\mathrm{ic}\,A) \to K_0(\mathbb{P}\mathrm{ic}\,A).$$

(Here, as noted in [3] p.460, F is necessarily E-surjective, so that the conditions of that theorem are fulfilled.) Moreover, elements of the relative K_0-group $K_0(F)$ consist of isomorphism classes of triples of the form $[M', \alpha, M]_F$ where M, M' are invertible A-modules and $\alpha : FM' \to FM$ is an A-module isomorphism. In particular, when α takes the form $F\alpha'$ for some isomorphism $\alpha' : M' \to M$ the triple $[M', \alpha, M]_F$ represents the isomorphism class of the zero element $[M, \mathrm{id}, M]_F$. Addition of classes obeys the following rules:

$$[M' \otimes_A N', \alpha \otimes \beta, M \otimes_A N]_F = [M', \alpha, M]_F + [N', \beta, N]_F \tag{6-1}$$

$$[M'', \alpha\beta, M]_F = [M', \alpha, M]_F + [M'', \beta, M']_F \tag{6-2}$$

In the particular case of interest, F is the nth power functor, sending an invertible A-module M to its nth tensor power. Here we write $\mathrm{Pic}(n, A)$ for $K_0(F)$ and $[M', \alpha, M]_\otimes$ for a typical element. Then as in [3] IX (Theorem 3.3) we obtain from the above (with first term omitted) the exact sequence

$$U(A) \overset{(\ )^n}{\to} U(A) \to \mathrm{Pic}(n, A) \to \mathrm{Pic}\,A \overset{\cdot n}{\to} \mathrm{Pic}\,A. \tag{6-3}$$

Here the first map sends a unit u to u^n, the second takes v to $[A, \cdot v, A]_\otimes$, the third (which we find convenient to take as the negative of Bass'), maps $[M', \alpha, M]_\otimes$ to $[M]_\otimes - [M']_\otimes \in \mathrm{Pic}\,A$, and the last is multiplication by n. We shall write ${}_n\,\mathrm{Pic}\,A$ for the kernel of this last homomorphism. In the case $n = 2$, the group $\mathrm{Pic}(2, A)$ has been studied as the group $\mathrm{Discr}(A)$ of isomorphism/isometry classes of discriminant modules of A [5].

Now observe that each matrix S in M_nA determines a triple $(A, \cdot(\det S), A\langle S\rangle)$. In general the ideal $A\langle S\rangle$ need not be invertible. The example $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ over $\mathbb{Z}$ shows that the condition that the fractional ideal $A\langle S\rangle$ is invertible, or even principal, does not imply that S is an intertwiner. Moreover, the A-homomorphism $\cdot(\det S) : A \to (A\langle S\rangle)^n$ need not be an isomorphism. It is a monomorphism precisely when $\det S$ is regular, and when it is an isomorphism $A\langle S\rangle$ must be invertible. From (iii) of (3.2), we thus immediately have the following.

Lemma 6.1. *For $S \in M_nA$, the triple $(A, \cdot(\det S), A\langle S\rangle)$ represents an element of $\mathrm{Pic}(n, A)$ if and only if S is an intertwiner.* □

Lemma 6.2. *For any $S, T \in \mathrm{Int}_n A$, we have*

$$A\langle ST\rangle = A\langle S\rangle\, A\langle T\rangle \cong A\langle S\rangle \otimes_A A\langle T\rangle\,.$$

Proof. The isomorphism is a consequence of the flatness of the ideals $A\langle S\rangle, A\langle T\rangle$. Since it suffices to prove the equality after localizing with respect to any maximal ideal of A, we may assume that A is a local ring. Then by (2.4), $A^\times \cdot \mathrm{GL}_n A = \mathrm{Int}_n A$. However, the equality clearly holds in $A^\times \cdot \mathrm{GL}_n A$ since for $S \in \mathrm{GL}_n A$, $A\langle S\rangle = A$. □

These two lemmas, and (6-1) above, allow the definition of a homomorphism

$$H : \mathrm{Int}_n A \to \mathrm{Pic}(n, A), \qquad S \mapsto [A, \cdot(\det S), A\langle S\rangle]_\otimes .$$

Evidently $H(S)$ is trivial when S has determinant 1. When S is a scalar matrix, the isomorphism $\cdot(\det S)$ is already the nth power of an isomorphism from A to $A\langle S\rangle$; so again $H(S)$ is trivial. Thus by (1.5)(c) H factors through the group $(\mathrm{Int}_n A)/(A^\times \cdot \mathrm{SL}_n A)$. We also use the notation H for homomorphisms induced on quotients of $\mathrm{Int}_n A$, and to quotients of $\mathrm{Pic}(n, A)$. In particular, we have $H : \mathrm{Int}_n A \to {}_n\mathrm{Pic}\, A$ given by $S \mapsto [A\langle S\rangle]_\otimes$. To determine its kernel, we recall that ${}_n\mathrm{Pic}\, A$ comprises differences of isomorphism classes of invertible modules whose nth tensor power is trivial. Now if S is invertible, $A\langle S\rangle = A$, while for $S \in A^\times$, $A\langle S\rangle \cong A$. Thus $A^\times \cdot \mathrm{GL}_n A$ is contained in the kernel. Conversely, an A-isomorphism $\phi : A \to A\langle S\rangle$ makes $A\langle S\rangle = A(\phi(1))$, so that $S = sS'$ where $s = \phi(1)$ is regular. By regularity of s, we have $A\langle S'\rangle = A$. Expansion of the equation $(A\langle S\rangle)^n = A(\det S)$ now reveals that $\det(S') \in U(A)$ and so $S' \in \mathrm{GL}_n A$. Thus the kernel is precisely $A^\times \cdot \mathrm{GL}_n A$. So in the diagram below the last vertical map is injective. A diagram chase now yields the injectivity of the other map H below, and our conclusions may be summarized as follows.

Theorem 6.3. *There is a commuting diagram*

$$\text{(6-4)} \qquad \begin{array}{ccccccc} \mu_n & \hookrightarrow & U(A) & \stackrel{\otimes I_n}{\to} & \mathrm{GL}_n A/\mathrm{SL}_n A & \to \\ \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{id}} & & \cong \downarrow^{\det} & \\ \mu_n & \hookrightarrow & U(A) & \stackrel{(\ )^n}{\to} & U(A) & \to \end{array}$$

$$\begin{array}{ccccc} \to & (\mathrm{Int}_n A)/(A^\times \cdot \mathrm{SL}_n A) & \twoheadrightarrow & (\mathrm{Int}_n A)/(A^\times \cdot \mathrm{GL}_n A) \\ & \downarrow^{H} & & \downarrow^{H} \\ \to & \mathrm{Pic}(n, A) & \twoheadrightarrow & {}_n\mathrm{Pic}\, A \end{array}$$

in which the rows are exact and all vertical arrows are injective. □

It follows from the Snake Lemma that the maps H in the above diagram have isomorphic cokernels. Here is an example in which the cokernels are nontrivial.

Example 6.4. Fix $n > 1$, and choose $k > n^2$. Over a field F, let A be the polynomial ring in $k + 1 + \binom{k+n-1}{n}$ variables $x_1, ..., x_k, d, x_{i_1\cdots i_n}$ $(1 \le i_1 \le \cdots \le i_n \le k)$, subject to the $\binom{k+n-1}{n} + 1$ relations

$$x_{i_1} \cdots x_{i_n} = d x_{i_1\cdots i_n}, \qquad \sum x_{i_1\cdots i_n} = 1.$$

Then the ideal I generated by $x_1, ..., x_k$ has $I^n = A(d) \cong A$, making $[I]_\otimes$ an element of ${}_n\mathrm{Pic}\, A$. However, I cannot be isomorphic to any ideal $A\langle S\rangle$ since it requires k generators, while any $A\langle S\rangle$ needs at most n^2 generators.

Let $\mathrm{Prin}\, A$ be the group $(A^\times)^{-1}A^\times/U(A)$ of principal fractional ideals of A, isomorphic to $(A^\times)^{-1}A^\times \mathrm{GL}_n A/\mathrm{GL}_n A$. The following is easily checked.

Lemma 6.5. *There is a commuting diagram*

$$\begin{array}{ccccc} \mathrm{GL}_n A & \rightarrowtail & (A^\times)^{-1}A^\times \cdot \mathrm{GL}_n A & \twoheadrightarrow & \mathrm{Prin}\, A \\ \downarrow^{=} & & \downarrow & & \downarrow \\ \mathrm{GL}_n A & \rightarrowtail & (A^\times)^{-1}\,\mathrm{Int}_n A & \twoheadrightarrow & (A^\times)^{-1}\,\mathrm{Int}_n A/\,\mathrm{GL}_n A \\ & & \downarrow & & \downarrow \\ & & \mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A) & = & \mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A) \end{array}$$

whose rows and columns are short exact sequences, and in which the top right square is bicartesian. □

In order to relate to homotopy groups of $B\,\mathrm{GL}_n A^+$ and thereby unstable K-theory, we now pass to classifying spaces. The above diagram gives a commuting diagram of fibrations. Since the rightmost groups above are all abelian, the fibrations must be plus-constructive [6], [8]. This gives the following.

Proposition 6.6. *There is a commuting diagram of fibrations*

$$\begin{array}{ccccc} B\,\mathrm{GL}_n A^+ & \to & B(A^\times \cdot \mathrm{GL}_n A)^+ & \to & B\,\mathrm{Prin}\, A \\ \downarrow^{=} & & \downarrow & & \downarrow \\ B\,\mathrm{GL}_n A^+ & \to & B\,\mathrm{Int}_n A^+ & \to & B((A^\times)^{-1}\,\mathrm{Int}_n A/\,\mathrm{GL}_n A) \\ & & \downarrow & & \downarrow \\ & & B(\mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A)) & = & B(\mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A)) \end{array}$$

□

Corollary 6.7. *Inclusion induces a natural map*

$$\pi_i(B\,\mathrm{GL}_n A^+) \to \pi_i(B\,\mathrm{Int}_n A^+)$$

which is an isomorphism for $i \geq 2$. For $i = 1$, this map is a monomorphism, and its abelian cokernel is given by a group extension

$$\mathrm{Prin}\, A \rightarrowtail \mathrm{Cokernel} \twoheadrightarrow \mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A),$$

whose quotient group is naturally isomorphic to a subgroup of $\mathrm{Pic}\, A$ *of exponent n.* □

To study this cokernel further, we relate it to the group $\mathrm{Cart}A$ of Cartier divisors of A (the group of fractional ideals of $A[(A^\times)^{-1}]$). Recall from [3]p.143 that there is a short exact sequence

$$\mathrm{Prin}\, A \rightarrowtail \mathrm{Cart}A \twoheadrightarrow \mathrm{Pic}\, A\,.$$

Now the map $(A^\times)^{-1}\,\mathrm{Int}_n A \to \mathrm{Cart}A$ sending $s^{-1}S$ to the fractional ideal $s^{-1}A\langle S\rangle$ is clearly a homomorphism, and trivial on $\mathrm{GL}_n A$. Thus, writing $\mathrm{Cart}_n A$ for the inverse image in $\mathrm{Cart}A$ of ${}_n\,\mathrm{Pic}\, A$ gives rise (by the preceding theorem) to a monomorphism of exact sequences

$$\begin{array}{ccccc} \mathrm{Prin}\, A & \rightarrowtail & (A^\times)^{-1}\,\mathrm{Int}_n A/\,\mathrm{GL}_n A & \twoheadrightarrow & \mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A) \\ \downarrow^{\mathrm{id}} & & \downarrow & & \downarrow^{H} \\ \mathrm{Prin}\, A & \rightarrowtail & \mathrm{Cart}_n A & \twoheadrightarrow & {}_n\,\mathrm{Pic}\, A\,. \end{array} \tag{6-5}$$

Observe that when A is a domain with quotient field K, the group $\mathrm{Prin}\, A$ becomes the group of divisibility $U(K)/U(A)$, and so $(\mathrm{Prin}\, A)/n$ reduces to the group $U(K)/(U(A)(U(K))^n)$.

