GROUPS WITH INFINITE HOMOLOGY

A. J. BERRICK AND P. H. KROPHOLLER

1. Introduction

We consider the reduced homology of a group G with coefficients in the trivial module :

$$\widetilde{H}(G) := \bigoplus_{n=1}^{\infty} H_n(G;\mathbb{Z}).$$

A group is said to be acyclic if its reduced homology vanishes. Many interesting classes of groups have been discovered having this property ([2] is a useful survey). This is an indication that the reduced homology carries limited information.

Here we obtain information about $\widetilde{H}(G)$ for a class of groups G that includes all locally finite groups and all soluble groups of finite rank. We show that non-locally-finite groups in the class cannot be acyclic, and that in fact their reduced homology is infinite.

Our main result is as follows.

Theorem 1.1. *Let G be a group having a series of finite length whose factors are either infinite cyclic or locally finite. Then the reduced homology $\widetilde{H}(G)$ is infinite or zero. Indeed, if $\widetilde{H}(G)$ is a torsion group, then G is locally finite, and either:*

- (i) *for some prime p occurring as the order of an element of G, and for infinitely many n, $H_n(G;\mathbb{Z})$ contains elements of order p; or*
- (ii) *G is acyclic.*

The proof brings together some techniques from homotopy theory and some Euler characteristic arguments. There is an interesting dichotomy between the cases of torsion-free and non-torsion-free groups. This shows up in the choice of prime field that results in infinite homology. When G is torsion-free it has finite cohomological dimension, and in this case one must prove that at least one of its rational homology groups is nonzero. At the other extreme, when G is locally finite it may be acyclic - the McLain group $M(\mathbb{Q},\mathbb{F}_p)$ is an example. (The McLain group $M(\mathbb{Q},\mathbb{F}_p)$ may be thought of as the group of upper unitriangular matrices with entries in $\mathbb{F}_p$, but with rows and columns indexed by the rational numbers rather than the natural numbers. See[1].) Otherwise, we show that for some prime p that is the order of an element of G the integral homology contains elements of order p in arbitrarily high dimensions; this has long been known to be the case for finite G. The phrase "for some prime" is best possible, in that the example of the direct product of a locally finite acyclic group and a group of prime order shows that there may be a unique prime that is detected by integral homology.

Note that for groups in general it is possible to have the second sentence of the theorem hold but neither (i) nor (ii). For instance, by [3] there is a perfect, torsion-generated group with nth integral homology group of order n whenever n is prime, and zero otherwise.

2. Two ingredients from homotopy theory

Our arguments rest on two results, of independent interest, which have a homotopy theoretic pedigree. They sharpen results of [3] which consider only integral coefficients. Let k be a commutative ring. We write $\mathrm{thd}_k(G)$ for the trivial homological dimension of G over k; this is the largest integer m for which $H_m(G;k)$ is nonzero, or infinity in case there is no such integer. We also use similar notation with G replaced by a topological space. First, here is a lemma that is a useful ancillary result for applications of Miller's Theorem [5]. It is used implicitly in [3].

Lemma 2.1. *If X has the homotopy type of a CW-complex and $\mathrm{thd}_{\mathbb{Z}}(X)$ is finite, then the suspension ΣX has the homotopy type of a finite CW-complex.*

Proof. Write $k = \mathrm{thd}_{\mathbb{Z}}(X)$.,When $k = 0$, X is acyclic, making ΣX contractible; so we may suppose that $k \geq 1$. Assuming also that X is actually a CW-complex, let $C(j)$ be the mapping cone of the inclusion $j : X^{(k)} \hookrightarrow X$ of its k-skeleton. Then the homology exact sequence of the pair $(X, X^{(k)})$ shows that $H_{k+1}(C(j))$ embeds in $H_k(X^{(k)})$, which is in turn a subgroup of the free abelian group of k-chains on $X^{(k)}$. Thus $C(j)$ has the homology of a wedge of $(k+1)$-spheres. Since $k \geq 1$, it follows from van Kampen's theorem that $C(j)$ is homotopy equivalent to $\bigvee S^{k+1}$. Therefore ΣX has the homotopy type of the mapping cone of some map $\bigvee S^{k+1} \to \Sigma X^{(k)}$, a CW-complex of dimension at most $k + 2$. □

