

ACYCLIC GROUPS AND WILD ARCS

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ABSTRACT. We discuss two classes of acyclic groups that are commutator subgroups of finitely presented groups with infinite cyclic abelianization. The first is algebraic and includes groups first exhibited by Baumslag & Gruenberg, of which it is shown that Epstein's acyclic group is a special case. The second class is geometric, and is shown to include a number of wild arc groups in the literature.

0. INTRODUCTION

This paper introduces unifying ideas to the study of acyclic groups of low dimension. Acyclic groups, those groups whose positive degree trivial-coefficients homology vanishes, occur in many areas of mathematics, so it is perhaps surprising that some systematization is possible.

It was shown in [15] that the automorphism groups of large mathematical objects tend to be acyclic. To prove this, one exploits the presence of a special structure, called a binate structure in [5]. A number of other historic examples exhibit binate structure (see [6] for a survey).

Examples of acyclic groups without binate structure have been more sporadic. The most famous is Higman's four-generator, four-relator group. In view of the fact that binate groups have no nontrivial finite quotients [1], it is noteworthy that, when originally introduced in [17], this group was also shown to have no nontrivial finite quotient; the proof of acyclicity followed only two decades later [10]. The first groups announced as acyclic were those of Baumslag-Gruenberg [4] and Epstein [11] in 1967. Whereas the former group is defined purely algebraically (as the perfect commutator subgroup of a two-generator, one-relator group with infinite cyclic abelianization), the latter is described as the fundamental group of an open manifold.

In this work, we expand these algebraic and geometric approaches, to highlight the following two classes of acyclic groups that are commutator subgroups of finitely presented groups with infinite cyclic abelianization.

Theorem A. *Let*

$$B = \langle x_n \mid r(x_n, x_{n+1}, \dots, x_{n+k}) \rangle_{n \in \mathbb{Z}}$$

where r is a word in the free group of rank $k + 1$. Then the following statements are equivalent.

- (a) r has exponent sum zero in k of its variables, and exponent sum ± 1 in the remaining variable.
- (b) B is a perfect group.
- (c) Define $G = \langle x, y \mid r(x, yxy^{-1}, \dots, y^kxy^{-k}) \rangle$. Then
 - (i) G_{ab} is infinite cyclic,
 - (ii) B is isomorphic to the commutator subgroup of G , and

(iii) B is acyclic.

We call groups B that satisfy the above conditions acyclic groups of *Baumslag-Gruenberg type*, and show that Epstein's group is an example.

Theorem B. *Let λ be a smooth knot in $X = S^2 \times S^1$ such that $[\lambda]$ generates $H_1(X)$. Then the connected infinite cyclic cover of $X \setminus \lambda$ is the complement of a wild arc κ in S^3 with the following properties.*

- (i) $S^3 \setminus \kappa$ is aspherical;
- (ii) $\pi_1(S^3 \setminus \kappa)$ is the commutator subgroup of $\pi_1(X \setminus \lambda)$ with

$$\pi_1(X \setminus \lambda) / \pi_1(S^3 \setminus \kappa) \cong \mathbb{Z};$$

- (iii) $\pi_1(S^3 \setminus \kappa)$ is acyclic.

Wild arc groups studied by Fox and Artin more than half a century ago [13] fit into this framework, as also do those studied recently by Freedman and Freedman [14]. In general, there is a rich class of acyclic wild arc groups that may be formed from a simple basic pattern that we call a *stitch*. Knot theory suggests guidelines for more systematic study of this class. We expect that many properties of knots and links, such as unknotting number, tunnel number, bridge index, symmetries, could be suitably translated and discussed in the context of stitches and wild arcs. A sample is the following.

Theorem C. *Let κ be the wild arc in S^3 corresponding to a tunnel number 1 stitch. Then $\pi_1(S^3 \setminus \kappa)$ is a group of Baumslag-Gruenberg type, in that $G = \pi_1(S^2 \times S^1 \setminus \lambda)$ and its commutator subgroup $B = \pi_1(S^3 \setminus \kappa)$ satisfy the equivalent conditions of Theorem A.*

The authors are grateful to K W Gruenberg and D B A Epstein for acquainting them with some of the early literature on the subject, and for encouragement provided by M Cencelj, who has discovered interesting linkages to gropes; they also thank J Howie and an anonymous referee for insightful comments. Several developments in this paper are pursued further in [8]. Contrasts between the low-dimensional acyclic groups introduced below and properties of binate groups are the subject of [7].

