PERFECT AND ACYCLIC SUBGROUPS OF FINITELY PRESENTABLE GROUPS

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Abstract. We consider acyclic groups of low dimension. To indicate our results simply, let G' be the nontrivial perfect commutator subgroup of a finitely presentable group G. Then $\mathrm{def}(G) \leq 1$. When $\mathrm{def}(G) = 1$, G' is acyclic provided that it has no integral homology in dimensions above 2 (a sufficient condition for this is that G' be finitely generated); moreover, G/G' is then Z or Z^2. Natural examples are the groups of knots and links with Alexander polynomial 1. We give a further construction based on knots in $S^2 \times S^1$. In these geometric examples, G' cannot be finitely generated; in general, it cannot be finitely presentable. When G is a 3-manifold group it fails to be acyclic; on the other hand, if G' is finitely generated it has finite index in the group of a $\mathbb{Q}$-homology 3-sphere.

0. Introduction

Historically, among the acyclic groups, namely those groups with the same untwisted homology as the trivial group, it is the 'large' examples that have received the most attention. These examples typically arise as automorphism groups of large mathematical structures, and occur naturally where such objects are found. Consequently they have been important to the study of foliations, algebraic K-theory, stable homotopy theory and algebraic topology (see [2]). Although the literature has for more than half a century contained examples of acyclic groups drawn from combinatorial group theory and low-dimensional topology, it was only in the recent paper [3] that a systematic study was initiated. This paper continues that investigation.

The acyclic groups of [3] are the commutator subgroups of groups described algebraically or geometrically. In the algebraic setting are the 'acyclic groups of Baumslag-Gruenberg type'. They are commutator subgroups of finitely presentable groups of deficiency 1, and have geometric dimension 2. On the other hand, there are acyclic commutator subgroups of fundamental groups of certain open 3-manifolds; such fundamental groups must have positive deficiency and, again, have geometric dimension 2. We therefore concentrate on finitely presentable

Date: 18/04/02.
1991 *Mathematics Subject Classification.* Primary 20J05. Secondary 57M07.

groups of positive deficiency and investigate the possibility that their commutator subgroups are acyclic. It seems that typically these acyclic groups are not finitely generated. The question as to whether they can be finitely generated relates, in the more general setting of finitely generated perfect subgroups, to the Whitehead conjecture on aspherical 2-complexes and so has independent interest.

After preparatory material in the first section below, we further explore the groups of [3], and obtain a natural generalization for acyclic groups of Baumslag-Gruenberg type. In the geometric situation, we exhibit a number of properties of the wild arc groups introduced there. Those acyclic groups are countable, and in many cases recursively presented. On the other hand, it turns out that there are severe constraints on the possible finitely generated perfect subgroups of groups arising in the low-dimensional setting. They are explored in the subsequent two sections, collated according as they are algebraically or geometrically defined. We obtain notable parallels between acyclicity results for the perfect commutator subgroup G' of a positive deficiency group G under the assumptions that G' is a wild arc group, G is of homological dimension 2, or G' is finitely generated. For example, we prove the following (part (a) is proved in Theorem 3.11, and part (b) in Lemma 1.1 below).

Theorem 0.1. *Let G be a finitely presentable group of positive deficiency whose commutator subgroup G' is perfect. Then the following hold.*

(a) *If G' is finitely generated, then G has homological dimension* 2*, or $G \cong Z$, the infinite cyclic group.*
(b) *If G has homological dimension* 2*, then G' is acyclic.* □

A significant novelty of this work is its application of recent developments in L^2-homology so as to obtain results wholly within the realm of group theory. As well as the above, this leads to a classification theorem (Theorem 3.7 below) that describes those finitely presentable groups of positive deficiency whose commutator subgroup contains an infinite, finitely presentable normal subgroup.

1. Notational conventions and the key lemma

We quickly review some terms and notation that we use. Homology is always with trivial integer coefficients, unless expressly stated otherwise. A *finitely presentable* group G is one that for some natural numbers m, n can be generated by m generators subject to n relators. This is linked to the homology of G by the inequality [12]

$$m - n \leq \text{rank} G_{\text{ab}} - s(H_2(G)),$$

where $s(H_2(G))$ denotes the minimum number of generators of $H_2(G)$, and the *abelianization* $G_{\text{ab}} = G/G' = H_1(G)$. We always write G'

for the *commutator subgroup* $[G, G]$ of G. Since finite presentability ensures that the terms on the right are both finite, the number $m - n$ achieves a maximum, the *deficiency* of G, written $\operatorname{def}(G)$. Then the above inequality yields that when G is *perfect* (that is, $G_{\text{ab}} = 0$) or finite (implying that G_{ab} is torsion) then $\operatorname{def}(G) \leq 0$. A group G is *superperfect* if it is perfect and $H_2(G) = 0$. Finitely presentable groups are FP_2 (see [4] for the definition).

Other numerical invariants of interest here are as follows. If X is an *aspherical* space (higher homotopy groups $\pi_i(X)$, $i \geq 2$, vanish) with $\pi_1(X) \cong G$ (in other words, X is a $K(\pi, 1)$) and is moreover a CW-complex of finite dimension, then G has finite *geometric dimension*, and $\operatorname{gd}(G)$ is the minimum possible dimension for such X. This number serves as an upper bound for the *homological dimension* $\operatorname{hd}(G)$ (again defined in [4] - here we take the dimension over the ring $\mathbb{Z}$), which in turn is an upper bound for the *trivial homological dimension* thd, namely the dimension of the highest nonzero trivial-coefficients homology group of G. Thus, an *acyclic* group is precisely one with trivial homological dimension zero. Likewise, the *cohomological dimension* $\operatorname{cd}(G)$ lies between $\operatorname{hd}(G)$ and $\operatorname{gd}(G)$. (Note that nontrivial cyclic subgroups of acyclic groups have $\operatorname{thd} > 0$, whereas $\operatorname{cd}(G)$ and $\operatorname{hd}(G)$ bound the corresponding dimensions of subgroups).

We have occasion to use the *L^2-Betti numbers* $\beta_i^{(2)}(G) \in [0, \infty]$ (for $i = 0, 1, \ldots$), which are invariants of a finitely presentable group G. These are defined in, for example, [25], while a brief survey of relevance here may be found in [20]. In matters concerning 3-manifolds, our basic reference is [16].

Since $\operatorname{thd}(G') \leq \operatorname{hd}(G)$ ([4](4.9)), part (b) of Theorem 0.1 is an immediate consequence of the following.