As an exercise for the reader, we indicate a strengthening of [37](IV Ex.9.2)[3]. Its final observation is by combination with Proposition 5.2.

[3]We are grateful to C. A. Weibel for bringing his exercise to our attention.

Corollary 6.8. (a) *For any commutative ring A, the assignment $S \mapsto \det S$ is a homomorphism* det *from* $\mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A)$ *to* $(\mathrm{Prin}\, A)/n$.

(b) *Let A be a domain. Then this homomorphism factorizes as θH, where $\theta : {}_n \mathrm{Pic}\, A \to (\mathrm{Prin}\, A)/n$ is the Snake Lemma connecting homomorphism of the commuting diagram*

$$\begin{array}{ccccc} \mathrm{Prin}\, A & \rightarrowtail & \mathrm{Cart} A & \twoheadrightarrow & \mathrm{Pic}\, A \\ \downarrow^{\cdot n} & & \downarrow^{\cdot n} & & \downarrow^{\cdot n} \\ \mathrm{Prin}\, A & \rightarrowtail & \mathrm{Cart} A & \twoheadrightarrow & \mathrm{Pic}\, A. \end{array}$$

(c) *Suppose that A is a noetherian domain such that for every maximal ideal $\mathfrak{m}$ the localization $A_{\mathfrak{m}}$ is factorial (for example, a Krull ring). Then by* [3](III, 7.17) Cart$A$ *is free abelian, so that:*

(i) *the map of exact sequences (6-5) is a monomorphism of free abelian group presentations;*
(ii) *θ is a monomorphism; and*
(iii) *there is a natural exact sequence of abelian groups*

$$\mu_n \hookrightarrow U(A) \overset{\otimes I_n}{\to} GL_n A/\mathrm{SL}_n A \to \mathrm{Int}_n A/(A^\times \cdot \mathrm{SL}_n A) \overset{\det}{\to}$$
$$\overset{\det}{\to} U(K)/(U(A)(U(K))^n) \to (\mathrm{Cart} A)/n \twoheadrightarrow (\mathrm{Pic}\, A)/n\ .$$

□

In Theorem 9.1 below, we show that the middle term of the last sequence above is isomorphic to $\mathrm{Pic}(n, A)$ when A is a domain of dimension 1. The behaviour of the fundamental groups in Corollary 6.7 above is studied further in Corollary 12.7 below.

7. A variant of Bass' Rosenberg-Zelinsky sequence

We now interpret another very natural exact sequence arising from the inclusion of $\mathrm{GL}_n A$ in $\mathrm{Int}_n A$ as a version of the sequence constructed by Rosenberg and Zelinsky [31]. This version will be seen to contain more precise information about the orders of elements constructed in $\mathrm{Pic}\, A$.

Definition 7.1. We introduce the notation $\mathrm{PInt}_n A$ for the projective group of intertwining matrices

$$\mathrm{PInt}_n A = (\mathrm{Int}_n A)/A^\times \cong ((A^\times)^{-1} \mathrm{Int}_n A)/((A^\times)^{-1} A^\times).$$

In the following, for an abelian group P, $\mathrm{Tor}_n P$ denotes the subgroup of those elements having order dividing a power of n.

Theorem 7.2. *There is a commuting diagram*

$$\begin{array}{ccccccc} U(A) & \overset{\otimes I_n}{\rightarrowtail} & \mathrm{GL}_n A & \to & \mathrm{PInt}_n A & \twoheadrightarrow & (\mathrm{Int}_n A)/(A^\times \cdot \mathrm{GL}_n A) \\ \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{End}} & & \downarrow^{H} \\ U(A) & \overset{\otimes I_n}{\rightarrowtail} & \mathrm{GL}_n A & \overset{\mathrm{End}}{\to} & \mathrm{Aut}(M_n A) & \overset{\varphi}{\to} & \mathrm{Tor}_n \mathrm{Pic}\, A \end{array} \tag{7-6}$$

in which the rows are exact and all vertical arrows are injective.

Proof. Exactness of the upper row follows from our work above, while the lower row is just the Rosenberg-Zelinsky sequence for the Azumaya algebra $M_n A$ over A, as described in [2](III.4.6), (III.4.7). Our effort is therefore directed at constructing the vertical arrow labelled End, and verifying that it fits into the claimed commuting squares.

Recall from Theorem 3.2 that any $S \in \operatorname{Int}_n A$ is both a right- and left-intertwiner. So by Section 1 above, there is a covariant endomorphism functor End sending S to $S_\# : M_nA \to M_nA$, with

$$(S_\# N)S = SN$$

for each $N \in M_nA$. Indeed, $S_\#$ is an automorphism with inverse $S^\#$. When S is a regular scalar matrix then these automorphisms are clearly the identity, and when S is invertible they coincide with conjugation by S and by S^{-1}. We do therefore have the claimed factorizations of conjugation-induced $\operatorname{End} : \operatorname{GL}_n A \to \operatorname{Aut}(M_nA)$ through $\operatorname{Int}_n A$ and further through $\operatorname{PInt}_n A$.

To examine commutativity of the rightmost square, we begin with $S \in \operatorname{Int}_n A$ and compare its two images in $\operatorname{Pic}(M_nA)$, exploiting the canonical isomorphism with $\operatorname{Pic} A$. The map $\varphi = \varphi_{M_nA}$ sends the automorphism $S_\#$ to the twisted invertible M_nA-bimodule, conventionally denoted ${}_1(M_nA)_{S_\#}$, whose actions are given by the formula

$$L \cdot M \cdot N = LM(S_\# N).$$

This bimodule is canonically isomorphic to the bimodule $(M_nA)S$ via the homomorphism $M \mapsto MS$. The actions are preserved under this map, since from the formulae above

$$(L \cdot M \cdot N)S = LM(S_\# N)S = L(MS)N.$$

Its bijectivity follows from regularity of S. On the other hand, the map H gives the class of invertible A-module $A\langle S\rangle$ in $\operatorname{Pic} A$, and thence the class represented by $M_nA \otimes_A A\langle S\rangle$ in $\operatorname{Pic}(M_nA)$. It therefore remains to construct isomorphisms between the bimodules $(M_nA)S$ and $M_nA \otimes_A A\langle S\rangle$. The map from $M_nA \otimes_A A\langle S\rangle$ to $(M_nA)S$ corresponds to scalar multiplication:

$$M \otimes s_{ij} \mapsto Ms_{ij} = M \sum_k E_{ki} S E_{jk} = \sum_k M E_{ki} (S_\# E_{jk}) S,$$

where as hitherto E_{pq} refers to a standard matrix unit. It clearly has a right inverse

$$MS \mapsto \sum_{p,q} M E_{pq} \otimes s_{pq}.$$

Since locally these maps reduce to maps between free modules of rank one, it follows that they are inverse isomorphisms, as required.

Finally, we must check that the two right-hand vertical arrows are injective. For the map H, this is done in Theorem 6.3. Then the remaining vertical map, End, is seen to be a monomorphism by a diagram chase (or Snake Lemma). □

Note that the elements in the lower right-hand group constructed by our methods have a lower bound on their orders than previously known. For by [2](III.4.7), there is a uniform bound on the power of n^2 that gives the orders of elements in the image of φ. In contrast, by Theorem 6.3, elements in the image of H have order dividing n itself.

8. Relation to K-groups

We now show how the group homomorphisms H above may be lifted to group derivations mapping to K-groups modulo n. These groups may be constructed directly as in [22] or via the machinery of [3] VII (Theorem 5.3) as before. For the latter approach one takes the functor on the category $\mathbb{P}\mathrm{roj}\,A$ of finitely generated projective (right) A-modules that sends each module P to its n-fold direct sum nP, and each homomorphism ϕ to its n-fold direct sum $n\phi$. The upshot is the exact sequence of abelian groups

(8-7) $$K_1(A) \xrightarrow{\cdot n} K_1(A) \xrightarrow{\rho} K_1(A;\mathbb{Z}/n) \xrightarrow{\delta} K_0(A) \xrightarrow{\cdot n} K_0(A).$$

The group $K_1(A;\mathbb{Z}/n)$ is generated by isomorphism classes of triples $[P',\alpha,P]_\oplus$ consisting of projective modules P,P' of rank m and an isomorphism $\alpha : nP' \to nP$. An isomorphism with a triple $[Q',\beta,Q]_\oplus$ corresponds to a pair of isomorphisms $f' : P' \to Q'$ and $f : P \to Q$ such that $(nf)\circ\alpha = \beta\circ(nf')$. Relations are given by the counterparts of Equations 6-1 and 6-2:

$$[P'\oplus Q',\alpha\oplus\beta,P\oplus Q]_\oplus = [P',\alpha,P]_\oplus + [Q',\beta,Q]_\oplus$$

$$[P'',\alpha\beta,P]_\oplus = [P',\alpha,P]_\oplus + [P'',\beta,P']_\oplus$$

Homomorphisms in the sequence have the following description. A projective module automorphism (or invertible matrix) representing a class in $K_1(A)$ is sent by $\cdot n$ to the class of its n-fold direct sum. Alternatively, by adding direct sums of identity automorphisms, we may represent any element of $K_1(A)$ by an automorphism α of a free A-module of rank a multiple of n. Then

$$\rho([\alpha : nrA \to nrA]) = [rA,\alpha,rA]_\oplus \in K_1(A;\mathbb{Z}/n).$$

Next,

$$\delta([P',\alpha,P]_\oplus) = [P]-[P'] \in K_0(A),$$

while in $K_0(A)$, $\cdot n[P] = [nP]$. The kernel of the homomorphism $\cdot n$ is denoted ${}_nK_0(A)$.

The tensor product of modules makes $K_0(A)$ into a commutative ring with ideal ${}_nK_0(A)$, and $K_1(A)$ and $K_1(A;\mathbb{Z}/n)$ into $K_0(A)$-modules via the action

$$[P',\alpha,P]_\oplus\cdot[R] = [P'\otimes R,\alpha\otimes\mathrm{id},P\otimes R]_\oplus.$$

The effect is that the homomorphisms in the above exact sequence (8-7) are all $K_0(A)$-module homomorphisms. This sequence can be mapped to the previous exact sequence (6-3) of Picard groups by means of determinant homomorphisms constructed via the exterior algebra, after the method of [3] (IX, 3). Thus $\det_1 : K_1(A) \to U(A)$ is the usual matrix determinant homomorphism, while $\det_0 : K_0(A) \to \operatorname{Pic} A$ sends $[Q]-[P]$ to $[\Lambda^q(Q)\Lambda^{-p}(P)]_\otimes$, where P and Q have ranks $p=[P:A]$ and $q=[Q:A]$ (not necessarily constant). Because $\det_0$ converts sums into products, it follows that it is a group homomorphism, and maps the kernel ${}_nK_0(A)$, of multiplication by n on $K_0(A)$, to ${}_n\operatorname{Pic} A$. Moreover, we have

$$\begin{aligned}[Q\otimes R]-[P\otimes R] &\mapsto [\Lambda^{qr}(Q\otimes R)\Lambda^{-pr}(P\otimes R)]_\otimes\\ &= [(\Lambda^q(Q))^r(\Lambda^p(P))^{-r}]_\otimes,\end{aligned}$$

so that $\det_0$ is invariant with respect to the action of elements of $K_0(A)$ represented by modules R of constant rank $r=1$, that is [3](III, 7.5), by invertible modules.