1. ACYCLIC GROUPS OF BAUMSLAG-GRUENBERG TYPE

In this section we first prove Theorem A characterizing the acyclic commutator subgroups of two-generator, one-relator groups. We then address the issue of when these groups have nontrivial finite quotients, a recurring theme for this work.

Proof of Theorem A. We write the abelianization of B additively, so that its defining relations become

$$\varepsilon_0 x_n + \varepsilon_1 x_{n+1} + \cdots + \varepsilon_k x_{n+k} = 0 \quad n \in \mathbb{Z}.$$

(a) \implies (b). For some h with $0 \leq h \leq k$, the relations reduce to $\pm x_{n+h} = 0$. So each generator maps to zero in the abelianization.

(b) \implies (c). Evidently, B sits inside G as $x_n = y^n x y^{-n}$. So because B is perfect, $x = x_0 \in B \subseteq [G, G]$. In fact, B is generated by the conjugates of x , hence normal in G , with G/B an infinite cyclic group generated by the image of y . Thus also $[G, G] \subseteq B$. So $B = [G, G]$ and $G_{ab} = G/B$. Given (b),(i) and (ii), then (iii) is immediate from [4] Theorem 4.

Of course, (c)(iii) implies (b), so it remains to show:

(b) \implies (a). Observe that there is an epimorphism from B onto the additive group of the ring

$$\mathbb{Z}[t, t^{-1}]/(\varepsilon_0 + \varepsilon_1 t + \cdots + \varepsilon_k t^k)$$

given by $x_n \mapsto t^n$. So when B is perfect its image must be zero. This group is zero precisely when the polynomial $\varepsilon_0 + \varepsilon_1 t + \cdots + \varepsilon_k t^k$ is a unit, and that occurs just when (a) holds. \square

Example 1.1. An especially straightforward example is the acyclic group B corresponding to the relator

$$r(x_n, x_{n+1}) = x_n[x_n^a, x_{n+1}^b].$$

An interesting property of this group, in contrast to other acyclic groups, given by the above theorem and met later, is that it has no nontrivial finite quotients. This may be seen by the following argument, modelled on that of [17]. First observe that, for $k \geq 1$,

$$y_n^{(a+1)^k} = y_{n+1}^{kb} y_n^{a^k} y_{n+1}^{-kb},$$

where y_i is the image of x_i , of order h_i say, in a finite quotient. Make the assumption that $h_n > 1$ for some n , whence, from this equation, $h_{n+1} > 1$ also. Write: $\text{lpf}(h)$ for the least prime factor of an integer $h \geq 2$; \bar{a} for the residue of $a \pmod{\text{lpf}(h_n)}$; and r_n for the order of $1 + \bar{a}^{-1}$ in the field with $\text{lpf}(h_n)$ elements. It follows from the equation above, with $k = h_{n+1}$, that h_n divides $(a+1)^{h_{n+1}} - a^{h_{n+1}}$ (whence $\bar{a} \neq 0$), and so r_n divides h_{n+1} . On the other hand, from Fermat's Little Theorem, r_n divides $\text{lpf}(h_n) - 1$. So

$$\text{lpf}(h_{n+1}) \leq r_n < \text{lpf}(h_n),$$

and iteration of this strict inequality gives the desired contradiction.

Observe that Higman's four-generator, four-relator group [17]

$$\langle x_n \mid x_n[x_n, x_{n+1}] \rangle_{n \in \mathbb{Z}/4}$$

is the quotient of this B (when $a = b = 1$) by the normal subgroup generated by $x_n x_{n+4}^{-1}$. From the theorem above, it is therefore the commutator subgroup of

$$\langle x, y \mid x[x, yxy^{-1}], [x, y^4] \rangle.$$

This prompts the search for a cyclicized version of the theorem above. However the situation seems less clear in this case. For example, the group

$$\langle x_n \mid x_n x_{n+2} x_{n+1}^{-1} \rangle_{n \in \mathbb{Z}/m}$$

with $(m, 6) = 1$ is known to be the fundamental group of a homology 3-sphere obtained as an m -fold cyclic cover of S^3 branched over the trefoil knot, and so perfect but not acyclic. See [18].