Lemma 1.1. *Let G be a finitely presentable group whose commutator subgroup G' is perfect. Then the following hold.*

(a) $\operatorname{def}(G) \leq 1$;
(b) *if* $\operatorname{def}(G) = 0$ *and G is perfect, then* $H_2(G') = H_2(G) = 0$;
(c) *if* $\operatorname{def}(G) = 1$, *then* $H_2(G') = 0$ *and G_{ab} is free abelian;*
(d) *if* $\operatorname{thd}(G) \leq 2$ *and* $H_2(G') = 0$, *then* $G_{\text{ab}} = 0$, Z *or* Z^2;
(e) *if* $\operatorname{def}(G) = 1$ *and* $\operatorname{thd}(G') \leq 2$, *then G' is acyclic and* $G_{\text{ab}} \cong Z$ *or* Z^2;
(f) *if* $\operatorname{gd}(G) \leq 2$, *then the following are equivalent.*
   (i) *The homomorphism $G \twoheadrightarrow G_{\text{ab}}$ induces an isomorphism on homology with arbitrary local coefficients induced from G_{ab}.*
   (ii) *The homomorphism $G \twoheadrightarrow G_{\text{ab}}$ induces an isomorphism on integral homology.*
   (iii) *Either* $\operatorname{def}(G) = 0$ *and G is perfect, or* $\operatorname{def}(G) = 1$.
   (iv) *G' is acyclic.*

Proof. Let X be the 2-complex with fundamental group G corresponding to a presentation of maximal deficiency. Then $\chi(X) = 1 - \mathrm{def}(G)$. Let A be the quotient of $G_{\mathrm{ab}} = G/G'$ by its torsion subgroup, and let K be the kernel of the projection of G onto A. Then $A \cong Z^\beta$, where $\beta = \beta_1(G;\mathbb{Z})$, and K/G' is a finite abelian group. The cellular chain complex $C_*(X_K\,;R)$ for the covering space X_K associated to K is a finite free $R[A]$-chain complex, and $R[A]$ is an integral domain if R is an integral domain (for example, if R is $\mathbb{Z}$ or a field).

(**a**) Let F be the field of fractions of $\mathbb{Q}[A]$. Counting ranks in $C_*(X_K;\mathbb{Q}) \otimes_{\mathbb{Q}[A]} F$ gives $\chi(X) = \Sigma(-1)^q \mathrm{dim}_F(H_q(X_K;\mathbb{Q}) \otimes_{\mathbb{Q}[A]} F) \geq 0$, since $H_1(X_K;\mathbb{Q}) = (K/G') \otimes \mathbb{Q} = 0$. Hence $\mathrm{def}(G) \leq 1$. (An alternative proof of this result appears in [5].)

(**b**) This is immediate from the inequality at the beginning of the section.

(**c**) The modules $H_0(X_K;R) = R$ and $H_1(X_K;R) \cong (K/G') \otimes R$ are torsion $R[A]$-modules, since $\beta \geq 1$, and so $H_2(X_K;R)$ has rank $\chi(X) = 0$. Since $H_2(X_K;R)$ is a submodule of $C_2(X_K;R)$ it is torsion-free and so is 0. Hence $H_2(K;R) = 0$ also, since it is a quotient of $H_2(X_K;R)$. In particular, $H_2(K) = 0$ and $H_2(K;\mathbb{F}_p) = 0$ for all primes p. Now recall the exact sequence of low degree from the LHS spectral sequence:

$$H_2(K;\mathbb{F}_p) \longrightarrow H_2(K/G';\mathbb{F}_p) \longrightarrow H_0(K/G';H_1(G';\mathbb{F}_p)).$$

Since $H_1(G';\mathbb{F}_p) = 0$, it follows that the finite abelian group K/G' has $H_2(K/G';\mathbb{F}_p) = 0$ for all primes p. Hence $K = G'$, and so we have that $H_2(G') = 0$ and $G_{\mathrm{ab}} \cong Z^\beta$ is free abelian.

(**d**) Observe that in the LHS spectral sequence for the extension

$$G' \hookrightarrow G \twoheadrightarrow G_{\mathrm{ab}},$$

nonzero elements of $H_3(G_{\mathrm{ab}};H_0(G'))$ would survive to $H_3(G)$, because $H_1(G_{\mathrm{ab}};H_1(G'))$ and $H_0(G_{\mathrm{ab}};H_2(G'))$ are both zero. However $H_3(G)$ is zero, by assumption. Thus the finitely generated abelian group G_{ab} has $H_3(G_{\mathrm{ab}}) = 0$, which forces G_{ab} to be free abelian, of rank at most two.

(**e**) If $\mathrm{def}(G) = 1$, then by (c) $H_2(G') = 0$, while $H_1(G') = 0$, by hypothesis. So if moreover $\mathrm{thd}(G') \leq 2$ then G' is acyclic. The LHS spectral sequence gives an isomorphism $H_2(G) \cong H_2(G_{\mathrm{ab}})$, which by (c) is just $H_2(Z^\beta)$. As $\binom{\beta}{2} = \beta_2(G;\mathbb{Z}) \leq \beta_2(X;\mathbb{Z}) = \beta - 1$ and $\beta \geq 1$ we must have $\beta = 1$ or 2.

(**f**) Clearly (i) implies (ii), while the equivalence of (i) and (iv) is an LHS spectral sequence argument (see [1]ch.4 for details). So it remains to show that (ii) implies (iii), and that (iii) yields (iv). We may assume that $X \simeq K(G,1)$, so that

$$\mathrm{def}(G) = 1 - \chi(X) = \mathrm{rank} G_{\mathrm{ab}} - \mathrm{rank} H_2(G).$$

Starting with (ii), we have $H_3(G_{\rm ab}) = H_3(G) = 0$. Then, as in (d) above, $G_{\rm ab} = 0$, Z or Z^2. In the first case G is perfect (indeed acyclic) and the above formula gives $\mathrm{def}(G) = 0$. On the other hand, when $G_{\rm ab}$ is trivial and $\mathrm{def}(G) = 0$, then from (b) $H_2(G) = 0$ too, making G acyclic. In the other cases, $G_{\rm ab} \cong Z$ or Z^2, the formula gives $\mathrm{def}(G) = 1$. Finally, we may apply (e) to obtain acyclicity of G' when $\mathrm{def}(G) = 1$, since $\mathrm{thd}(G') \leq \mathrm{gd}(G) \leq 2$ □

Remarks 1.2. (1) For the proof of (f), it suffices to assume that $\mathrm{hd}(G) \leq 2$ and G is *efficient*, meaning that $\mathrm{def}(G) = \mathrm{rank}G_{\rm ab} - s(H_2(G))$.

(2) The hypotheses in (a)-(e) cannot be weakened much further, as the examples Z, $Z/2Z$, Z^3 and Z^3 (respectively) show.

(3) The alternatives in (f)(iii) are exemplified by $G = 1$ and Z. There are also nonabelian groups G of geometric dimension 2 with G' acyclic and $G_{\rm ab} = 0$, Z or Z^2, respectively. The groups of Theorem 2.2 below are examples of the second kind; see Remark 2.9 for examples of the third type. An algebraic construction of groups of all three types appears in the next example.