There is also a group homomorphism $\det_1^{(n)} : K_1(A;\mathbb{Z}/n) \to \operatorname{Pic}(n,A)$ that sends $[P,\alpha,Q]_\oplus$ to $[\Lambda^p(P),\Lambda^{np}(\alpha),\Lambda^p(Q)]_\otimes$. (Necessarily $p=q$ here.) Again, this map is invariant with respect to the action of invertible modules, because

$$\begin{aligned}[P,\alpha,Q]_\oplus\cdot[R] &= [P\otimes R,\alpha\otimes\mathrm{id},Q\otimes R]_\oplus\\ &\longmapsto [\Lambda^{pr}(P\otimes R),\Lambda^{npr}(\alpha\otimes\mathrm{id}),\Lambda^{pr}(Q\otimes R)]_\otimes\\ &= [(\Lambda^p(P))^r(\Lambda^r(R))^p,(\Lambda^{np}(\alpha))^r,(\Lambda^p(Q))^r(\Lambda^r(R))^p]_\otimes\\ &= [(\Lambda^p(P))^r,(\Lambda^{np}(\alpha))^r,(\Lambda^p(Q))^r]_\otimes.\end{aligned}$$

To define the desired lifting $\bar{H}$ we need a counterpart to Lemma 6.1.

Lemma 8.1. *For $S \in M_nA$, the triple $(A,S\cdot,A\langle S\rangle)$ represents an element of $K_1(A;\mathbb{Z}/n)$ if and only if S is an intertwiner.*

Proof. For any $S \in M_n A$, the action of S on the column space $A^n = nA$ defines an A-module map $S\cdot$ from nA to $nA\langle S\rangle$. It is a monomorphism precisely when the matrix S is regular. When S is further an intertwiner, then locally it is scalar-times-invertible, making $S\cdot$ locally an isomorphism and so an isomorphism as sought. In the other direction, when $(A, S\cdot, A\langle S\rangle)$ represents an element of $K_1(A;\mathbb{Z}/n)$, application of $\det_1^{(n)}$ yields the element $[A, \cdot(\det S), A\langle S\rangle]_\otimes$ of $\mathrm{Pic}(n, A)$. Then Lemma 6.1 completes the proof. □

In order to use this result to define derivations from quotients of $\mathrm{Int}_n A$ to ${}_nK_0(A)$ and $K_1(A;\mathbb{Z}/n)$, we need to know the actions of $\mathrm{Int}_n A$ on ${}_nK_0(A)$ and $K_1(A;\mathbb{Z}/n)$. These are given by the module actions of the ring $K_0(A)$ described above, where $T \in \mathrm{Int}_n A$ acts as $[A\langle T\rangle] \in K_0(A)$.

Proposition 8.2. *There are derivations of groups*

$$\begin{aligned}\bar{H}_0 : (\mathrm{Int}_n A)/(A^\times \cdot \mathrm{GL}_n A) &\longrightarrow {}_nK_0(A)\\ S &\longmapsto [A\langle S\rangle] - [A]\end{aligned}$$

and

$$\begin{aligned}\bar{H}_1 : (\mathrm{Int}_n A)/A^\times &\longrightarrow K_1(A;\mathbb{Z}/n)\\ S &\mapsto [A, S\cdot, A\langle S\rangle]_\oplus\ ,\end{aligned}$$

with $\bar{H}_0$ injective, which make the following commute. (To save space, the symbol A is omitted here in $\mathrm{GL}_n A, \mathrm{SL}_n A, \mathrm{Int}_n A$.)

$$\text{(8-8)}\qquad \begin{array}{ccccccc}
U(A) & & \xrightarrow{\otimes I_n} & \mathrm{GL}_n & & \longrightarrow & (\mathrm{Int}_n)/A^\times\\
& \searrow^{\mathrm{id}} & & & \searrow & & \\
& & U(A) & & \longrightarrow & \mathrm{GL}_n/\mathrm{SL}_n & \\
\downarrow & & & \downarrow & & & \downarrow^{\bar{H}_1}\\
K_1(A) & & \xrightarrow{\cdot n} & K_1(A) & & \xrightarrow{\rho} & K_1(A;\mathbb{Z}/n)\\
& \searrow^{\det_1} & \downarrow^{\mathrm{id}} & & \searrow^{\det_1} & \downarrow & \\
& & U(A) & & \xrightarrow{(\ )^n} & U(A) &
\end{array}$$

$$\begin{array}{ccccc}
(\mathrm{Int}_n)/A^\times & & \twoheadrightarrow & & (\mathrm{Int}_n)/(A^\times\cdot\mathrm{GL}_n)\\
& \searrow & & & \\
& \longrightarrow & (\mathrm{Int}_n)/(A^\times\cdot\mathrm{SL}_n) & & \\
\downarrow^{\bar{H}_1} & & & & \downarrow^{\bar{H}_0}\\
K_1(A;\mathbb{Z}/n) & & \overset{\delta}{\twoheadrightarrow} & & {}_nK_0(A)\\
& \searrow^{\det_1^{(n)}} & \downarrow^{H} & & \\
& \longrightarrow & \mathrm{Pic}(n,A) & &
\end{array}$$

$$\begin{array}{ccc}
(\mathrm{Int}_n)/(A^\times\cdot\mathrm{GL}_n) & & \\
& \searrow^{\mathrm{id}} & \\
& \twoheadrightarrow & (\mathrm{Int}_n)/(A^\times\cdot\mathrm{GL}_n)\\
\downarrow^{\bar{H}_0} & & \\
{}_nK_0(A) & & \\
& \searrow^{\det_0} & \downarrow^{H}\\
& \twoheadrightarrow & {}_n\,\mathrm{Pic}\,A
\end{array}$$

Proof. We have already noted that $(\mathrm{Int}_n A)/(A^\times \cdot \mathrm{GL}_n A)$ is an abelian group. Likewise, that $(\mathrm{Int}_n A)/A^\times$ is a group isomorphic to $((A^\times)^{-1}\,\mathrm{Int}_n A)/((A^\times)^{-1}A^\times)$

also follows from (5.1) assisted by (1.5), (3.8). To check that $\bar{H}_i$ is a derivation we observe that, for $S, T \in \mathrm{Int}_n A$, in ${}_nK_0(A)$

$$[A\langle ST\rangle] - [A] = ([A\langle T\rangle] - [A]) + ([A\langle S\rangle] - [A]) \cdot [A\langle T\rangle]$$

by (6.2), while in $K_1(A; \mathbb{Z}/n)$

$$[A, ST, A\langle ST\rangle]_\oplus = [A, T, A\langle T\rangle]_\oplus + [A, S, A\langle S\rangle]_\oplus \cdot [A\langle T\rangle].$$

Now $A^\times$ and $A^\times \cdot \mathrm{GL}_n A$ are normal submonoids of $\mathrm{Int}_n A$, so we may apply (1.5)(c), since evidently $[A\langle S\rangle] = [A]$ when $S \in A^\times \cdot \mathrm{GL}_n A$, and $[A, S\cdot, A\langle S\rangle]_\oplus$ is trivial when $S \in A^\times \subseteq \mathrm{Int}_n A$ (for such S has the form $s \otimes I_n$ for some $s \in A^\times$). Commutativity is easily checked. Then injectivity of $\bar{H}_0$ follows from injectivity of H (6.3). □

9. Surjectivity of H

Theorem 9.1. *Suppose that A is a commutative domain, with field of fractions K, each of whose invertible modules can be generated by at most two elements. (For example, suppose that A is 1-dimensional [21].) Then, for all $n \geq 2$, H induces isomorphisms*

$$\mathrm{Int}_n A/(A^\times \cdot \mathrm{SL}_n A) = (K^\times \cdot \mathrm{Int}_n A)/(K^\times \cdot \mathrm{SL}_n A) \xrightarrow{\cong} \mathrm{Pic}(n, A)$$

and

$$\mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A) = (K^\times \cdot \mathrm{Int}_n A)/(K^\times \cdot \mathrm{GL}_n A) \xrightarrow{\cong} {}_n\mathrm{Pic}\, A.$$

Proof. After (6.3), it suffices to show that the second of these maps is surjective. Since A is a domain, the canonical map $\mathrm{Pic}(A, A^\times) \to \mathrm{Pic}(A)$ is surjective, and so every element of $\mathrm{Pic}(A)$ is representable by a fractional ideal of A [3]p.137. By assumption, we may further represent any nonzero element of ${}_n\mathrm{Pic}\, A$ as the class of a fractional ideal $A(af^{-1}, bf^{-1})$ with just two generators, such that $(A(af^{-1}, bf^{-1}))^n = A(cg^{-1})$, say. Then, since both

$$cg^{-1} = \sum_i r_i(af^{-1})^{n-i}(bf^{-1})^i \quad \text{and each} \quad (af^{-1})^{n-j}(bf^{-1})^j = s_j cg^{-1}$$

for suitable r_i, s_j, we find that each $a^{n-j}b^j = s_j \sum_i r_i a^{n-i} b^i$. Thus $(A(a,b))^n = A(d)$ where

$$d = r_0 a^n + r_1 a^{n-1} b + \cdots + r_{n-1} a b^{n-1} + r_n b^n.$$

Then the matrix

$$S = \begin{bmatrix} a & -b & 0 & \dots & 0 & 0 \\ 0 & a & -b & & & \\ & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & a & -b & 0 \\ 0 & 0 & \cdots & 0 & a & -b \\ r_n b & r_{n-1} b & \cdots & & r_2 b & r_0 a + r_1 b \end{bmatrix}$$

has determinant d. It follows that $(A\langle S\rangle)^n = A(\det S)$. So by (3.2) S is an intertwiner. Evidently, $H(S) = [A(a,b)]_\otimes = [A(af^{-1}, bf^{-1})]_\otimes$, as required. □

Note that S above is Sylvester's matrix (which explains our notation) for the ordered sets

$$a, -b \quad \text{and} \quad r_n b, ..., r_2 b, r_0 a + r_1 b.$$

Its determinant is therefore the resultant of the polynomials

$$ax - b \quad \text{and} \quad r_n b x^{n-1} + \cdots + r_2 b x + r_1 b + r_0 a.$$

See [15]§37 for a discussion of the applicability of the 2-generator condition above.

From Lemma 8.1, if $S \in \mathrm{Int}_2 A$, then $A \oplus A \cong A\langle S\rangle \oplus A\langle S\rangle$, whence $A\langle S\rangle$ can be generated by at most two elements. This shows that, in the case $n = 2$, the condition that each invertible ideal have at most two generators is both necessary and sufficient for H to be an isomorphism. Note that, from consideration of the second exterior power, one has that an invertible module I can be generated by two elements if and only if the sum $I \oplus I^{-1}$ is free (of rank 2).

Example 9.2. The following example, kindly pointed out by J-P Serre, shows that the condition of Theorem 9.1 cannot be relaxed to allow each *divisor* class to be representable by a fractional ideal with two generators, a condition that holds in any Krull ring [13]p.83.