2. EPSTEIN'S GROUP AS A WILD ARC GROUP

We turn now to the acyclic group described by Epstein [11]. The theme here is that a careful study enables it to be placed in a wider setting.

Let Γ be the 1-complex in S^3 defined in [11]; that is, Γ consists of two circles Γ_- , Γ_+ and an arc Γ_0 joining them, as in Figure 1. A regular neighbourhood N of Γ is a union of a regular neighbourhood N^- of Γ_- , a regular neighbourhood N^+ of Γ_+ and a regular neighbourhood N^0 of Γ_0 such that $N^- \cap N^0$ and $N^+ \cap N^0$ are disks. Let $M = S^3 \setminus N$.

A Wirtinger presentation of $\pi_1(M)$ can be obtained as follows. Consider Γ as lying on the x - y plane with double points at the crossings and linkage points where Γ_0 intersects Γ_- and Γ_+ . For each overpass with its endpoints either a double point or a linkage point, pick an oriented loop starting at the base point, going around the overpass once and ending at the base point. These loops labelled as a, b, z, p, q, r, s are shown in Figure 1. By abuse of notation, we shall denote the homotopy class of a loop by the loop itself.

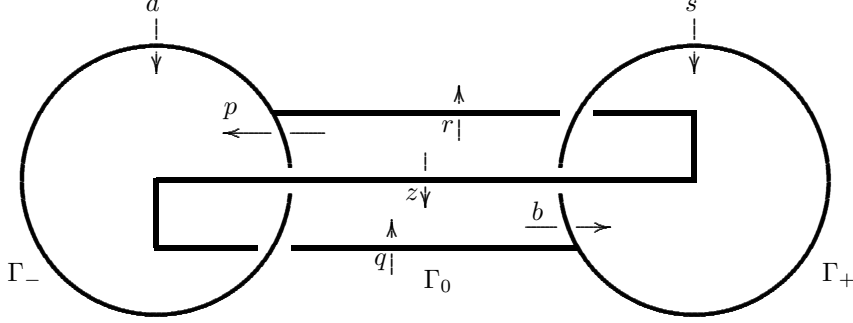


Figure 1. The 1-complex Γ

The group $\pi_1(M)$ is generated by a, b, z, p, q, r and s with relations:

$$ra = p, pz = za, qa = az, rs = sz, zb = sz \text{ and } qb = s.$$

Each relation is obtained as a consequence of a trivial loop under either a crossing or a linkage point of Γ . By eliminating p, q, r and s , we obtain a single relation $aza^{-1} = [z, b]$. Hence $\pi_1(M) = \langle a, b, z \mid aza^{-1} = [z, b] \rangle$.

It is shown in [11] that $\pi_1(\partial N^- \setminus N^0)$ and $\pi_1(\partial N^+ \setminus N^0)$ inject into $\pi_1(M)$. In our presentation of $\pi_1(M)$, the subgroups $\pi_1(\partial N^- \setminus N^0)$ and $\pi_1(\partial N^+ \setminus N^0)$ are free groups generated by a, z and b, z respectively. Here the pair of loops a, z (respectively b, z^{-1}) form a pair consisting of an oriented longitude and meridian of the torus ∂N^- (respectively ∂N^+).