Theorem 2.2. *Let p be a fixed prime and let G be a group with $\mathrm{thd}_{\mathbb{F}_p}(G)$ finite. Then any homomorphism $G \to \mathrm{GL}(\mathbb{C})$ is trivial on all elements of p-power order.*

Proof. Since the argument involves only a tweaking of that given in [3] Theorem 3, we indicate it briefly. As in [3], for any cyclic subgroup C of p-power order in G, by change of basis the homomorphism can be made to restrict to a unitary representation on C, giving rise to the commuting diagram

$$\begin{array}{ccccc} BC & \longrightarrow & BU & \xrightarrow{\simeq} & \Omega^2 BU \\ \downarrow & & \downarrow^{\simeq} & & \\ BG & \longrightarrow & B\,\mathrm{GL}(\mathbb{C}) & & \end{array}$$

We focus on the adjunction $\Sigma^2 BC \to \Sigma^2 BG \to BU$. Since $\Sigma^2 BG$ is simply-connected with $\mathrm{thd}_{\mathbb{F}_p}(\Sigma^2 BG)$ finite, by Theorems C, D of [5] $\mathrm{map}_*(BC, \Sigma^2 BG)$ is weakly contractible. This makes $[\Sigma^2 BC, \Sigma^2 BG] = 0$. Then the result follows as in [3] by appeal to Atiyah's embedding of the complex representation ring of C in $\mathbb{Z}\times[BC, BU]$. □

Theorem 2.3. *Let p be a fixed prime and let G be a locally finite group with $k = \mathrm{thd}_{\mathbb{F}_p}(G)$ finite. Then G is $\mathbb{F}_p$-acyclic, that is, $\mathrm{thd}_{\mathbb{F}_p}(G) = 0$.*

Proof. As ΣBG is simply-connected, we pass to its p-localization $(\Sigma BG)_{(p)}$, which is also simply-connected. By definition, it is $\mathbb{F}_q$-acyclic whenever $q \neq p$, while its homology with other prime field coefficients is that of ΣBG. Hence $\mathrm{thd}_{\mathbb{F}_p}((\Sigma BG)_{(p)})$ is finite. On the other hand, since G is locally finite it is the direct limit of finite subgroups. So its homology is the direct limit of the homology of those finite subgroups; thus $\widetilde{H}((\Sigma BG)_{(p)}; \mathbb{Q}) = 0$. Therefore, by universal coefficients, $\mathrm{thd}_{\mathbb{Z}}((\Sigma BG)_{(p)})$ is finite, and the lemma thus implies that $(\Sigma BG)_{(p)}$ has the homotopy type of a finite-dimensional complex. It follows from Miller's Theorem that $[\Sigma BG, (\Sigma BG)_{(p)}] = 0$, whence by its universal property $(\Sigma BG)_{(p)}$ is contractible. This makes zero the reduced homology of ΣBG, hence of G, with coefficients in the local ring $\mathbb{Z}_{(p)}$. Since $\mathbb{F}_p$ is a $\mathbb{Z}_{(p)}$-module, it follows by universal coefficients that $\widetilde{H}(G; \mathbb{F}_p) = 0$ too. □

3. Relevant groups, and notation

For any group G, let $\tau(G)$ denote the unique largest normal locally finite subgroup (for example, see [6] p 418). We consider the class of groups having a series of finite length whose factors are either infinite cyclic or locally finite. Then the number of infinite cyclic factors in such a series is an invariant of the group, known as the torsion-free rank or Hirsch length $h(G)$ (cf. [6] p 407). Recall also that the Fitting subgroup, or nilpotent radical, of a group is defined to be the product of all its nilpotent normal subgroups. We shall use the following folklore characterization of this class.