Example 1.3. Let $H = \langle x_n \mid x_n[x_n, x_{n+1}]\rangle_{n\in\mathbb{Z}/4\mathbb{Z}}$ be Higman's acyclic group. Then $\mathrm{def}(H) = 0$ and $\mathrm{gd}H = 2$ [11]. Let A be a free abelian group of rank $\beta = 0, 1$ or 2. Then the free product $G = H * A$ has abelianization Z^β; it is finitely presentable, of deficiency $0, 1$ or 1 respectively; and it has geometric dimension at most 2. Its commutator subgroup is just the normal closure of H in G, and so perfect. So by the key lemma it is acyclic.

These examples lead to the answer to a well-known question about the possible cohomological dimension of an acyclic group.

Theorem 1.4. *Let n be an integer > 1. Then there is an acyclic group of cohomological dimension n with a finite n-dimensional Eilenberg-Mac Lane complex.*

Proof. The finite 2-complex associated to the presentation for H given above is aspherical, so the assertion holds for $n = 2$.

Let $G = (H \times Z) *_Z H$, where the amalgamated subgroup Z is identified with the direct factor in $H \times Z$ on the left and with any infinite cyclic subgroup of H on the right. Then there is a finite 3-dimensional $K(G, 1)$ complex, and $H \times Z < G$, so $\mathrm{cd}G = 3$. A Mayer-Vietoris argument shows that G is acyclic. Thus the assertion holds for $n = 3$.

If $k \geq 2$ the groups H^k and $H^{k-1} \times G$ are acyclic, by the Künneth Theorem, and have cohomological dimensions $2k$ and $2k + 1$, by a spectral sequence corner argument (see Theorem 5.5 of [4] and page

223 of [9]). Since the corresponding cartesian products of Eilenberg-Mac Lane spaces are finite complexes of dimensions $2k$ and $2k+1$, respectively, the theorem is true in general. □

Since groups of cohomological dimension 1 are free there are no acyclic groups of dimension 1.

A related remark: let π be a high-dimensional knot group, with weight element (normal generator) t, and let a be one of the generators for H (in the standard presentation). Then $H \times \pi$ is a high-dimensional knot group, with weight element ta^{-1}. Hence $H^n \times \pi$ is a high-dimensional knot group, for all $n \geq 1$. Starting with $\pi = Z$ and π any nontrivial classical knot group, we get high-dimensional knot groups with cohomological dimension n, for all $n \geq 1$.

2. Wild arc groups and generalizations

One of the main themes of [3] was to provide a geometric setting for examples of what are there called acyclic groups of Baumslag-Gruenberg type. Such groups are characterized by the following theorem.

Theorem 2.1. [3] *Let*

$$B = \langle x_n \mid w(x_n, x_{n+1}, \ldots, x_{n+k}) \rangle_{n \in \mathbb{Z}}$$

where w is a word in the free group of rank $k+1$. Then the following statements are equivalent.

(a) *w has exponent sum zero in k of its variables, and exponent sum ± 1 in the remaining variable.*
(b) *B is a perfect group.*
(c) *Define $G = \big\langle x, y \mid w(x, yxy^{-1}, \ldots, y^k xy^{-k}) \big\rangle$. Then*
  (i) *G_{ab} is infinite cyclic,*
  (ii) *B is isomorphic to the commutator subgroup of G, and*
  (iii) *B is acyclic.*

□

A fruitful source of examples of such acyclic groups is the following geometric situation.

Theorem 2.2. [3] *Let λ be a smooth knot in $S^2 \times S^1$ such that $[\lambda]$ generates $H_1(S^2 \times S^1)$. Then the connected infinite cyclic cover of $(S^2 \times S^1) \setminus \lambda$ is the complement of a wild arc κ in S^3 with the following properties.*

(i) *$S^3 \setminus \kappa$ is aspherical;*
(ii) *$\pi_1(S^3 \setminus \kappa)$ is the commutator subgroup π' of $\pi = \pi_1((S^2 \times S^1) \setminus \lambda)$ with $\pi/\pi' \cong Z$;*
(iii) *$\pi_1(S^3 \setminus \kappa)$ is acyclic.*

□

We record a number of interesting facts about the groups defined in this way. In particular, (d) below solves a problem raised in [3].

Theorem 2.3. *Let λ be a smooth knot in $S^2 \times S^1$ such that $[\lambda]$ generates $H_1(S^2 \times S^1)$, and let M be the closed complement of a tubular neighbourhood of λ in $S^2 \times S^1$. Then*

(a) *M is an aspherical Haken manifold, and is an integral homology circle;*
(b) *$\pi = \pi_1(M)$ is finitely presentable,* $\mathrm{def}\pi = 1$ *and* $\mathrm{gd}\pi \leq 2$*;*
(c) *π is locally indicable and in particular contains no nontrivial finitely generated perfect subgroups;*
(d) *π is residually finite;*
(e) *all L^2-Betti numbers of π vanish.*

Proof. **(a)** M is aspherical because, by the theorem above, its cover is. In particular, every embedded 2-sphere in M bounds a 3-cell, which must be standard as $M \subset S^2 \times S^1$. Therefore M is irreducible. Since π' is acyclic, the integral homology of M is just that of the quotient $\pi/\pi' = Z$. Therefore M is Haken, since it is irreducible and $\beta_1(M;\mathbb{Z}) > 0$.

(b) This follows from (a). As every compact 3-manifold with nonempty boundary collapses to a finite 2-complex, π is finitely presentable and $\mathrm{gd}\pi \leq 2$. As $\chi(M) = 0$, the presentation derived from such a 2-complex has deficiency 1, and as $\pi/\pi' \cong Z$ this is best possible.

(c) This follows from [21](6.2), since M is irreducible and $H_3(M) = 0$ (so M is not a rational homology 3-sphere).

(d) This follows from Theorem 1.1 of [17], since M is Haken.

(e) Since M is a $K(\pi, 1)$, the L^2-Betti numbers of π are those of M, and these are all 0, by Theorem 0.1 of [24]. □

Now recall that Example 1.2 of [3] exhibits groups of Baumslag-Gruenberg type that have no nontrivial finite quotient. Accordingly, by (d) of the theorem above such groups cannot be derived from wild arcs constructed as in Theorem 2.2. On the other hand, [3] also gives an example of a group π as above that does not appear to be a 1-relator group (see Example 2.6 below). Nevertheless, by (b) of the theorem above, any such group π must be a group G as in the next result. This theorem generalizes Theorem 2.1 above (the case $m = n = 1$) because in the 1-relator case it is guaranteed that G, and so B, has homological dimension at most 2 [26].