For each $i \in \mathbb{Z}$, let M_i be a copy of M , with N_i^-, N_i^+, N_i^0 corresponding to N^-, N^+, N^0 , and write $\pi_1(M_i)$ as $\langle a_i, b_i, z_i \mid a_i z_i a_i^{-1} = [z_i, b_i] \rangle$. Also, let $f_i : \partial N_i^- \setminus N_i^0 \rightarrow \partial N_{i+1}^+ \setminus N_{i+1}^0$ be the homeomorphism that switches the longitude and the meridian. Hence, $f_{i*}(a_i) = z_{i+1}^{-1}$ and $f_{i*}(z_i) = b_{i+1}$. The space L in [11] is defined to be the adjunction space

$$\cdots \cup_{f_{i-1}} M_i \cup_{f_i} M_{i+1} \cup_{f_{i+1}} \cdots$$

Proposition 2.1. *L has the following properties.*

(i) *L is aspherical and*

$$\pi_1(L) = \langle a_i, b_i, z_i \mid a_i z_i a_i^{-1} = [z_i, b_i], a_i = z_{i+1}^{-1}, z_i = b_{i+1} \rangle_{i \in \mathbb{Z}}.$$

(ii) *L is the infinite cyclic cover of a 3-manifold X' , and $\pi_1(L)$ is the commutator subgroup of*

$$\pi_1(X') = \langle z, t \mid z = [z, tz^{-1}t^{-1}][z, t^{-1}zt] \rangle.$$

(iii) *$\pi_1(L)$ is an acyclic group.*

In particular, $\pi_1(L)$ is an acyclic group of Baumslag-Gruenberg type. A proof of this proposition is presented in the Appendix below.

Interestingly, $\pi_1(L)$ affords an example of a group of Baumslag-Gruenberg type with a nontrivial finite quotient, in contrast to those of Example 1.1. For, it admits a nontrivial representation in \mathfrak{A}_5 given by sending z_i to the 5-cycle (12345), z_{i+1} to (21435) and z_{i+2} to (41325), where $i \equiv 0 \pmod{3}$.

Next, we turn to a more geometric study of Epstein's group, and show that it is the 'knot group' of a wild arc κ in S^3 .

To begin, fix two points $\mathbf{p} = (\frac{6}{5}, 0, 0)$ and $\mathbf{q} = (-\frac{6}{5}, 0, 0)$ in $\mathbb{R}^3 \cup \{\infty\} = S^3$. These will be the two limiting endpoints of κ . First, we express $S^3 \setminus \{\mathbf{p}, \mathbf{q}\}$ as a union of two descending nested sequences of solid tori such that the intersection of each sequence is point \mathbf{p} or \mathbf{q} respectively.

For each $n \in \mathbb{Z}^+$, let $a_n = \frac{6}{5} - \frac{1}{5} \frac{1}{6^n}$ and $b_n = \frac{6}{5} + \frac{4}{5} \frac{1}{6^n}$. Note that both (a_n) and (b_n) converge to the x -coordinate of \mathbf{p} . Let H_n be the solid torus obtained by revolving the disk

$$\left\{ (x, 0, z) \in \mathbb{R}^3 \mid \left(x - \frac{a_n + b_n}{2}\right)^2 + z^2 \leq \frac{1}{4} \left(\frac{1}{6^n}\right)^2 \right\}$$

about the axis $\lambda_n = \{(2b_n - a_n, 0, z) \mid z \in \mathbb{R}\}$. We regard H_n as sitting inside $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Then $H_{n+1} \subset \overset{\circ}{H}_n$ and $\mathbf{a}_n = (a_n, 0, 0)$ is a point on ∂H_n , with the sequence (\mathbf{a}_n) converging to \mathbf{p} . Let $N_n = H_n \setminus \overset{\circ}{H}_{n+1}$. Then

$$(S^3 \setminus \overset{\circ}{H}_1) \cup \bigcup_{n=1}^{\infty} N_n = S^3 \setminus \left(\bigcap_{n=1}^{\infty} \overset{\circ}{H}_n \right) = S^3 \setminus \{\mathbf{p}\}.$$

Similarly, for each $n \in \mathbb{Z}^-$, let H_n be the solid torus obtained by reflecting each H_{-n} about the yz -plane. Also let $N_n = H_n \setminus \overset{\circ}{H}_{n-1}$. Then

$$S^3 \setminus \{\mathbf{q}\} = (S^3 \setminus \overset{\circ}{H}_{-1}) \cup \bigcup_{n=-\infty}^{-1} N_n.$$

Hence,

$$S^3 \setminus \{\mathbf{p}, \mathbf{q}\} = (S^3 \setminus (\overset{\circ}{H}_{-1} \cup \overset{\circ}{H}_1)) \cup \bigcup_{n \neq 0} N_n.$$