Let P be the projective plane over a field k, and let Γ be a nonsingular conic in P that has no k-rational point. Let X be the affine surface $P - \Gamma$ and A its affine ring. If $f(x, y, z)$ is an equation of Γ, we may view A as the component of degree 0 of the graded ring $B = k[x, y, z, 1/f]$. It is easy to see (for example, geometrically) that $\mathrm{Pic}A$ is cyclic of order 2. The component of degree 1 of B is an invertible module I that represents the nonzero element of $\mathrm{Pic}A$ and is generated by the three elements x, y, z. If I could be generated by two elements c, d, defining divisors C, D of X, then C, D would be curves of odd degree, with empty intersection $C \cdot D$ (in X). By Bezout, this would imply that the intersection (in P) of their projective closures is a k-rational zero-cycle of Γ of odd degree, contrary to the assumption made on Γ. As an example, we may take $k = \mathbb{R}$ and $f = x^2 + y^2 + z^2$; in that case, the fact that $I \oplus I$ is not free can also be proved topologically by showing that the corresponding real vector bundle has a nonzero second Stiefel-Whitney class.

By combining Theorem 9.1 with the exact sequences of (8.2), we immediately obtain the following result, which makes no explicit reference to intertwining matrices. Previously it had been known that $\det_0 : \tilde{K}_0(A) \longrightarrow \mathrm{Pic}\, A$ below is an epimorphism [3]p.466.

Corollary 9.3. *When A is a domain each of whose invertible modules can be generated by at most two elements, then for all $n \geq 2$*

$$\det_1^{(n)} : K_1(A; \mathbb{Z}/n) \longrightarrow \mathrm{Pic}(n, A)$$

is an epimorphism, and

$$\det_0 : \mathrm{Tor}\, \tilde{K}_0(A) \longrightarrow \mathrm{Tor}\, \mathrm{Pic}\, A$$

is a split epimorphism. □

As an application of [3](IV.3.8), (III.7.5), (IX.3.8), in the following circumstance we have both that the condition of Theorem 9.1 holds and $\det_0$ is an isomorphism.

Corollary 9.4. *Suppose that A is a commutative domain whose maximal spectrum is a Noetherian space that is a union of a finite number of subspaces of dimension at most 1. Then, for all $n \geq 2$, H induces isomorphisms*

$$\mathrm{Int}_n\, A/(A^\times \cdot \mathrm{SL}_n\, A) \xrightarrow{\cong} \mathrm{Pic}(n, A)$$

and

$$H = \det_0 \circ \bar{H}_0 : \mathrm{Int}_n\, A/(A^\times \cdot \mathrm{GL}_n\, A) \xrightarrow{\cong} {}_n K_0(A) \xrightarrow{\cong} {}_n \mathrm{Pic}\, A.$$

□

We now specialize to the case where A is a Dedekind domain with fraction field K.

Definition 9.5. Adopting the notation of [22](2.1) and [32](6.3.2), we write as $\mathcal{U}(A;\,\mathbb{Z}/n)$ the subgroup of $U(K)/(U(K))^n$ comprising those classes $[x]$ where there is a fractional ideal I of A with $I^n = xA$.

In fact, as noted in [32], it is possible to represent each element by an $x \in A^\times$ and I integral. The next result shows how to do this.

Corollary 9.6. *Let A be a Dedekind domain, and let $n \geq 2$.*
(a) *The homomorphism*

$$\begin{aligned} \operatorname{Int}_n A &\rightarrow \mathcal{U}(A;\,\mathbb{Z}/n) \\ S &\longmapsto [\det S] \end{aligned}$$

induces isomorphisms

$$\operatorname{Int}_n A/(A^\times \cdot \operatorname{SL}_n A) \xrightarrow{\cong} \operatorname{Pic}(n, A) \xrightarrow{\cong} \mathcal{U}(A;\,\mathbb{Z}/n).$$

(b) *If further $U(A)$ is infinite and the fraction field K is a global field, then there are isomorphisms*

$$\operatorname{Int}_n A/(A^\times \cdot \operatorname{SL}_n A) \xrightarrow{\bar{H}_1} K_1(A;\,\mathbb{Z}/n) \xrightarrow{\det_1^{(n)}} \operatorname{Pic}(n, A) \xrightarrow{\cong} \mathcal{U}(A;\,\mathbb{Z}/n).$$

Proof. (a) The definition exploits the fact that for an intertwiner S, we have $A(\det S) = (A\langle S\rangle)^n$ by Theorem 3.2. Using Theorem 9.1, we clearly have the maps well-defined and the former an isomorphism. We then have from (8-8), [22](1.1) and [3](IX, 3.8) an isomorphism of short exact sequences

$$\begin{array}{ccccc} U(A)/(U(A))^n & \rightarrowtail & \operatorname{Pic}(n, A) & \twoheadrightarrow & {}_n\operatorname{Pic} A \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \cong \uparrow{\scriptstyle \det_0} \\ U(A)/(U(A))^n & \rightarrowtail & \mathcal{U}(A;\,\mathbb{Z}/n) & \twoheadrightarrow & {}_nK_0(A) \end{array}$$

(b) To establish the first two isomorphisms, consider the map of short exact sequences

$$\begin{array}{ccccc} (K_1(A))/n & \overset{\rho}{\rightarrowtail} & K_1(A;\,\mathbb{Z}/n) & \overset{\delta}{\twoheadrightarrow} & {}_nK_0(A) \\ \downarrow{\scriptstyle \det_1} & & \downarrow{\scriptstyle \det_1^{(n)}} & & \downarrow{\scriptstyle \det_0} \\ U(A)/(U(A))^n & \rightarrowtail & \operatorname{Pic}(n, A) & \twoheadrightarrow & {}_n\operatorname{Pic} A \end{array}$$

Again, by [3](IX, 3.8) the rightmost vertical arrow is an isomorphism. However, from [4], the further hypothesis makes the leftmost vertical arrow an isomorphism. So by the Five Lemma, and the preceding theorem, we have as isomorphisms both $\det_1^{(n)}$ and H in the commuting square co

$$\begin{array}{ccc} (\operatorname{Int}_n A)/A^\times & \longrightarrow & (\operatorname{Int}_n A)/(A^\times \cdot \operatorname{SL}_n A) \\ \downarrow{\scriptstyle \bar{H}_1} & & \downarrow{\scriptstyle H} \\ K_1(A;\,\mathbb{Z}/n) & \xrightarrow{\det_1^{(n)}} & \operatorname{Pic}(n, A) \end{array}$$

The result follows. □

10. Quadratic field examples

In order to give the flavour of calculations that follow from the above theory, we consider some simple examples corresponding to rings of integers in imaginary quadratic number fields.

In general, in order to exhibit an intertwining matrix representing a nonzero element of the Picard group of a number field, its ideal class group, one first finds a two-element representation $A(a, b)$ of a prime ideal representative $\mathfrak{p}$. Algorithms for this are well-known (for example, [29](6.3), [14]Algorithm 4.7.10). One can of course even choose a or b as $p \in \mathbb{N}$ where $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$. Then one applies the method of Theorem 9.1 above to obtain $S = S_{(a,b)}$ with $H(S) = [\mathfrak{p}]_\otimes \in \mathrm{Pic}(A)$. We note that the ideal $\mathfrak{p}$ is not uniquely determined by its class; neither is the pair (a, b) uniquely determined by $\mathfrak{p}$; nor the matrix $S_{(a,b)}$ uniquely determined by (a, b), since there are choices available for the elements $r_0, r_1, \ldots, r_n$, subject only to the condition that

$$r_0 a^n + r_1 a^{n-1} b + \cdots + r_{n-1} a b^{n-1} + r_n b^n$$

generate the principal ideal $(A(a, b))^n$. On the other hand, the number n is determined as the order of the class $[\mathfrak{p}]_\otimes$; while by Corollary 9.6 $\det S$ is determined to within multiplication by the n th power of a nonzero element of the field. Although, in principle, exhibiting an intertwiner could involve n^2 calculations of its entries, the method above results in at most $n + 2$ distinct nonzero entries (in fact, as we shall see, often considerably fewer).

Example 10.1. The ‘first’ non-pid typically encountered is the ring of integers $A = \mathbb{Z}[\sqrt{-5}]$ in the number field $\mathbb{Q}(\sqrt{-5})$. Its class group has order 2, with nonzero element represented by the prime ideal $A(2, 1 + \sqrt{-5})$ above the ramified prime 2. Thus $(A(2, 1 + \sqrt{-5}))^2 = A(2)$. So the equation

$$2 = r_0 \cdot 2^2 + r_1 \cdot 2(1 + \sqrt{-5}) + r_2(1 + \sqrt{-5})^2$$

yields $r_2 = c$ say, $r_1 = -c$ and $r_0 a = 2r_0 = 1 + 3c$, so that

$$S = \begin{bmatrix} 2 & -1 - \sqrt{-5} \\ c(1 + \sqrt{-5}) & 1 + c(2 - \sqrt{-5}) \end{bmatrix},$$

of determinant 2. Observe that $S^2/2$ is an invertible matrix.

Example 10.2. Consider the ring of integers $A = \mathbb{Z}[\omega]$ in $\mathbb{Q}(\sqrt{-23})$, where $\omega = (1+\sqrt{-23})/2$. In this case, the prime 2 is completely split, with the prime ideals $2_1 = A(2, \omega)$ and $2_2 = A(1 + \omega, -2)$ representing the two distinct nonzero ideal classes, which therefore have order 3. The method of Theorem 9.1 now gives corresponding intertwiners

$$S_1 = \begin{bmatrix} 2 & -\omega & 0 \\ 0 & 2 & -\omega \\ -\omega & -\omega & -4 - \omega \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 1 + \omega & 2 & 0 \\ 0 & 1 + \omega & 2 \\ -2 & -6 & (1 + \omega) - 4 \end{bmatrix}$$

respectively. The fact that the ideals 2_1 and 2_2 represent inverse classes corresponds to the matrix equation

$$S_1 S_2 = 2T$$

where $T = \begin{bmatrix} 1 + \omega & 5 - \omega & -\omega \\ \omega & 1 + 4\omega & 5 + \omega \\ 7 & 15 + \omega & 9 - 2\omega \end{bmatrix} \in \mathrm{SL}_3 A.$

Example 10.3. For an example with a non-cyclic class group, we look at the ring of integers $A = \mathbb{Z}[\omega]$ in the field $\mathbb{Q}(\sqrt{-231})$, where $\omega = (1 + \sqrt{-231})/2$. This example occurs, from a different perspective discussed below, in the extended (preprint) version of [22]. The two non-inert primes are 2 (split) and 3 (ramified). Thus 3 is below a prime ideal $A(1 + \omega, -3)$ that represents the generator of a subgroup of order 2 in the class group. A corresponding intertwiner is

$$\begin{bmatrix} 1 + \omega & 3 \\ -21 & -3 + (1 + \omega) \end{bmatrix}.$$

The class group is the product of this subgroup with a subgroup of order 6 generated by the class of the prime ideal $2_1 = A(2 + \omega, -2)$. By the method of Theorem 9.1, an associated intertwiner is

$$\begin{bmatrix} 2+\omega & 2 & 0 & 0 & 0 & 0 \\ 0 & 2+\omega & 2 & 0 & 0 & 0 \\ 0 & 0 & 2+\omega & 2 & 0 & 0 \\ 0 & 0 & 0 & 2+\omega & 2 & 0 \\ 0 & 0 & 0 & 0 & 2+\omega & 2 \\ 33542822 & 162830 & 0 & 0 & 0 & 32566-(2+\omega) \end{bmatrix}$$

where $33542822 = 2 \times 16\,771\,411$ and $162830 = 5 \times 32566 = 2 \times 5 \times 19 \times 857$; its determinant is $2+\omega$. The square of the class of 2_1 has order $n = 3$, and so corresponds to an element of $\mathrm{Int}_3 A$:

$$\begin{bmatrix} 2+\omega & -4 & 0 \\ 0 & 2+\omega & -4 \\ 52 & 0 & 8+(2+\omega) \end{bmatrix};$$

again the determinant is $2+\omega$. This gives the nonzero element detected by [22] Proposition 2.8; for our

$$a = 2+\omega = (5+\sqrt{-231})/2 = (\alpha+\sqrt{\delta})/2$$

and $b = 4$ satisfy the conditions that $3 \mid (5^2 - 4 \cdot 4^3) = -231 = \delta$, in other words, $n \mid (\alpha^2 - 4 \cdot b^n) = \delta$. On the other hand, the cube of 2_1 corresponds to an element of $\mathrm{Int}_2 A$:

$$\begin{bmatrix} 2+\omega & 8 \\ 24 & 16-3(2+\omega) \end{bmatrix},$$

with the same determinant.