Theorem 2.4. *Let*

$$B = \langle x_{i,r} \mid w_j(x_{1,r}, \ldots, x_{1,r+k}, \ldots, x_{m,r}, \ldots, x_{m,r+k}) \rangle_{r\in\mathbb{Z};\, i\in\{1,\ldots,m\};\, j\in\{1,\ldots,n\}}$$

where each w_j is a word in the free group of rank $m(k+1)$. Then the following statements are equivalent.

(a) *Write the abelianization of B additively, so that its defining relations become*

$$\sum_{i=1}^{m}\sum_{q=0}^{k}\alpha_{ij}^{q}x_{i,r+q}=0, \qquad r\in\mathbb{Z}; \quad j\in\{1,...,n\}.$$

Then $m\le n$, and the $m\times m$ minors of the $m\times n$ matrix with (i,j)-entry $\sum_{q=0}^{k}\alpha_{ij}^{q}t^{q}$ in $\mathbb{Z}[t,t^{-1}]$ generate $\mathbb{Z}[t,t^{-1}]$ as an ideal.

(b) *B is a perfect group.*

If further $m=n$ and $\mathrm{thd}B\le 2$ *then each statement is equivalent to the following.*

(c) *Define*

$$G=\left\langle x_i, y \mid w_j(x_1,\dots,y^kx_1y^{-k},\dots,x_m,\dots,y^kx_my^{-k})\right\rangle_{i,j\in\{1,\dots,m\}}.$$

Then

(i) *G_{ab} is infinite cyclic,*

(ii) *B is isomorphic to the commutator subgroup of G, and*

(iii) *B is acyclic.*

Proof. (a)⇔(b). To check first that the condition $m\le n$ is necessary for (b), we form the group

$$G=\left\langle x_i, y \mid w_j(x_1,\dots,y^kx_1y^{-k},\dots,x_m,\dots,y^kx_my^{-k})\right\rangle_{i\in\{1,\dots,m\};\, j\in\{1,\dots,n\}}$$

and observe that B sits inside G as $x_{i,r}=y^rx_iy^{-r}$. So because B is perfect, $x_i=x_{i,0}\in B\subseteq G'$. In fact, B is generated by the conjugates of the x_i, hence normal in G, with G/B an infinite cyclic group generated by the image of y. Thus also $G'\subseteq B$. So $B=G'$. Then by the proof of Lemma 1.1(b)(i), when B is perfect $\mathrm{def}(G)\le 1$ and so $m\le n$. Write $\Lambda=\mathbb{Z}[t,t^{-1}]$ and let Λ^m denote the free right Λ-module generated by $e_1,...,e_m$ say. Then, by analogy with the construction of the Alexander module, we observe that the abelianization of B is a Λ-module, the t-action corresponding to the shift $x_{i,r}\mapsto x_{i,r+1}$. Thus the correspondence $x_{i,q}\mapsto t^qe_i$ makes B_{ab} isomorphic to the quotient of Λ^m by the submodule generated by the n expressions

$$\rho_j=\sum_{i=1}^{m}\sum_{q=0}^{k}\alpha_{ij}^{q}t^{q}e_i.$$

Therefore the ideal generated by the $m\times m$ minors of the $m\times n$ matrix with (i,j)-entry $\sum_{q=0}^{k}\alpha_{ij}^{q}t^{q}$ in Λ is the zero-th elementary or determinantal ideal (or first Fitting ideal) of the Λ-module B_{ab}. So, by a standard result (e.g.[6] p.573) $B_{\mathrm{ab}}=0$ if and only if this ideal is the full ring Λ.

Of course, (c)(iii) implies (b).

(b)$\Longrightarrow$(c). We have already shown that (ii) holds and so $G_{\mathrm{ab}} = G/B$, which was observed to be infinite cyclic. Part (iii) now follows from Theorem 0.1(b). □

Remark 2.5. Let X be the 2-complex corresponding to the presentation of G given in part (c) of the theorem, and let $Y = X \cup_f e^2$ where $f : S^1 \to X$ represents $y \in \pi_1(X) = G$. If G is normally generated by y, then $\pi_1(Y) = 1$, and so Y is contractible (since it is 2-dimensional and $\chi(Y) = \chi(X) + 1 = 1$). Thus, *if the Whitehead Conjecture is true,* X is aspherical, and so $\mathrm{gd}G \leq 2$. In particular, the hypothesis $\mathrm{thd}B \leq 2$, of part (c) above, holds. (The condition that G be the normal closure of y can be paraphrased in terms of B and the generators $x_{i,r}$.)

Example 2.6. The Borromean stitch introduced in [3] leads to a wild arc whose group B has the presentation with two families of generators a_r and b_r and the two families of relators

$$a_{r+1}b_{r+1}a_rb_ra_r^{-1}b_{r+1}^{-1}a_rb_r^{-1}a_r^{-1},$$
$$a_{r+1}b_{r+1}a_rb_rb_{r+1}^{-1}a_{r+1}^{-1}b_{r+1}a_r^{-1}b_{r+1}^{-1},$$

where $r \in \mathbb{Z}$. So the corresponding presentation of its abelianization has two families of generators a_r and b_r and the two families of relations

$$0a_r + 1a_{r+1} + 0b_r + 0b_{r+1} = 0,$$
$$0a_r + 0a_{r+1} + 1b_r + 0b_{r+1} = 0.$$

Here $k = 1$, $m = n = 2$ and $x_{1,r} = a_r$, $x_{2,r} = b_r$. Therefore in this instance the matrix with (i, j)-entry $\sum_{q=0}^{k} \alpha_{ij}^q t^q$ is just $\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$, which is obviously invertible in $\mathbb{Z}[t, t^{-1}]$.

Example 2.7. When the relators are all of the form $x_{j,r}s_j$ where the words s_j involve only generators $x_{i,t}$ with $r < t \leq r + k$ the subgroup of B generated by the images of $\{x_{i,r+1}, \ldots, x_{i,r+k}\}_{i\in\{1,\ldots,m\}}$ is free, and B is an increasing union of such subgroups. Thus B is locally free, and so of homological dimension 1. For example, with $m = n = 1$, $k = 2$, we may take

$$B = \langle x_r \mid x_r = [x_{r+1},\, x_{r+2}]\rangle_{r\in\mathbb{Z}}\,,$$

the commutator subgroup of

$$G = \left\langle x, y \mid x = \left[yxy^{-1},\, y^2xy^{-2}\right]\right\rangle.$$

This G is also obtainable as an ascending HNN-extension of the free group of rank 2 (via the monomorphism that sends the first generator to the second, and the second to the commutator of them both). The 2-complex corresponding to this 1-relator presentation is aspherical [26], and so G has geometric dimension 2. (This may be seen more directly as a $K(G, 1)$ complex may be constructed as the mapping torus of a self map of $S^1 \vee S^1$).