Identify $S^3 \setminus (N^- \cup N^+)$ with $S^3 \setminus (\overset{\circ}{H}_{-1} \cup \overset{\circ}{H}_1)$ by means of a homeomorphism ϕ_0 such that $\phi_0(\Gamma_0 \cap \partial N^-) = \mathbf{a}_{-1}$ and $\phi_0(\Gamma_0 \cap \partial N^+) = \mathbf{a}_1$. We denote the arc $\phi_0[\Gamma_0]$ in $S^3 \setminus (\overset{\circ}{H}_{-1} \cup \overset{\circ}{H}_1)$ by κ_0 . See Figure 2.

Also, $S^3 \setminus (N^- \cup N^+) = (S^3 \setminus N^-) \setminus N^+$ is homeomorphic to a solid torus $S^3 \setminus N^-$ with the trivially embedded open solid torus N^+ deleted. For each $n \in \mathbb{Z}^+$, fix a homeomorphism ϕ_n from $(S^3 \setminus N^-) \setminus N^+$ onto N_n so that $\phi_n(\Gamma_0 \cap \partial N^-) = \mathbf{a}_n$ and $\phi_n(\Gamma_0 \cap \partial N^+) = \mathbf{a}_{n+1}$. Denote the image of the arc Γ_0 by κ_n , an arc in N_n joining \mathbf{a}_n to \mathbf{a}_{n+1} .

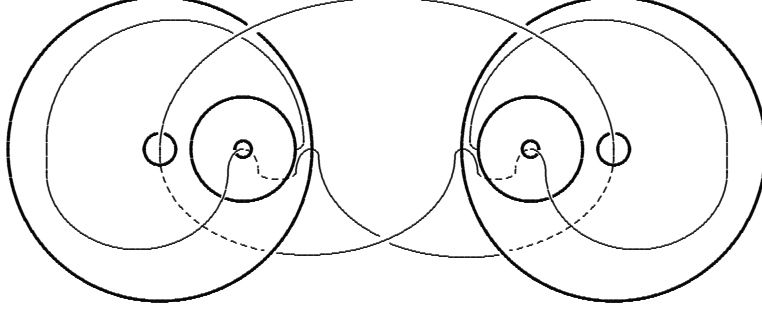


Figure 2. The first two stages of the construction

Similarly, $(S^3 \setminus N^+) \setminus N^-$ is homeomorphic to a solid torus with the trivially embedded open solid torus N^- deleted. We take the mirror image of the above construction. For each $n \in \mathbb{Z}^-$, fix a homeomorphism ϕ_n from $(S^3 \setminus N^+) \setminus N^-$ onto N_n so that $\phi_n(\Gamma_0 \cap \partial N^+) = \mathbf{a}_n$ and $\phi_n(\Gamma_0 \cap \partial N^-) = \mathbf{a}_{n-1}$. Again, denote the image of the arc Γ_0 by κ_n . Then

$$\kappa = \{\mathbf{p}\} \cup \bigcup_{n=-\infty}^{\infty} \kappa_n \cup \{\mathbf{q}\}$$

is a wild arc in S^3 . See the first two stages of the construction in Figure 2 and the resulting wild arc in Figure 3.