Proposition 3.1. *Suppose that G admits a finite series*

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

in which the factors G_i/G_{i-1} are either infinite cyclic or locally finite.

Let H denote the subgroup containing $\tau(G)$ such that $H/\tau(G)$ is the Fitting subgroup of $G/\tau(G)$. Let K denote the subgroup containing H such that $K/H = \tau(G/H)$. Then

(1) *there is a finite dimensional $\mathbb{C}$-linear representation of G with kernel equal to $\tau(G)$,*
(2) *$H/\tau(G)$ is torsion-free nilpotent and of finite Hirsch length,*
(3) *K/H is finite, and*
(4) *G/K is a Euclidean crystallographic group.*

Thus, schematically, G has the following decomposition.

$$
\begin{array}{ll}
G & \\
| & \text{maximal abelian, by finite} \\
K & \\
| & \text{finite nilpotent} \\
H & \\
| & \text{torsion-free nilpotent} \\
\tau(G) & \\
| & \text{locally finite} \\
1 &
\end{array}
\left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} \text{linear}
$$

Proof. (Outline) Professor Wehrfritz has kindly indicated some steps in the argument here. Such a group is locally finite, by soluble with a finite series with abelian torsion-free factors of finite rank, by finite. The proof is by induction on the Hirsch length, noting that such G with no normal torsion are linear over the rationals [11], and such a G lying in $\mathrm{GL}_n(\mathbb{Q})$ must be soluble by finite [9]. Now by a theorem of Mal'cev [6] p. 436, soluble groups linear over $\mathbb{Q}$ are torsion-free nilpotent by abelian by finite, while by Gruenberg [10] p. 102 their Fitting subgroups are nilpotent. Finally, the deduction that G/K is actually maximal abelian by finite, in other words crystallographic, may be seen from an argument of Zassenhaus [6] p. 435. □

Observe that this particular class of groups is closed under passage to subgroups, quotients and extensions.

For G in this class, an easy spectral sequence argument shows that the rational homology groups $H_n(G;\mathbb{Q})$ are all finite-dimensional over $\mathbb{Q}$, and that $\mathrm{thd}_{\mathbb{Q}}(G) \le h(G) < \infty$ (cf. [8]). We shall use the notation $\bar{\chi}(G)$ for the naive Euler characteristic,

$$\bar{\chi}(G) := \sum_{n=0}^{\infty} (-1)^n \dim_{\mathbb{Q}} H_n(G;\mathbb{Q}).$$

This definition suits our purpose although it is not always easily or closely connected with the classical Euler characteristic. For example, the infinite dihedral group D_∞ has Euler characteristic zero whereas $\bar{\chi}(D_\infty) = 1$. We refer the reader to Brown's book [4] for a detailed account of Euler characteristics and cohomological methods.

4. Proof of the main theorem

Our strategy is to use the naive Euler characteristic to set up contradictions that reduce to consideration of locally finite groups. Throughout we assume that G satisfies the hypotheses of Proposition 3.1.

Lemma 4.1. *If G is nontrivial and torsion-free, then $\overline{\chi}(G) = 0$.*

Proof. Let H and K be the subgroups defined using Proposition 3.1. Since G is nontrivial and torsion-free, we know that $\tau(G)$ is trivial and H is infinite. Let L be any subgroup containing H as a normal subgroup of finite index. Now L is torsion-free and nilpotent-by-finite, of finite Hirsch length equal to the Hirsch length $h(H)$ of H. Let L_0 be a finitely generated subgroup of L that contains a transversal to H in L and which has full rank, that is $h(L_0) = h(H)$. Let H_0 denote $H \cap L_0$. Then the inclusion of H_0 into H induces isomorphisms in rational homology. Moreover the short exact sequence