Here is the appropriate generalization of the last example above ($n = 3$) that relates to nonzero class group elements detected by [22].

Proposition 10.4. *Let A be the ring of integers in the quadratic number field $\mathbb{Q}(\sqrt{\delta})$ of discriminant δ ($\neq -3, -4$). Suppose that there exist $\alpha, b, n \in \mathbb{Z}$ ($n \geq 3$) such that*

$$\delta = \alpha^2 - 4b^n \qquad \text{and} \qquad (\alpha, b) = 1;$$

thus we may write $u = (\alpha+\sqrt{\delta})/2$ and choose $s, t \in \mathbb{Z}$ to satisfy

$$s\alpha b^{n-2} + t(\alpha^2 - b^n)b^{n-3} = 1.$$

Then
(a) the matrix

$$S_n = \begin{bmatrix} u & -b & 0 \\ 0 & u & -b \\ (sb+t\alpha)b & 0 & sb+tu \end{bmatrix} \qquad n = 3$$

$$S_n = \begin{bmatrix} u & -b & 0 & 0 \\ 0 & u & -b & 0 \\ 0 & 0 & u & -b \\ (sb+t\alpha)b^2 & 0 & sb & tb \end{bmatrix} \qquad n = 4$$

$$S_n = \begin{bmatrix} u & -b & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & -b & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & u & -b & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & u & -b & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & u & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & u & -b \\ (sb+t\alpha)b^{n-2} & 0 & sb & tb & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad n \geq 5$$

is an element of $\mathrm{Int}_n A$*; and*

(b) if further n *is an odd integer, then the element of the ideal class group of* A *represented by* S_n *has order* n*, provided that either:*

(i) *for every integer* β *and proper divisor* m *of* n *the number* $\delta\beta^2 \pm 4b^m$ *is not a perfect square; or*

(ii) $n \mid \delta$*, and that moreover when* $\delta > 0$ *then the fundamental unit* $(\varepsilon_1+\varepsilon_2\sqrt{\delta})/2$ *of* A *has* $n \mid \varepsilon_2$*.*

Proof. **(a)** From $u(\alpha - u) = (\alpha^2 - \delta)/4 = b^n$, we obtain

$$\alpha u = u^2 + b^n$$
$$(\alpha^2 - b^n)u = u^3 + \alpha b^n.$$

The definition of s, t yields

$$u = su^2b^{n-2} + tu^3b^{n-3} + (sb^{n-2} + tb^{n-3}\alpha)b^n,$$

from which (a) follows by Theorem 9.1.

(b) In the specified cases it is shown in [22] Propositions 1.9, 2.8 that the class obtained in (a) has nontrivial Dennis trace invariant (with $\mathbb{Z}/n$ coefficients), and is consequently of order n. □

III Stabilization

11. Direct sum stabilization

The next result shows that the usual kind of stabilization is unsuited to the analysis of intertwining matrices.

Lemma 11.1. *Let* $S \in M_nA$*. The following are equivalent.*

(i) $S \oplus dI_1$ *is an intertwiner.*

(ii) S *is an intertwiner with* $A\langle S\rangle = A(d)$*.*

(iii) $S = dS'$ *with* $d \in A^\times$*,* $S' \in \mathrm{GL}_n A$*.*

Proof. (i) ⇒ (ii): Over each localization $A_\mathfrak{m}$, there exists $e \in A_\mathfrak{m}$ with $S \oplus dI_1 = eS'' \oplus e\alpha I_1$ where $S'' \oplus \alpha I_1$ is invertible. Thus S'' is invertible and $\alpha \in U(A_\mathfrak{m})$. Substituting $e = d\alpha^{-1}$ gives $S = d(\alpha^{-1}S'')$ with $\alpha^{-1}S''$ invertible. So S is locally scalar-times-invertible and thus an intertwiner. Also, locally $A_\mathfrak{m}\langle S\rangle = A_\mathfrak{m}(d)$. Hence $A\langle S\rangle = A(d)$.

(ii) ⇒ (iii): Because $A\langle S\rangle = A(d)$ we may write $S = dS'$, so that $A(d) = A\langle S\rangle = A(d)A\langle S'\rangle$. Now since S is intertwining and d^n is associate to $\det S$, d is regular. This makes $A\langle S'\rangle = A$. Since $\operatorname{adj} S = d^{n-1}\operatorname{adj} S'$, substitution in (3.2)(i) gives

$$A(d^n)A\langle \operatorname{adj} S'\rangle = A(d^n \det S').$$

Thus $\det S'$ divides each entry of $\operatorname{adj} S'$, making S' invertible, as desired.

(iii) ⇒ (i): $S \oplus dI_1 = d(S' \oplus I_1)$ is scalar-times-invertible, hence certainly locally scalar-times-invertible. Moreover, $\det S$ is associate to d^n and therefore regular. So $\det(S \oplus dI_1) = d \det S$ is also regular. □

The usual stabilization of $\mathrm{GL}_n A$ is obtained by including $M_n A$ in $M_{n+1}A$ via $B \mapsto B \oplus I_1$, giving $\mathrm{GL}\, A = U(M_{\mathrm{fte}}A)$ where $M_{\mathrm{fte}}A = \bigcup_n M_n A$. Notation for the inclusion of $M_n A$ in $M_{\mathrm{fte}}A$ is $B \mapsto B \oplus I_\omega$, where I_ω denotes the identity matrix with respect to a basis of order type ω. Perhaps surprisingly, we conclude that $\mathrm{Int}_{\mathrm{fte}}\, A = \bigcup_n \mathrm{Int}_n\, A$ is not in general a submonoid of $M_{\mathrm{fte}}A$.

Proposition 11.2. $\mathrm{Int}_{\mathrm{fte}}\, A$ *is a submonoid of* $M_{\mathrm{fte}}A$ *if and only if* $\mathrm{Int}_{\mathrm{fte}}\, A = \mathrm{GL}\, A$.

Proof. To prove the nonobvious direction, take any noninvertible intertwiner $S \in M_n A$. In view of the fact that $S \oplus S$ is also a noninvertible intertwiner (compare Lemma 12.2 below), it suffices to show that $S^2 \oplus S \oplus I_\omega = (S \oplus S \oplus I_\omega)(S \oplus I_\omega) \notin \mathrm{Int}_{\mathrm{fte}}\, A$. By the above lemma, if $S^2 \oplus S \oplus I_\omega \in \mathrm{Int}_{\mathrm{fte}}\, A$, then $S^2 \oplus S \in \mathrm{Int}_{2n}\, A$. Then by Corollary 3.7(a), it follows that $S \oplus I_\omega \in \mathrm{Int}_{2n}\, A$. However, by the previous lemma again, this is impossible. □

Now the domains A with $A^\times = U(A)$ are precisely the fields, and they have $\tilde{K}_0(A) = 0$, so that, by (8.2), each $\mathrm{Int}_n\, A = \mathrm{GL}_n\, A$.

Corollary 11.3. *Let A be a domain. Then* $\mathrm{Int}_{\mathrm{fte}}\, A$ *is a submonoid of* $M_{\mathrm{fte}}A$ *if and only if A is a field.* □

Similar localization arguments to those of (11.1) above lead to the following.

Lemma 11.4. *Every* $S \in \mathrm{Int}_n\, A$ *has* $S \oplus \mathrm{adj}\, S \in \mathrm{Int}_{2n}\, A$ *if and only if* $n = 2$. □

We shall see in (14.2) below that considering $S \otimes \mathrm{adj}\, S$ instead of $S \oplus \mathrm{adj}\, S$ is far more fruitful.

12. Tensor product stabilization

Lemma 12.1. *Tensor product of matrices defines a monoid homomorphism*

$$\begin{array}{ccc} \tau : \mathrm{Int}_m\, A \times \mathrm{Int}_n\, A & \to & \mathrm{Int}_{mn}\, A \\ (S, T) & \mapsto & S \otimes T \end{array}$$

such that there is a commuting square of homomorphisms

$$\begin{array}{ccc} \mathrm{Int}_m\, A \times \mathrm{Int}_n\, A & \to & \mathrm{Int}_{mn}\, A \\ \downarrow H_m \times H_n & & \downarrow H_{mn} \\ {}_m\operatorname{Pic} A \times {}_n\operatorname{Pic} A & \to & {}_{mn}\operatorname{Pic} A \end{array}$$

Proof. Given that $S \in \mathrm{Int}_m\, A$ and $T \in \mathrm{Int}_n\, A$, so that $(A\langle S\rangle)^m = A(\det S)$ and $(A\langle T\rangle)^n = A(\det T)$, it is easy to check that $(A\langle S \otimes T\rangle)^{mn} = A(\det(S \otimes T))$.
In fact, $A\langle S \otimes T\rangle = A\langle S\rangle A\langle T\rangle$, so that

$$\begin{aligned} A\langle S \otimes T\rangle^{mn} &= (A(\det S))^n (A(\det T))^m \\ &= A((\det S)^n (\det T)^m) = A(\det(S \otimes T)). \end{aligned}$$

τ is a monoid homomorphism because $\tau(S,T) \cdot \tau(S',T') = \tau(SS', TT')$. Finally, the square commutes because $A\langle S \otimes T\rangle = A\langle S\rangle A\langle T\rangle$. □

The case of tensoring with an identity matrix merits special attention.

Lemma 12.2. *Let $k \in \mathbb{N}$ be arbitrary. Then* $S \in \mathrm{Int}_n\, A$ *if and only if* $I_k \otimes S \in \mathrm{Int}_{kn}\, A$.