Corollary 2.8. *For a knot λ in S^3, the following are equivalent.*

(i) *The Alexander polynomial for λ is ± 1.*
(ii) *The commutator subgroup of the knot group $\pi_1(S^3 - \lambda)$ is perfect.*
(iii) *The commutator subgroup of the knot group $\pi_1(S^3 - \lambda)$ is acyclic.*

Proof. It is well-known that the knot group G has m relators, $m+1$ generators and infinite cyclic abelianization, for some $m \geq 0$. Since G (and so $B = G'$) has homological dimension at most 2 [4](7.8), we may apply the preceding theorem, to obtain the equivalence of (ii) and (iii) above.

On the other hand, B_{ab} viewed as a $\mathbb{Z}[G/B]$-module is the $\mathbb{Z}[G/B]$-torsion submodule of the Alexander module of λ. Because it vanishes exactly when the Alexander polynomial is invertible, the equivalence of (i) and (ii) follows. □

Remark 2.9. If $G = \pi_1(S^3 - L)$ is the group of a nontrivial μ-component classical link L then $\mathrm{def}(G) \geq 1$, $\mathrm{gd}(G) = 2$ and $G_{\mathrm{ab}} \cong Z^\mu$. The commutator subgroup G' is perfect if and only if the μ-variable Alexander polynomial is a unit in the integral Laurent polynomial ring on μ variables. By Lemma 1.1, this can only happen if $\mu \leq 2$. The simplest nontrivial knot with Alexander polynomial 1 is the Kinoshita-Terasaka knot [22], which has an 11 crossing projection. The (2-component) Hopf link has group Z^2. One may construct other 2-component examples from knots with Alexander polynomial 1 by adjoining a second component bounding a small disc that the knot meets transversely in one point. (The linking number of a 2-component link L with Alexander polynomial $\Delta_L(x,y)$ is $\Delta_L(1,1)$, by the Torres conditions [34], and so is 1 if $\Delta_L(x,y) \equiv 1$.)

Consideration of how the groups in such examples are related leads to the following purely algebraic construction. Let G be a group with a deficiency 1 presentation $\langle x_1, \ldots, x_g, y \mid r_1, \ldots, r_g \rangle$, where the generators x_i represent elements of G', and G' is perfect. Then G satisfies the hypotheses of Theorem 2.4(c) (and every such group has such a presentation). Let $G^* = \langle G, u \mid yu = uy \rangle$ be the HNN extension with base G and associated subgroups the infinite cyclic group generated by the image of y. Then $G^*_{\mathrm{ab}} \cong Z^2$ and $G^{*\prime}$ is perfect. If moreover $\mathrm{cd}(G) = 2$ then $\mathrm{cd}(G^*) = 2$ (and similarly for hd - see page 83 of [4]), while if $\mathrm{gd}(G) = 2$ then $\mathrm{gd}(G^*) = 2$ (since $K(G^*,1) \simeq K(G,1) \cup_{S^1} (S^1 \times S^1)$).

3. Finitely generated subgroups of positive deficiency groups

In this section we consider finitely presentable groups with positive deficiency, and show that under rather mild additional conditions on suitable subgroups in such a group G then $\mathrm{def}(G) = 1$ and $\mathrm{gd}G \leq 2$. Theorem 0.1(a) follows easily from the following results.

Lemma 3.1. *Let $L = \bigcup_{n\geq 0} L_n$ be a group of cohomological dimension 2 which is the union of a strictly increasing sequence of subgroups such that L_0 is FP_2 and $[L_{n+1} : L_n]$ is finite for all $n \geq 0$. Then $L_n \cong Z$ for all $n \geq 0$ and L is a rank 1 abelian group.*

Proof. Since L_1 is torsion-free and nontrivial it is infinite, and so L_0 is also infinite. Moreover the subgroups L_n are all FP_2, as they are finite extensions of L_0. It follows from the Kurosh subgroup theorem that the number of indecomposable factors in such groups must be strictly decreasing unless one is indecomposable (in which case all are). (See Lemma 1.4 of [30]). If some L_m has one end then so does L_n for all $n \geq 0$. It then follows from Theorem 3.3 of [15] that $\mathrm{cd}L = 3$, contrary to the hypothesis. Therefore L_n has two ends, and so $L_n \cong Z$, for all $n \geq 0$. The final assertion is clear. □

Theorem 3.2. *Let G be a finitely presentable group with $\mathrm{def}(G) > 0$, and which has an infinite subgroup M. Suppose that either*

(1) *M is amenable and ascendant; or*
(2) *M is finitely generated, subnormal and of infinite index in G.*

Then $\mathrm{def}(G) = 1$, $\mathrm{gd}G = 2$ (unless $G \cong Z$ in case (1)), and $\chi(G) = 0$.

Proof. Suppose first that M is amenable and let $\{M_\alpha\}$ be an ascendant series for $M = M_0$ in $G = M_\nu$. It then follows by transfinite induction that for all α we have $\beta_i^{(2)}(M_\alpha) = 0$ for all $i \geq 0$, by recourse to [25] Theorem 3.3(1),(2),(3) in turn when α is respectively 0, a successor ordinal, and a limit ordinal.

Suppose next that M is finitely generated, $[G : M] = \infty$ and $L_0 = M < \cdots < L_n = G$ is a subnormal chain. Then $[L_{i+1} : L_i]$ is infinite for some $0 \leq i < n$. Let $j = \min\{i \mid [L_{i+1} : L_i] = \infty\}$. Then M has finite index in L_j, and so L_j is finitely generated. Hence $\beta_i^{(2)}(L_{j+1}) = 0$ for $i \leq 1$ [13]. Using Theorem 3.3(2) of [25] and proceeding by induction up the subnormal chain gives $\beta_1^{(2)}(G) = 0$ again.

In either case, by Theorem 2 of [19], $\mathrm{def}(G) = 1$ and the 2-complex associated to a presentation of minimal deficiency is aspherical, so $\mathrm{gd}G \leq 2$ and $\chi(G) = 1 - \mathrm{def}(G) = 0$. Moreover G is not free, since free groups do not have subgroups M of either type (apart from the exception noted). Hence $\mathrm{gd}G = 2$. □

Theorem 3.3. *Let G be a countable group with $\mathrm{cd}G \leq 2$, and let M be an ascendant subgroup which is FP_2. Then M is free or of finite index in G.*

Proof. Let us assume that M is neither free nor of finite index in G, and derive a contradiction. By this assumption, $\mathrm{cd}M = \mathrm{cd}G = 2$. We consider in turn various special cases.

(i) First, Corollary 8.6 of [4] handles the case where M is normal in G and G is finitely generated.

(ii) Now consider the case where M is normal in G, without further restriction on G. If J is a finitely generated subgroup of G that contains M, then $[J : M]$ is finite, by (i); and so G, being countable, is the union of a strictly increasing sequence of finite extensions of M. Thus we are in the situation of Lemma 3.1.