$$H_0 \rightarrowtail L_0 \twoheadrightarrow H/L$$

embeds naturally into

$$H \rightarrowtail L \twoheadrightarrow H/L$$

and comparison of the associated Lyndon-Hochschild-Serre spectral sequences shows that the inclusion of L_0 into L induces isomorphisms in rational homology. Hence $\bar{\chi}(L_0) = \bar{\chi}(L)$. Since L_0 is torsion-free, virtually nilpotent and finitely generated, it is a Poincaré duality group of type FP and our naive Euler characteristic coincides with the genuine classical Euler characteristic which is zero in this case (see [4] pp. 201, 213, 224, [7]). Thus $\bar{\chi}(L)$ is zero.

Since G/K is a crystallographic group it admits a proper cocompact action on a Euclidean space X which we may suppose to be endowed with a CW-structure. In this way we have a contractible cocompact G-CW-complex X in which each stabilizer is a finite extension of K (see [12] (3.1.2), (3.1.3)). Consider the Leray spectral sequence ([4] VII.7.10)

$$E^1_{p,q} := \bigoplus_{\dim\sigma = p} H_q(G_\sigma;\mathbb{Q}) \qquad \Longrightarrow \qquad H_{p+q}(G;\mathbb{Q})$$

where σ runs through a set of orbit representatives of cells in X. Reading Euler characteristics, we have

$$\begin{aligned}\bar{\chi}(G) &= \sum_{p,q}(-1)^{p+q}\dim E^1_{p,q}\\ &= \sum_{\sigma}(-1)^{\dim\sigma}\left(\sum_q(-1)^q\dim H_q(G_\sigma;\mathbb{Q})\right)\\ &= \sum_\sigma(-1)^{\dim\sigma}\bar{\chi}(G_\sigma).\end{aligned}$$

The stabilizers G_σ are all groups of the form L considered above and so their naive Euler characteristics are always zero. Hence $\bar{\chi}(G)$ is zero too. □

Lemma 4.2. *If $\widetilde{H}(G)$ is torsion then $\overline{\chi}(G/\tau(G)) = 1$.*

Proof. The universal coefficient theorem shows that

$$H_n(G; \mathbb{Q}) = H_n(G; \mathbb{Z}) \otimes \mathbb{Q} = 0$$

for all $n \geq 1$, and hence $\overline{\chi}(G) = 1$. The subgroup $\tau(G)$ makes no impact on the rational homology with trivial coefficients and therefore $\overline{\chi}(G/\tau(G)) = 1$ also. □

Lemma 4.3. *If $\mathrm{thd}_{\mathbb{F}_p}(G) < \infty$ for all primes p occurring as orders of elements of G, then $G/\tau(G)$ is torsion-free.*

Proof. Using Proposition 3.1 (1), there is a finite dimensional $\mathbb{C}$-linear representation ρ of G having kernel $\tau(G)$. Theorem 2.2 shows that every element of finite order in G belongs to the kernel of ρ, and hence $G/\tau(G)$ is torsion-free. □

To prove the main theorem, suppose now that $\widetilde{H}(G)$ is torsion, yet for all primes p occurring as orders of elements of G only finitely many $H_n(G; \mathbb{Z})$ contain p-torsion. Then by universal coefficients, for all such p, $\mathrm{thd}_{\mathbb{F}_p}(G) < \infty$. Bringing Lemmas 4.2 and 4.3 into play, we see that $G/\tau(G)$ is torsion-free and has naive Euler characteristic equal to 1. Applying Lemma 4.1 to this quotient shows that it is trivial, and so $G = \tau(G)$ is locally finite. Therefore, as in the proof of Theorem 2.3, its reduced rational homology is zero. By Theorem 2.3 and the Universal Coefficient Theorem, we conclude that G is acyclic.

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Department of Mathematics, National University of Singapore, Kent Ridge 117543, SINGAPORE
E-mail address: berrick@math.nus.edu.sg

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, ENGLAND UK
E-mail address: P.H.Kropholler@qmw.ac.uk