Proof. This is most easily seen from Theorem 3.2(ii), since $I_k \otimes S = S \oplus \cdots \oplus S$ is evidently locally scalar-times-invertible just when S itself is. □

We now define $M_\otimes A$ to be the colimit of the monoid homomorphisms

$$\iota_{kn,n} : M_n A \to M_{kn} A$$
$$S \longmapsto I_k \otimes S,$$

with canonical homomorphisms $\iota_n : M_n A \to M_\otimes A$. We use the same notation for the maps restricted to $\mathrm{Int}_n A$, $\mathrm{GL}_n A$, $\mathrm{SL}_n A$ and $E_n A$, with colimits denoted by $\mathrm{Int}_\otimes A = (M_\otimes A)^\times$, $\mathrm{GL}_\otimes A = U(M_\otimes A)$, $\mathrm{SL}_\otimes A$ and $E_\otimes A$. The above lemma may be restated as $\iota_{kn,n}^{-1}(\mathrm{Int}_{kn} A) = \mathrm{Int}_n A$, while $\iota_{kn,n}^{-1}(\mathrm{GL}_{kn} A) = \mathrm{GL}_n A$, $\iota_{kn,n}(\mathrm{SL}_n A) \subseteq \mathrm{SL}_{kn} A$, and $\iota_{kn,n}(E_n A) \subseteq E_{kn} A$. On occasion we omit these maps, in effect identifying matrices with their images under the monomorphisms. If we again let $A^\times$ represent the scalar matrices in $\mathrm{Int}_\otimes A$, then the enveloping group of $\mathrm{Int}_\otimes A$ is clearly $(A^\times)^{-1} \cdot \mathrm{Int}_\otimes A$.

Lemma 12.3. *If $\phi : \mathrm{Int}_\otimes A \to M$ (or $\phi : (A^\times)^{-1} \cdot \mathrm{Int}_\otimes A \to M$) is a monoid homomorphism with $\phi(S) = \phi(I_k \otimes S) = \phi(S \otimes I_k)$ for all k, then ϕ has abelian image.*

Proof. On $\mathrm{Int}_n A$,

$$\begin{aligned}\phi(S)\phi(T) &= \phi((I_n \otimes S)(T \otimes I_n)) = \phi(T \otimes S) \\ &= \phi((T \otimes I_n)(I_n \otimes S)) = \phi(T)\phi(S).\end{aligned}$$

Since $A^\times$ is central in $\mathrm{Int}_\otimes A$, the alternative formulation follows. □

Lemma 12.4. *With (e_i) as basis for the standard free A-module, let $A^m \otimes A^n$ have basis $(e_i \otimes e_j)$ lexicographically ordered. Then the permutation $\sigma : im + j \longmapsto jn + i$ (where $0 \le i < n$, $0 \le j < m$) has corresponding $mn \times mn$ permutation matrix σ inducing*

$$\sigma^{-1}(S \otimes T)\sigma = T \otimes S,$$

and has sign $(-1)^{\nu(\sigma)}$ where $\nu(\sigma) = \binom{m}{2}\binom{n}{2}$.

Proof. It is well-known that σ has the claimed effect. Its sign may be calculated by reference to the number $\nu(\sigma)$ of its inversions, an inversion occurring whenever we have $p_1 < p_2$ but $\sigma(p_1) > \sigma(p_2)$. With $p_h = i_h m + j_h$, this event corresponds to $0 \le i_1 < i_2 < n$ and $0 \le j_2 < j_1 < m$. There are evidently $\binom{m}{2}\binom{n}{2}$ such occurrences. □

(The case $m = n$ of the above is considered in [2]p.120.) It follows that we can force σ to be an even permutation by arranging to have m or n congruent to 0 or 1 modulo 4. For the next proof, though, a simpler device is to observe that the matrix $I_2 \otimes \sigma$ necessarily corresponds to an even permutation. The resulting theorem may be considered as the 'Whitehead Lemma' for our K-theory. We write $G' = [G, G]$ for the commutator subgroup of any group G. As before, $\mathcal{P}G$ stands for the perfect radical of G.

Theorem 12.5. $\mathcal{P}((A^\times)^{-1}\,\mathrm{Int}_\otimes A) = ((A^\times)^{-1}\,\mathrm{Int}_\otimes A)' = E_\otimes A$.

Proof. We first check that $E_\otimes A$ is perfect. This follows from the same property for $E_n A$ when n is sufficiently large (in fact, $n \ge 3$). For then $\iota_n E_n A = \iota_n (E_n A)' = (\iota_n E_n A)' \le (E_\otimes A)'$. Therefore

$$E_\otimes A \le \mathcal{P}((A^\times)^{-1}\,\mathrm{Int}_\otimes A) \le ((A^\times)^{-1}\,\mathrm{Int}_\otimes A)',$$

and it remains to show that $((A^\times)^{-1}\,\mathrm{Int}_\otimes A)' \le E_\otimes A$. Now by (3.8), $A^\times \cdot E_n A$ is normal in $\mathrm{Int}_n A$. So by (1.1), $A^\times \cdot E_\otimes A$ is normal in $\mathrm{Int}_\otimes A$. From the fact that

$\mathrm{Int}_\otimes A[(A^\times \cdot E_\otimes A)^{-1}]$ is the enveloping group $(A^\times)^{-1}\,\mathrm{Int}_\otimes A$ of $\mathrm{Int}_\otimes A$, it follows from (1.5) that

$$(A^\times)^{-1}A^\times \cdot E_\otimes A = A^\times \cdot E_\otimes A[(A^\times \cdot E_\otimes A)^{-1}]$$

is a normal subgroup of $(A^\times)^{-1}\,\mathrm{Int}_\otimes A$. However, $E_\otimes A$ is evidently the commutator subgroup of $(A^\times)^{-1}A^\times \cdot E_\otimes A$ and thus a characteristic subgroup of it. This makes it also a normal subgroup of $(A^\times)^{-1}\,\mathrm{Int}_\otimes A$. Hence we may consider the group epimorphism

$$\phi : (A^\times)^{-1}\,\mathrm{Int}_\otimes A \to ((A^\times)^{-1}\,\mathrm{Int}_\otimes A)\big/\, E_\otimes A.$$

Take $S \in \mathrm{Int}_n A$ and arbitrary k. Then (with $\phi\iota_m$ written simply as ϕ), we have

$$\phi(S) = \phi(I_k \otimes S) = \phi(I_2 \otimes I_k \otimes S).$$

On the other hand,

$$\begin{aligned}\phi(S \otimes I_k) &= \phi(I_2 \otimes S \otimes I_k)\\ &= \phi(I_2 \otimes \sigma^{-1}(I_k \otimes S)\sigma)\\ &= \phi(I_2 \otimes \sigma)^{-1}\phi(I_2 \otimes I_k \otimes S)\phi(I_2 \otimes \sigma)\\ &= \phi(I_2 \otimes I_k \otimes S),\end{aligned}$$

since because $I_2 \otimes \sigma$ is an even permutation it lies in $E_{2kn}A$ and so has $\phi(I_2 \otimes \sigma)$ trivial in $((A^\times)^{-1}\,\mathrm{Int}_\otimes A)\big/\, E_\otimes A$. Hence by Lemma 12.3 ϕ has abelian image. This means that $E_\otimes A$ contains the commutator subgroup of $((A^\times)^{-1}\,\mathrm{Int}_\otimes A)$, and the proof is complete. □

This result enables a refinement of Proposition 5.4.

Corollary 12.6. *For $n > 2$, the commutator subgroup of $(A^\times)^{-1}\,\mathrm{Int}_n A$ modulo $E_n A$ is a torsion subgroup of $\mathrm{SL}_n A/E_n A$.*

Proof. Consider the group extension

$$\mathrm{SL}_n A/E_n A \rightarrowtail ((A^\times)^{-1}\,\mathrm{Int}_n A)\big/\, E_n A \twoheadrightarrow ((A^\times)^{-1}\,\mathrm{Int}_n A)\big/\, \mathrm{SL}_n A.$$

Since the right-hand group is abelian, $((A^\times)^{-1}\,\mathrm{Int}_n A)'\big/\, E_n A$ lies in $\mathrm{SL}_n A/E_n A$. On the other hand, from the above theorem $((A^\times)^{-1}\,\mathrm{Int}_\otimes A)'\big/\, E_\otimes A$ is trivial, so that it further lies in the kernel of the stabilization map from $\mathrm{SL}_n A/E_n A$ to $\mathrm{SL}_\otimes A/E_\otimes A$. However, the usual Whitehead Lemma shows that, modulo $E_{mn}A$, for any $S \in \mathrm{SL}_n A$,

$$S^m \equiv I_m \otimes S.$$

So this kernel is a torsion group. □

Under favourable circumstances, we can now take Corollary 6.7 further.

Corollary 12.7. (a) *Suppose that $n > \max(2, \mathrm{sr}A)$ and that $SK_1 A$ is torsion-free. Then $E_n A$ is the derived subgroup of $(A^\times)^{-1}\,\mathrm{Int}_n A$.*

(b) *Suppose moreover that the condition of Corollary 6.8 (c) holds for A (for example, A is the ring of integers in a number field, satisfying the conditions of Corollary 9.6(b)). Then*

$$\pi_1(B\,\mathrm{Int}_n A^+) \cong \pi_1(B\,\mathrm{GL}_n A^+) \oplus ((A^\times)^{-1}\,\mathrm{Int}_n A)\big/\, \mathrm{GL}_n A.$$

Proof. **(a)** In the present situation, $\mathrm{SL}_n A/E_n A \cong SK_1 A$, so the result is immediate from the previous corollary.

(b) When (a) holds, the short exact sequence

$$\mathrm{GL}_n\, A/E_n A \rightarrowtail (A^\times)^{-1}\,\mathrm{Int}_n A\big/\, E_n A \twoheadrightarrow (A^\times)^{-1}\,\mathrm{Int}_n A\big/\, \mathrm{GL}_n\, A$$

becomes an abelian group extension, which now splits because in the setting of Corollary 6.8 (c) the quotient group $(A^\times)^{-1}\operatorname{Int}_n A/\operatorname{GL}_n A$ is free abelian. Since E_nA is the maximal perfect subgroup of $(A^\times)^{-1}\operatorname{Int}_n A$, we have

$$\pi_1(B\operatorname{Int}_n A^+) = (A^\times)^{-1}\operatorname{Int}_n A/\,E_nA,$$

and likewise for $\pi_1(B\operatorname{GL}_n A^+)$. So the result follows. □

13. Automorphisms of Azumaya algebras

Here we turn our attention to the tensor product stabilization of Bass' Rosenberg-Zelinsky sequence. As in [35], it may be written

$$U(A) \;\rightarrowtail\; \operatorname{GL}_\otimes A \;\overset{\text{End}}{\to}\; \operatorname{Aut}(\operatorname{Az} A) \;\overset{\varphi}{\twoheadrightarrow}\; \operatorname{Tor}\operatorname{Pic} A.$$

The group $\operatorname{Aut}(\operatorname{Az} A)$ may be constructed by restricting attention to the cofinal subcategory of the category of all Azumaya A-algebras, and their A-algebra automorphisms, consisting of those Azumaya algebras that are endomorphism rings of free A-modules. In other words, it is the colimit of the groups $\operatorname{Aut}(M_nA)$, the structure maps for the directed set being given by tensor product with identity matrices. So, although one can use Theorem 4.3 to construct, for each faithfully projective A-module P, a generalized map $\operatorname{End}: \operatorname{PInt} P \to \operatorname{Aut}(\operatorname{End} P)$ as in Theorem 7.2, it suffices to concentrate on the case where P has the form A^n.

We saw in Lemma 12.1 that $H : \operatorname{Int}_n A \to \operatorname{Tor}\operatorname{Pic} A$ respects tensor products. So it induces a map $H : \operatorname{Int}_\otimes A \to \operatorname{Tor}\operatorname{Pic} A$ which is easily seen to factor through the groups $\operatorname{PInt}_\otimes A = (\operatorname{Int}_\otimes A)/A^\times$ and $(\operatorname{Int}_\otimes A)/(A^\times \cdot \operatorname{GL}_\otimes A)$. The map $\operatorname{End} : \operatorname{PInt}_n A \to \operatorname{Aut}(M_nA)$, constructed in Theorem 7.2 as an extension of the usual $\operatorname{End} : \operatorname{GL}_n A \to \operatorname{Aut}(M_nA)$, also respects the tensor product structure. For, if S and T are intertwining matrices of the same size as matrices M and N respectively, then

$$S_\# M \otimes T_\# N = (M \otimes N)(S \otimes T),$$

from which it follows, by regularity, that

$$(S \otimes T)_\# = S_\# \otimes T_\#,$$

as required. Thus we may form the tensor product stabilization of the diagram of Theorem 7.2, giving the following result.