(iii) Turning now to the general case, let $\{M_\alpha\}$ be an ascendant series for $M = M_0$ in $G = M_\nu$, and, as $[M_\nu : M]$ is supposed infinite, let β be the minimum ordinal with $[M_\beta : M]$ infinite. By Lemma 3.1 β cannot be a limit ordinal, while by (ii) β cannot be a successor ordinal. So again we have a contradiction. □

As an immediate consequence of these theorems we may exclude finitely presentable perfect subnormal subgroups.

Corollary 3.4. *If M is an infinite perfect subnormal subgroup of a finitely presentable group G with* $\mathrm{def}(G) > 0$ *then M is not FP_2.* □

Example 3.5. This result fails for G of zero deficiency, as seen by the group $G = \langle x, y \mid x[x, yxy^{-1}], [x, y^4]\rangle$. For $G_{\mathrm{ab}} = Z$ and $G' \cong H$, Higman's acyclic group, which is finitely presentable. (See Example 1.3 above.)

Corollary 3.6. *Let G be a finitely presentable group with* $\mathrm{def}(G) > 0$ *and let M be the subgroup generated by all elements of finite order. If M is finitely generated or is locally finite then it is finite.*

Proof. The subgroup M is normal, and has infinite index in G, since M/M' is a torsion group and G/G' has positive rank. If M is infinite then $\mathrm{cd}G = 2$ is finite, by Theorem 3.2, and so G must be torsion-free. This contradicts the assumption that M is infinite. □

In this situation must M be trivial? (Some hypothesis on M is neeeded, as $\mathrm{def}(Z*(Z/2Z)) = 1$ and the normal subgroup of $Z*(Z/2Z)$ generated by elements of finite order is infinite.)

If S and T are groups and $\alpha : T \to \mathrm{Aut}(S)$ is a homomorphism let $S \rtimes_\alpha T$ denote the semidirect product of T and S, with T acting on S via α. Let $\sqrt{G}$ denote the Hirsch-Plotkin radical of G (the maximal locally-nilpotent normal subgroup).

Theorem 3.7. *Let G be a finitely presentable group with* $\mathrm{def}(G) > 0$ *and with an infinite subnormal subgroup $M \leq G'$ which is FP_2. If there is a subnormal chain $L_0 = M < \cdots < L_n = G$ in which all the terms are finitely generated then either:*

1. *$G \cong Z \rtimes_{-1} Z$;*
2. *G is virtually $F \times A$, where F is free of finite rank ≥ 2, $A \cong Z$ and M has finite index in A; or*
3. *$G \cong F \rtimes_\alpha Z$, where F is free of finite rank ≥ 2, $\alpha \in \mathrm{Aut}(F)$ and M has finite index in F.*

Proof. Since $\mathrm{def}(G) > 0$ the abelianization $G_{\mathrm{ab}} = G/G'$ has positive rank, and so $[G : M]$ is infinite. By Theorems 3.2 and 3.3, G is finitely generated, $\mathrm{gd}G = 2$ and M is free of finite rank $r \geq 1$.

Suppose first that $r = 1$. That is, $M \cong Z$, and so $M \leq \sqrt{L_1}$. A finite induction now shows that $M \leq \sqrt{G}$. If N is a finitely generated subgroup of $\sqrt{G}$ then N is nilpotent, and has Hirsch length $= \mathrm{cd}N \leq 2$, by Theorem 7.14 of [4]. Therefore N is abelian, and so $A = \sqrt{G}$ is abelian. Let $C = C_G(A)$, the centralizer of A in G.

If G is virtually soluble, then, by Theorem 7.10 of [4], G has Hirsch length at most 2. Since $M \cong Z$ and Hirsch length is additive in extensions, all but one of the (finitely generated) subquotients L_i/L_{i-1} (with $1 \leq i \leq n$) must be finite. It follows easily that G is virtually Z^2, and hence, since G is torsion-free and $G' \neq 1$, that $G \cong Z \rtimes_{-1} Z$. This is case (1).

Next, assume that G is not virtually soluble. Then in particular $[G : A]$ is infinite, so by Theorem 3.3 either A is not finitely generated and $\mathrm{cd}A = 2$, or else A is free (in other words $A \cong Z$).

In the former case, by Theorems 7.14 and 8.8 of [4], A has rank 1 and $C = A$ (by maximality of A). But then G/A embeds in $\mathrm{Aut}(A)$, which is abelian, contradicting the hypothesis that G is not virtually soluble.

Therefore we may further assume that $A \cong Z$. Then $[G : C] \leq 2$, and so C is finitely presentable. Since G is not virtually soluble, neither C nor C' can be abelian. Hence, by Theorem 8.8 of [4], C' must be free nonabelian. Thus $C' \cap A = 1$. Since C, and so C_{ab}, is finitely generated, we may then choose an epimorphism $\phi : C \to Z$ that maps A injectively onto a subgroup of finite index in Z. Let $D = \phi^{-1}(\phi(A)) = FA$ where $F = \mathrm{Ker}\phi$. Then A is a direct factor of D, so $D \cong F \times A$. Since D has finite index in C (and hence in G) it is finitely presentable. Therefore F is also finitely presentable, and so is free of finite rank, by [4](8.6) again. The rank must be at least 2 since $C' \leq F$ and C' is nonabelian. This gives case (2).

Suppose now that $r > 1$. Let $j = \min\{i \mid [L_{i+1} : L_i] = \infty\}$. Then L_j is free of finite rank, since it is torsion-free and $[L_j : M] < \infty$. On the other hand L_{j+1} is not free, since it has a nontrivial finitely generated normal subgroup of infinite index. Therefore $\mathrm{cd}L_i = 2$ for $i > j$. The quotient L_{j+1}/L_j is virtually free of finite rank, by Theorem 8.4 of [4]. Hence L_{j+1} is finitely presentable, and so $[G : L_{j+1}]$ is finite, by Theorem 3.3. Moreover L_{j+1}/L_j has a well-defined Euler characteristic $\chi(L_{j+1}/L_j) \in \mathbb{Q}$, and $\chi(L_{j+1}) = \chi(L_j)\chi(L_{j+1}/L_j)$ [9]p.250. Now $\chi(L_{j+1}) = [G : L_{j+1}]\chi(G) = 0$, while $[L_j : M]\chi(L_j) = \chi(M) = 1 - r < 0$. Hence $\chi(L_{j+1}/L_j) = 0$ and so L_{j+1}/L_j is virtually Z. Since G/G' has positive rank and $M < G'$ there is an epimorphism $\theta : G \to Z$ such that $F = \mathrm{Ker}\theta$ contains M as a subgroup of finite index. Then

$G \cong F \rtimes_\alpha Z$. As F is torsion-free and $[F : M]$ is finite F is free of finite rank. □

Does the theorem remain true without the hypothesis that the intermediate subgroups L_j be finitely generated?

Example 3.8. The nonsoluble possibilities allowed by the theorem may be illustrated by taking G to be the group of the trefoil knot and $M = \mathcal{Z}(G) \cong Z$ (in case (2)) or $M = G'$, which is free of rank 2 (in case (3)).

In case (2) of the theorem the subgroup C is a semidirect product $F \rtimes_\alpha Z$; moreover the image of α in the outer automorphism group $\mathrm{Out}(F)$ has finite order. However G itself need not be a semidirect product of Z with a free normal subgroup, as in case (3).

Example 3.9. Let $G = \langle x, y, t \mid xtx^{-1} = t^{-1},\ ty = yt\rangle$ and $K = \langle t\rangle$. Since t represents an element of order 2 in G/G' any epimorphism $\theta : G \to Z$ must factor through the nonabelian free group $G/K \cong \langle x, y\rangle$, and so $\mathrm{Ker}\theta$ is neither finitely generated nor free. (Note also that $C_G(K) \cong F \times K$, where F is free of rank 3, and that G *is* a semidirect product $K \rtimes_\varepsilon (G/K)$, where $\varepsilon(x) = -1$ and $\varepsilon(y) = 1$ in $\mathrm{Aut}(K) = \{\pm 1\}$).

If m is a nonzero integer we shall let $Z*_m$ denote the group with presentation $\langle a, t \mid tat^{-1} = a^m\rangle$. These groups are ascending HNN extensions with infinite cyclic base and associated subgroups, and are soluble of geometric dimension at most 2. In particular, $Z*_1 \cong Z^2$ and $Z*_{-1} \cong Z \rtimes_{-1} Z$, the fundamental group of the Klein bottle.

Corollary 3.10. *Let G be a finitely presentable group with* $\mathrm{def}(G) > 0$. *If $G^{(n)}$ is FP_2 for some $n \geq 2$ then $G'' = 1$.*

Proof. It suffices to establish that G is soluble with $\mathrm{cd}G \leq 2$, for then $G \cong Z$ or $Z*_m$, for some nonzero integer m [14], whence $G'' = 1$. Let $M = G^{(n)}$, where n is the least integer ≥ 2 such that $G^{(n)}$ is FP_2.

If M is finite then G is elementary amenable, so Theorem 3.2 yields that $\mathrm{gd}G \leq 2$. Hence G is torsion-free, so $M = 1$ and G is soluble. Otherwise, M is infinite, so that we are in the situation of Theorem 3.7. If $M \cong Z$ then $G^{(n+1)} = 1$. The only other possibility allowed by Theorem 3.7 is case (3), in which G is a semidirect product $F \rtimes_\alpha Z$ with F free of finite rank. Therefore $G' \leq F$ and so $G^{(n)} \leq F'$. But finitely generated normal subgroups of infinite index in free groups are trivial. Hence once more we conclude that G must be soluble. □

The condition $n \geq 2$ is necessary, as the commutator subgroups of the groups of nontrivial fibred classical knots are free of finite rank ≥ 2.

Theorem 3.11. *Let G be a finitely presentable group with* $\mathrm{def}(G) > 0$, *and such that G' is finitely generated and perfect. Then the following hold.*

(a) $\mathrm{def}(G) = 1$ *and* $\mathrm{gd}(G) \leq 2$.
(b) G_{ab} *is* Z *or* Z^2.
(c) G' *is acyclic.*
(d) *The following are equivalent:*
  (i) G' *is* FP_2,
  (ii) G *is* Z *or* Z^2 *(and* G' *is trivial),*
  (iii) *the centre* $\mathcal{Z}(G)$ *is nontrivial,*
  (iv) G *has a presentation with one relator.*

Proof. If G' is finite then G is elementary amenable and (a) follows as in Corollary 3.10. If G' is infinite (a) follows from Theorem 3.2 with $M = G'$. Parts (b) and (c) then follow from Lemma 1.1, since $\mathrm{hd}(G) \leq \mathrm{gd}(G) \leq 2$. Thus G is torsion-free. If G' is FP_2 it must be trivial, by Theorem 3.7 (with $M = G'$). If $\mathcal{Z}(G)$ is nontrivial then G' is free, by Corollary 8.9 of [4], and so must be trivial. If G has a presentation with one relator then it is locally indicable [7]. Therefore the finitely generated subgroup G' is indicable, and so, being perfect, is trivial. It is clear that if $G \cong Z$ or Z^2 the other parts of (d) hold. □

Remark 3.12. It would be very interesting to know whether there exist examples of groups G as in the theorem, other than the degenerate cases Z or Z^2. Note that [8] excludes many groups G having few relators.

Example 3.13. Let G be the 2-generator, 1-relator group of Example 2.7 above. While the other assumptions hold, G' fails to be finitely generated. Although conclusions (a), (b), (c) and (d)(iv) hold, (d)(i),(ii) and (iii) are false.

Example 3.14. Examples of finitely presentable groups with finitely presentable perfect commutator subgroup, but deficiency not positive, are provided by the symmetric group $\mathfrak{S}_n$ $(n \geq 5)$, the braid group B_n $(n \geq 5)$ [23], and the general linear group $\mathrm{GL}_n(\mathbb{Z})$ $(n \geq 3)$, with abelianizations $Z/2Z$, Z and $Z/2Z$ respectively.

Theorem 3.15. *Let* G *be a finitely presentable group with* $\mathrm{def}(G) > 0$, *and such that* G_{ab} *is torsion-free and* $H_2(G) = 0$. *If* $G^{(n)}$ *is perfect for some* $n \geq 0$ *then* $\mathrm{def}(G) = 1$, G'' *is superperfect and* $G/G'' \cong Z$ *or* $Z*_2$.

Proof. Let d be the rank of G_{ab} and let F be the free group of rank d. Let $f : F \to G$ be a homomorphism which induces an isomorphism on abelianization. Then f induces isomorphisms on the nilpotent quotients, $F/F_q \cong G/G_q$, for all $q \geq 1$ [32]. If $d > 1$ the derived length of F/F_q is unbounded as $q \to \infty$. Hence we must have $d = 1$, and so $\mathrm{def}(G) = 1$ also. Thus G is a finitely presentable E-group, in the sense of [33], and so $G^{(n)} = \cap_{j \geq 0} G^{(j)}$ is superperfect and $\mathrm{cd}(G/G^{(n)}) \leq 2$, by

Theorem A of [33]. Since $G/G^{(n)}$ is finitely generated, soluble and nontrivial it is Z or $Z*_m$ for some nonzero integer m [14]. Since $G/G' \cong Z$ we must have $m = 2$ in the latter case. In either case $G^{(n)} = G''$. □

Example 3.16. Let π be the group of a classical knot with Alexander polynomial 1 (so that π' is perfect), and let $G = (Z*_2) *_Z \pi$, where the generator of the amalgamated subgroup is identified with the generator t of $Z*_2$ and with a meridian in π. Then G has deficiency 1, G'' is perfect and $G/G'' \cong Z*_2$. Moreover cd$G = 2$ and so G'' is acyclic, by the theorem. If the knot is nontrivial $G'' \neq 1$.