Theorem 13.1. *There is a commuting diagram*

$$\begin{array}{ccccccc} U(A) & \rightarrowtail & \operatorname{GL}_\otimes A & \to & \operatorname{PInt}_\otimes A & \twoheadrightarrow & (\operatorname{Int}_\otimes A)/\,(A^\times \cdot \operatorname{GL}_\otimes A) \\ \downarrow^{\text{id}} & & \downarrow^{\text{id}} & & \downarrow^{\text{End}} & & \downarrow^{H} \\ U(A) & \rightarrowtail & \operatorname{GL}_\otimes A & \overset{\text{End}}{\to} & \operatorname{Aut}(\operatorname{Az} A) & \overset{\varphi}{\to} & \operatorname{Tor}\operatorname{Pic} A \end{array} \tag{13-9}$$

in which the rows are exact and all vertical arrows are injective. □

For certain rings A, this combines with Theorem 9.1.

Corollary 13.2. *When A satisfies the conditions of Theorem 9.1, the natural monomorphism*

$$\operatorname{End} : \operatorname{PInt}_\otimes A \to \operatorname{Aut}(\operatorname{Az} A)$$

is an isomorphism. □

To pass to K-theory, we consider the group extension

$$U(A) \;\rightarrowtail\; \operatorname{GL}_\otimes A \;\twoheadrightarrow\; \operatorname{PGL}_\otimes A$$

and map (indeed, pull-back) of group extensions

$$\begin{array}{ccccc} \mathrm{PGL}_{\otimes} A & \to & \mathrm{PInt}_{\otimes} A & \twoheadrightarrow & (\mathrm{Int}_{\otimes} A)/\,(A^{\times} \cdot \mathrm{GL}_{\otimes} A) \\ \downarrow^{\mathrm{id}} & & \downarrow^{\mathrm{End}} & & \downarrow^{H} \\ \mathrm{PGL}_{\otimes} A & \overset{\mathrm{End}}{\to} & \mathrm{Aut}(\mathrm{Az}\, A) & \overset{\varphi}{\to} & \mathrm{Tor}\,\mathrm{Pic}\, A \end{array}$$

Since the first is a central extension, while the other two have abelian quotients, all three extensions have their associated classifying space fibration plus-constructive [6]. In the first case, the homotopy exact sequence gives a calculation of the homotopy groups $\pi_*(B\,\mathrm{PGL}_{\otimes} A^+)$ in terms of $\pi_*(B\,\mathrm{GL}_{\otimes} A^+)$, which, following [36] after [25], is just $\mathbb{Q}{\otimes}K_*A$ (positive dimensions only). (An alternative argument uses [2](I.6) to compare $\pi_*(B\,\mathrm{GL}_{\otimes} A^+)$ and $\pi_*(B\,\mathrm{GL}\, A^+)$.) This calculation is performed in [35]. The map of fibrations reduces to covering projections

$$B\,\mathrm{PGL}_{\otimes} A^+ \longrightarrow B\,\mathrm{PInt}_{\otimes} A^+ \overset{B\,\mathrm{End}^+}{\longrightarrow} B\,\mathrm{Aut}(\mathrm{Az}\, A)^+.$$

Since $B\,\mathrm{Aut}(\mathrm{Az}\, A)^+$ is known to be an infinite loop space [35], we obtain the following.

Theorem 13.3. *The spaces $B\,\mathrm{PInt}_{\otimes} A^+$ and $B\,\mathrm{Int}_{\otimes} A^+$ are infinite loop spaces.* □

Using information in low dimensions provided by [2](III.6.8), and noting the splittings afforded by divisible subgroups, we thereby obtain the following conclusion. Its last assertion follows from Theorem 12.5.

Theorem 13.4. *For $i \geq 3$, there are natural isomorphisms*

$$\pi_i(B\,\mathrm{PInt}_{\otimes} A^+) \cong \pi_i(B\,\mathrm{Aut}(\mathrm{Az}\, A)^+) \cong \pi_i(B\,\mathrm{PGL}_{\otimes} A^+) \cong \mathbb{Q}{\otimes}K_iA.$$

For $i = 2$,

$$\pi_2(B\,\mathrm{PInt}_{\otimes} A^+) \cong \pi_2(B\,\mathrm{Aut}(\mathrm{Az}\, A)^+) \cong \pi_2(B\,\mathrm{PGL}_{\otimes} A^+) \cong \mu(A) \oplus (\mathbb{Q}{\otimes}K_2A),$$

where $\mu(A)$ is the subgroup of $U(A)$ comprising all roots of unity in A.

For $i = 1$, there is a natural short exact sequence

$$\pi_1(B\,\mathrm{PInt}_{\otimes} A^+) \rightarrowtail \pi_1(B\,\mathrm{Aut}(\mathrm{Az}\, A)^+) \twoheadrightarrow \mathrm{Coker}\, H$$

and isomorphism

$$\pi_1(B\,\mathrm{PInt}_{\otimes} A^+) \cong (\mathbb{Q}/\mathbb{Z} \otimes U(A)) \oplus (\mathbb{Q} \otimes SK_1(A)) \oplus \mathrm{Im}\, H,$$

where $\pi_1(B\,\mathrm{PInt}_{\otimes} A^+) = (\mathrm{Int}_{\otimes} A)/\,(A^{\times} \cdot EA)$. □

To translate this into a computation of $\pi_*(B\,\mathrm{Int}_{\otimes} A^+)$, we can start with the central extension

$$(A^{\times})^{-1}A^{\times} \rightarrowtail (A^{\times})^{-1}(\mathrm{Int}_{\otimes} A) \twoheadrightarrow \mathrm{PInt}_{\otimes} A,$$

and derive (using [6] again, and of course Lemma 1.4) the middle row of the commuting diagram, in which each row and column is a fibration:

$$\begin{array}{ccccc} BU(A) & \to & B\,\mathrm{GL}_{\otimes} A^+ & \to & B\,\mathrm{PGL}_{\otimes} A^+ \\ \downarrow & & \downarrow & & \downarrow \\ B((A^{\times})^{-1}A^{\times}) & \to & B\,\mathrm{Int}_{\otimes} A^+ & \to & B\,\mathrm{PInt}_{\otimes} A^+ \\ \downarrow & & \downarrow & & \downarrow \\ B\,\mathrm{Prin}\, A & \to & B(\,(A^{\times})^{-1}\,\mathrm{Int}_{\otimes} A/\,\mathrm{GL}_{\otimes} A) & \to & B\,\mathrm{Im}\, H \end{array}$$

This leads to the following stabilization of Corollary 6.7 (which could of course also have been obtained by direct limit arguments from that result).

Corollary 13.5. *For $i \geq 2$, there is a natural isomorphism*

$$\pi_i(B\operatorname{Int}_{\otimes} A^+) \cong \mathbb{Q}{\otimes}K_iA;$$

while for $i = 1$ the natural map $\mathbb{Q}{\otimes}K_1A \to \pi_1(B\operatorname{Int}_{\otimes} A^+)$ is a split monomorphism, with abelian cokernel given by a group extension

$$\operatorname{Prin} A \rightarrowtail \operatorname{Cokernel} \twoheadrightarrow \operatorname{Im} H.$$

. □

14. Action of $\tilde{K}_0(A)$ on K-theory

An attractive feature of the intertwining matrix approach is the way it naturally displays the action of $\tilde{K}_0(A)$ on the groups $K_i(A)$, $i \geq 0$. We show how this action is induced from the matrix action $S^{\#}$ of an intertwiner S. Essentially, this is because there is an induced action on each $\pi_i(B\operatorname{GL}_n A^+)$, and thence, by direct sum stabilization, on each $\pi_i(B\operatorname{GL} A^+)$. Alternatively, one may consider the action on each $\pi_i(B\operatorname{Int}_n A^+)$. Tensor product stabilization then gives an action on $\pi_i(B\operatorname{Int}_{\otimes} A^+)$ that we show is trivial.

Since the action of $S \in \operatorname{Int}_n A$ on M_nA is an automorphism, it preserves both $\operatorname{GL}_n A$ and, for $n \geq 3$, E_nA (cf. (3.7), (3.8)). To stabilize, we exploit Lemma 12.2, and let the result of S acting on $B \in M_mA$ be

$$(B \oplus I_{m(n-1)})^{(I_m \otimes S)} \in M_{mn}A.$$

Now if $B \in E_mA$, then $B \oplus I_{m(n-1)} \in E_{mn}A$, so that we obtain the induced action

$$\operatorname{GL}_m A/\, E_mA \times \operatorname{Int}_n A \longrightarrow \operatorname{GL}_{mn} A/\, E_{mn}A.$$

Furthermore, this is consistent with the usual stabilization $B \mapsto B \oplus I$ of the groups GL and E, and with our stabilization $S \mapsto I{\otimes}S$ of Int. It therefore produces natural actions

$$K_1(A) \times \operatorname{Int}_n A \longrightarrow K_1(A)$$

and

$$K_1(A) \times \operatorname{Int}_{\otimes} A \longrightarrow K_1(A).$$

Of course, when $S \in I_n \otimes A^{\times}$, $S^{\#}$ is just the identity automorphism. Also observe that, if $S, B \in \operatorname{GL}_n A$, then

$$B^S = S^{-1}BS \equiv B \pmod{E_{2n}A},$$

by a well-known precursor to the usual Whitehead Lemma. Thus, from the monomorphisms

$$H : (\operatorname{Int}_n A)/(A^{\times} \cdot \operatorname{GL}_n A) \to {}_n\operatorname{Pic} A$$

and their colimit

$$(\operatorname{Int}_{\otimes} A)/\,(A^{\times} \cdot \operatorname{GL}_{\otimes} A) \to \operatorname{Tor}(\operatorname{Pic} A),$$

we obtain a natural action of $\operatorname{Im} H$ on $K_1(A)$.

To make further progress, it is helpful to consider the general case. Let M be a monoid. Then the above arguments show that there is an action

$$U(M) \times M^{\times}/(M^{\times} \cap \mathcal{Z}(M)) \to U(M),$$

which, for any characteristic subgroup K of the unit group $U(M)$, induces an action

$$U(M)/K \times M^{\times}/(M^{\times} \cap \mathcal{Z}(M)) \to U(M)/K.$$

In particular, when K is the commutator subgroup of $U(M)$, we obtain the further action

$$U(M)_{\mathrm{ab}} \times M^{\times}/\operatorname{Sen}(M) \to U(M)_{\mathrm{ab}},$$

which generalizes the one specified above, where $M = \bigcup M_nA$. An alternative generalization is to consider as characteristic subgroup the perfect radical $\mathcal{P}U(M)$.

This case provides a useful extension of the action, by the following device, which we apply first in the present situation where each monoid Σ_n is actually a group.