4. Finitely generated perfect subgroups of 3-manifold groups

We now obtain strong restrictions on the existence of finitely generated perfect normal subgroups of 3-manifold groups. First, in contrast to results above, we are able to exclude the acyclic possibility.

Theorem 4.1. *If a finitely generated and acyclic group π is the fundamental group of a 3-manifold, then π is trivial.*

*Proof***.** Let M be a 3-manifold with $\pi_1(M) = \pi$. Because π is perfect, it cannot map nontrivially onto the group of order two. Therefore M is orientable. Since π is finitely generated, by Scott's core theorem [29] there is a (possibly bounded) compact submanifold N such that the inclusion induces an isomorphism of fundamental groups. Thus N is also orientable and the exact sequence

$$H_2(N, \partial N) \to H_1(\partial N) \to H_1(N)$$

combines with Poincaré duality $H_2(N, \partial N) \cong H^1(N)$ and the triviality of $H_1(N)$ to force $H_1(\partial N)$ to be zero. Now ∂N is a disjoint union of surfaces; since each has first homology zero it must be S^2. By capping each such S^2 with a 3-ball we obtain a closed orientable 3-manifold P with fundamental group π. Then, since prime decomposition corresponds to free product of fundamental groups, the prime factors of P each have finitely generated, acyclic fundamental group, and we may as well assume that P itself is prime. Hence either P is aspherical or $P \simeq S^1 \times S^2$ or π is finite and the universal cover $\tilde{P}$ is a homotopy 3-sphere (see [16] p.171). If P is aspherical, then $H_3(\pi) \cong H_3(P) \cong Z$, while if $P \simeq S^1 \times S^2$ then $H_1(\pi) \cong Z$. So its acyclicity forces π to be finite. However, arguing for example from the Gysin sequence for the spherical fibration $\tilde{P} \to P \to K(\pi, 1)$, we have that $H_3(\pi) \cong Z/|\pi|Z$, and hence π is trivial. □

One interpretation of this result is that one cannot improve on the well-known fact that the fundamental group of any homology 3-sphere is superperfect, whereas any finitely presentable acyclic group can be the fundamental group of a high-dimensional homology sphere [2].

Next we show that there are serious constraints imposed even by the weaker assumption that the finitely generated subgroup is perfect normal. As in [16] p.25, for a connected 3-manifold M we denote by $\hat{M}$ the manifold obtained from M by capping off each 2-sphere component of its boundary with a 3-cell. These two manifolds have the same fundamental group.

Theorem 4.2. *Let M be a compact connected 3-manifold without P^2-component in its boundary. If the fundamental group π contains a nontrivial finitely generated perfect normal subgroup ν, then*

(i) *the covering $\hat{M}_\nu$ of $\hat{M}$ associated to ν is an integral homology 3-sphere,*
(ii) *$\hat{M}$ is a rational homology 3-sphere and*
(iii) *π/ν is a finite group of periodic cohomology, with period 1, 2 or 4.*

Proof. Since ν is nontrivial and perfect, it cannot be the fundamental group of a compact 2-manifold. Therefore ν must have finite index in π, by [16](11.1). This makes $\hat{M}_\nu$ a finite cover of $\hat{M}$, and so compact too. Because ν is perfect, $\hat{M}_\nu$ is orientable too, and its boundary components are surfaces. Now by Poincaré duality the homology exact sequence for the pair $(\hat{M}_\nu, \partial\hat{M}_\nu)$ contains an exact sequence

$$H_2(\partial\hat{M}_\nu) \to H_2(\hat{M}_\nu) \to H^1(\hat{M}_\nu) \to H_1(\partial\hat{M}_\nu) \to H_1(\hat{M}_\nu).$$

So each of the boundary component surfaces must have perfect fundamental group and thus be a 2-sphere. Therefore $\partial\hat{M}_\nu$ is empty. The exact sequence above now yields that $H_2(\hat{M}_\nu) = 0$. Since $\hat{M}_\nu$ is orientable, we deduce (i).

By the Lefschetz fixed point theorem, since the covering transformations act on $\hat{M}_\nu$ without fixed points they must each have degree 1. So $\hat{M}$ is also orientable. As $H_1(\hat{M};\mathbb{Q}) = 0$ and $\partial\hat{M}$ is empty, $\hat{M}$ is a rational homology 3-sphere. Finally, (iii) follows from the fact that the finite group π/ν acts fixed-point freely on an integral homology 3-sphere [10](12.11), (16.9). □

Remarks 4.3. (1) The need to avoid a P^2 in the boundary of M is shown by the example of $M = \hat{M} = \Sigma\#(P^2 \times I)$, where Σ is an integral homology 3-sphere. The orientable double cover is a punctured $\Sigma\#\Sigma$ and therefore not an integral homology 3-sphere, although it has perfect fundamental group.

(2) On the other hand, if the boundary of M itself contains a P^2, then the theorem above applies to its orientable double cover, since ν is still a normal subgroup of any index 2 subgroup of π.

(3) It follows from the classification of groups of periodic cohomology that either:

(i) ν is the maximum perfect subgroup of π (corresponding to π/ν being soluble); or else

(ii) π/ν is the direct product of the binary icosahedral group $\mathrm{SL}(2,5)$ with a finite cyclic group Z/mZ (possibly trivial) [27], [31] (see also [28]). In this case we have that π has perfect commutator subgroup, $H_1(\hat{M}) = Z/mZ$, and $\pi'/\nu = \mathrm{SL}(2,5)$.

(4) If one relaxes the assumption that ν is normal in π, then by [21](6.2) M has an irreducible rational homology 3-sphere summand $\breve{M}$ whose fundamental group $\breve{\pi}$ has a nontrivial finitely generated perfect subgroup. Since this subgroup has finite index in $\breve{\pi}$ (as in the proof of the theorem), it has only finitely many conjugates and so lies inside a nontrivial finitely generated perfect normal subgroup. Hence the theorem applies to $\breve{M}$.

(5) In Lemma 2 of [18] it is shown that if M is a closed orientable 3-manifold and ν is a perfect normal subgroup of $\pi = \pi_1(M)$ such that $\rho = \pi/\nu$ has finitely many ends then either: ρ is finite and has cohomological period dividing 4 (as in the above theorem); or $\rho \cong Z$ or $D_\infty = (Z/2Z) * (Z/2Z)$; or ρ is an orientable Poincaré duality group of formal dimension 3. In particular, if the derived series for π terminates after finitely many steps then the maximal soluble quotient of π is either finite with cohomological period dividing 4, Z, D_∞ or is polycyclic of Hirsch length 3.

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