Lemma 14.1. *Suppose that a monoid Σ is a nested union of submonoids*

$$\Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

such that, for each n, $(\Sigma_n)^\times = \Sigma_n \subseteq \mathcal{P}G \cdot C_\Sigma(\Sigma_n)$, where G denotes the enveloping group $\Sigma[\Sigma^{-1}]$, with perfect radical $\mathcal{P}G$, and C denotes the centralizer. Then, for each $s \in \Sigma$, the endomorphism $s^\#$ of Σ induces the identity map on the homotopy groups of $B\Sigma^+$.

Proof. We implicitly use Lemmas 1.1(d) and 1.4. By hypothesis, for $s \in \Sigma_m$, the action of $s^\#$ on any Σ_n with $n \geq m$ coincides with conjugation γ_σ by some element σ of $\mathcal{P}G$. Then by an easy generalization of [23](1.1.9), each map $B\Sigma_n^+ \to B\Sigma^+$ induced by γ_σ is homotopic to that induced by inclusion. Since the plus-construction is a localization [9], we have $B\Sigma^+$ as the inductive limit of $B\Sigma_n^+$. However, the possibility of phantom maps thwarts the conclusion that $B\gamma_\sigma^+ : B\Sigma^+ \to B\Sigma^+$ is homotopic to the identity. Nevertheless, since homotopy groups preserve direct limits, each $\pi_i(B\gamma_\sigma^+)$ is the identity. □

In the case at hand, any $G \in \mathrm{GL}_n A$ may be written in $\mathrm{GL}\, A$ as

$$\begin{aligned} G &= (G \oplus G^{-1})(I_n \oplus G) \\ &\in E_{2n}A \cdot (I_n \oplus \mathrm{GL}_n A) \\ &\subseteq \mathcal{P}\,\mathrm{GL}\, A \cdot C_{\mathrm{GL}\, A}(\mathrm{GL}_n A), \end{aligned}$$

again by standard Whitehead Lemma arguments. Thus, by the above lemma, the action on each $\pi_i(B\,\mathrm{GL}\, A^+)$ induced from G-conjugation is the identity. Hence, for each $n \geq 2$, $i \geq 1$, we obtain a natural action of $\mathrm{Int}_n A/(A^\times \cdot \mathrm{GL}_n A)$ on K_iA. Moreover, since the action of $S \in \mathrm{Int}_n A$ is defined via the action of $I_m \otimes S$ for sufficiently large m, these actions are coherent with respect to the inclusions $\mathrm{Int}_n A \hookrightarrow \mathrm{Int}_{mn} A$, $S \mapsto I_m \otimes S$. The injective derivations

$$\begin{aligned} \bar{H}_0 : (\mathrm{Int}_n A)/(A^\times \cdot \mathrm{GL}_n A) &\longrightarrow {}_n\tilde{K}_0(A) \\ S &\longmapsto [A\langle S\rangle] - [A] \end{aligned}$$

are evidently also coherent with respect to these inclusions, and so define an injective derivation

$$\bar{H}_0 : (\mathrm{Int}_\otimes A)/\,(A^\times \cdot \mathrm{GL}_\otimes A) \longrightarrow \mathrm{Tor}\, \tilde{K}_0(A).$$

Therefore there is induced an action of $\mathrm{Im}\, \bar{H}_0$ on each K_iA, $i \geq 1$.

We now use Lemma 14.1 to obtain information about the action. For this application we require the following result.

Lemma 14.2. *If $S \in \mathrm{Int}_n A$, then*

$$\begin{aligned} I_2 \otimes (\mathrm{adj}\, S \otimes S) &= (I_2 \otimes \mathrm{adj}\, S \otimes I_n)(I_{2n} \otimes S) \\ &\in A^\times \cdot E_{2n^2} A \cap C_{\mathrm{Int}_{2n^2} A}(\mathrm{Int}_n A) \cdot (I_{2n} \otimes S). \end{aligned}$$

Proof. Let $s = \det S \in A^\times$. Then from the ideal equation $A\langle S\rangle\, A\langle \mathrm{adj}\, S\rangle = A(\det S)$, we have that $\mathrm{adj}\, S \otimes S = sS_0$ for some $S_0 \in M_{n^2}A$. After localization, $S_0 = S^{-1} \otimes S$. This makes S_0 invertible, with inverse given by $S \otimes \mathrm{adj}\, S = sS_0^{-1}$. So

$$(\mathrm{adj}\, S \otimes S) \oplus (S \otimes \mathrm{adj}\, S) = s(S_0 \oplus S_0^{-1}) \in A^\times \cdot E_{2n^2}A.$$

Now

$$S \otimes \mathrm{adj}\, S = \sigma^{-1}(\mathrm{adj}\, S \otimes S)\sigma$$

by Lemma 12.4, where σ is a permutation in $\mathrm{GL}_{n^2} A$. Thus

$$I_2 \otimes (\operatorname{adj} S \otimes S) = (I_{n^2} \oplus \sigma)^{-1}(I_2 \otimes \operatorname{adj} S \otimes S)(I_{n^2} \oplus \sigma),$$

which lies in $A^\times \cdot E_{2n^2}A$ since $E_{2n^2}A$ is permutation-invariant. Finally, because $\mathrm{Int}_n A$ embeds in $\mathrm{Int}_{2n^2} A$ via $T \mapsto I_2 \otimes I_n \otimes T$, so evidently

$$I_2 \otimes \operatorname{adj} S \otimes I_n \in C_{\mathrm{Int}_{2n^2} A}(\mathrm{Int}_n A),$$

as claimed. □

From the above two lemmas it now follows that, for each $S \in \mathrm{Int}_n A$, $S^\#$ induces the identity map on all groups $\pi_i(B\,\mathrm{Int}_\otimes A^+)$ and hence on all $\mathbb{Q}\otimes K_iA$, $i \geq 1$. In particular, this affords a direct proof that $B\,\mathrm{Int}_\otimes A^+$ is a simple space, which is of course also a consequence of Theorem 13.3.

Acknowledgements. I am pleased to acknowledge the stimulus given to this long-gestating work by W. van der Kallen and M. Karoubi, after I gave an Oberwolfach lecture about its precursor [11]. I would like to thank M. Karoubi for his continuing encouragement, and him and T. Lambre for many useful conversations. Helpful discussions with J.-P. Serre and C. A. Weibel are also gratefully acknowledged.

References

[1] D Arlettaz: Algebraic K-theory of rings from a topological viewpoint, *Publ Mat* **44** (2000), 3-84.
[2] H Bass: *Lectures on Topics in Algebraic K-Theory*, Tata Inst Fund Research (Bombay, 1967).
[3] H Bass: *Algebraic K-Theory*, Benjamin (New York, 1968).
[4] H Bass, J Milnor & J-P Serre: Solution of the congruence subgroup problem for SL_n ($n \geq 3$), and Sp_{2n} ($n \geq 2$), *Inst Hautes Études Publ Math* **33** (1967).
[5] H. Bass: Clifford algebras and spinor norms over a commutative ring, *Amer. J. Math.* **96** (1974), 156–206.
[6] A J Berrick: The plus-construction and fibrations, *Quart J Math Oxford Ser (2)* **33** (1982), 149-157.
[7] A J Berrick: *An Approach to Algebraic K-Theory*, Pitman Research Notes in Math 56 (London, 1982).
[8] A J Berrick: Characterisation of plus-constructive fibrations, *Advances in Math* **48** (1983), 172-176.
[9] A J Berrick: The plus-construction as a localization, *Algebraic K-Theory and its Applications: Procs ICTP 1997,* World Scientific (Singapore, 1999), 313-336.
[10] A J Berrick: Algebraic K-theory and algebraic topology, *Algebraic K-Theory and its Applications: ICTP Lecture Notes,* ICTP (Trieste, 2003), 1-94.
[11] A J Berrick, K L Boey & K H Leung: Row versus column operations, *J Algebra* **184** (1996), 897-911.
[12] A J Berrick & M E Keating: *Modules and Categories, with K-Theory in View*, Cambridge Studies in Advanced Math 67, Cambridge Univ Press (Cambridge, 2000).
[13] N Bourbaki: *Algèbre Commutative, Chap 7: Diviseurs*, Hermann (Paris, 1965).
[14] H Cohen: *A Course in Computational Algebraic Number Theory*, Grad Texts Math 138, Springer (Berlin, 1993).
[15] C W Curtis & I Reiner: *Methods of Representation Theory - With Applications to Finite Groups and Orders, Vol I*, John Wiley & Sons (New York, 1981).
[16] H Flanders: Tensor and exterior powers, *J Algebra* **7** (1967), 1-24
[17] P Gabriel & M Zisman: *Calculus of Fractions and Homotopy Theory*, Ergeb Math 35, Springer-Verlag (New York, 1967).
[18] C H Giffen: Loop spaces for the Q-construction, *J Pure Appl Algebra* **52** (1988), 1–30.
[19] H Gillet & D R Grayson: The loop space of the Q-construction, *Illinois J Math* **31** (1987), 574–597.
[20] D R Grayson: Higher algebraic K-theory: II (after D Quillen), in: Algebraic K-Theory, Evanston 1976, Springer Lect Notes in Math 551 (Berlin, 1976), 217-240.
[21] R C Heitmann: Generating ideals in Prüfer domains, *Pacific J Math* **62** (1976), 117-126.

[22] M Karoubi & T Lambre: Quelques classes caractéristiques en théorie des nombres, *(Crelle's) J reine angew Math* **543** (2002), 169-186.
[23] J-L Loday: K-Théorie algébrique et représentations de groupes, *Ann Sci Ec Norm Sup* (4^e) **9** (1976), 309-377.
[24] S Mac Lane: *Categories for the Working Mathematician*, Graduate Texts in Math 5, Springer-Verlag (New York, 1971).
[25] J P May: *E_∞ Ring Spaces and E_∞ Ring Spectra*, Lect Notes in Math 577, Springer-Verlag (Berlin, 1977).
[26] D McDuff & G Segal: Homology fibrations and the "Group-completion" Theorem, *Inventiones Math* **31** (1976), 279-284.
[27] J Milnor: *Introduction to Algebraic K-Theory*, Ann Math Studies 72, Princeton Univ Press (Princeton NJ, 1971).
[28] S Mitchell: On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint, *Algebraic Topology and Its Applications*, Math. Sci. Res. Inst. Publ. 27, Springer (New York, 1994), 163–240.
[29] M. Pohst & H. Zassenhaus: *Algorithmic Algebraic Number Theory*, Cambridge Univ. Press (Cambridge, 1989).
[30] D Quillen: Higher Algebraic K-theory I, in: *Algebraic K-Theory I*, Springer Lect Notes in Math 341, (Berlin, 1973), 85–147.
[31] A Rosenberg & D Zelinsky:Automorphisms of separable algebras, *Pacific J Math* **11** (1961), 1109-1117.
[32] J Rosenberg: *Algebraic K-Theory and its Applications*, Graduate Texts in Math 147, Springer-Verlag (New York, 1994).
[33] L Rowen: *Ring Theory, Vol I*, Pure & Applied Math 127, Academic Press (San Diego, 1988).
[34] L N Vaserstein: Normal subgroups of the general linear groups over Banach algebras, *J Pure & Applied Algebra* **41** (1986), 99-112.
[35] C A Weibel: K-theory of Azumaya algebras, *Proc Amer Math Soc* **81** (1981), 1-7.
[36] C A Weibel: KV-theory of categories, *Trans Amer Math Soc* **267** (1981), 621-635.
[37] C A Weibel: *Algebraic K-Theory*, http://math.rutgers.edu/~weibel/Kbook.html

Department of Mathematics, National University of Singapore, Kent Ridge 117543, SINGAPORE,
E-mail address: berrick@math.nus.edu.sg