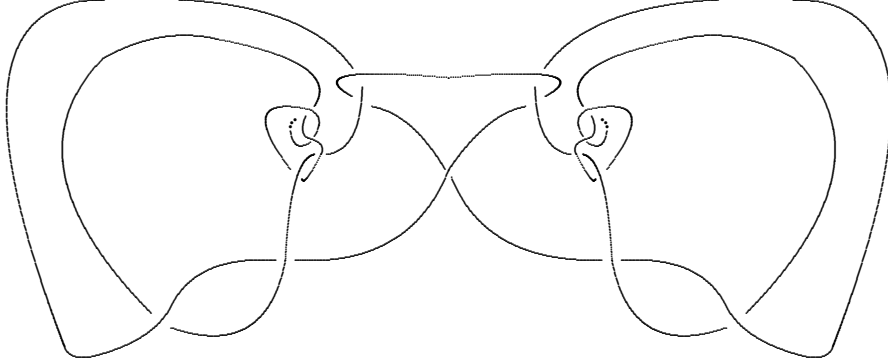


Figure 3. The wild arc

Fix a pair of oriented loops α, β on ∂N^- and α', β' on ∂N^+ . Denote the corresponding images of α, β, α' and β' in N_n under ϕ_n by $\alpha_n, \beta_n, \alpha'_n$ and β'_n respectively. If we take $M_n = N_n \setminus \kappa_n$, then $S^3 \setminus \kappa$ is simply the adjunction space $L = \bigcup_{n=-\infty}^{\infty} M_n$, in which α'_n is identified with β_{n+1} and β'_n is identified with α_{n+1} . Therefore the space L is in fact the complement of the wild arc κ in S^3 , and $\pi_1(S^3 \setminus \kappa)$ is Epstein's group.

3. THE GENERAL CONSTRUCTION

We now show how to generalize the construction of the previous section, leading to a whole class of geometrically defined acyclic groups. In general, we start with a closed connected orientable 3-manifold X with infinite cyclic universal cover $S^2 \times \mathbb{R}$.

Lemma 3.1. *Let X be a closed connected orientable 3-manifold with infinite cyclic cover $S^2 \times \mathbb{R}$. Then X is homeomorphic to $S^2 \times S^1$.*

Proof. By the prime factorization of compact 3-manifolds [16], we may write $X = X_0 \# X_1 \# \cdots \# X_k$, where X_0, \dots, X_k are closed prime 3-manifolds. As $\pi_1(X) = \mathbb{Z}$, one of these prime 3-manifolds, say X_0 , has fundamental group \mathbb{Z} and the rest are all homotopy 3-spheres. Let $X' = X_1 \# \cdots \# X_k$. Then we have $X = X_0 \# X'$, where X_0 is a closed connected orientable prime 3-manifold with infinite cyclic fundamental group and X' is a homotopy 3-sphere. Since X is a homotopy $S^2 \times S^1$, X_0 is also a homotopy $S^2 \times S^1$ so that $\pi_2(X_0)$ is non-trivial. By the Sphere Theorem, there exists an embedded 2-sphere representing a nontrivial element of $\pi_2(X_0)$. Hence, X_0 is not irreducible. By Lemma 3.13 in [16], X_0 is a 2-sphere bundle over S^1 . As X_0 is orientable, it must be $S^2 \times S^1$. Therefore, $X = (S^2 \times S^1) \# X'$. Now the infinite cyclic cover \tilde{X} of X can be constructed by taking the connected sum of the infinite cyclic cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$ with a countable number of copies of X' . Since we assume the infinite cyclic cover of X is also $S^2 \times \mathbb{R}$, \tilde{X} is homeomorphic to $S^2 \times \mathbb{R}$. However, $S^2 \times \mathbb{R}$ is embeddable in S^3 which contains no fake 3-cell. Therefore, X' must be S^3 and X is $S^2 \times S^1$. \square

Let $X = S^2 \times S^1$. Corresponding to the projection $p : X \rightarrow S^1$ we have the fundamental group isomorphism $p_* : \pi_1(X) \rightarrow \mathbb{Z}$. Let λ be a smooth loop in X such that $p_*(\lambda)$ is a generator of \mathbb{Z} . Since λ generates $H_1(X)$, there exists λ^* in $H^1(X) = \text{Hom}(H_1(X), \mathbb{Z})$ such that $\lambda^*(\lambda) = 1$. Then, since X is orientable, by Poincaré duality there exists a closed orientable surface \mathcal{F} in X intersecting λ algebraically at one point. Write Z for the space obtained by cutting X open along \mathcal{F} . This means that Z is obtained by deleting an open collar about \mathcal{F} . The boundary of Z consists of two copies \mathcal{F}^+ and \mathcal{F}^- of \mathcal{F} . Let $\phi : \mathcal{F}^+ \rightarrow \mathcal{F}^-$ be the homeomorphism which identifies \mathcal{F}^+ with \mathcal{F}^- to get X . To construct the infinite cyclic cover, take countably many copies Z_n of Z , $n \in \mathbb{Z}$ and form the adjunction space

$$\cdots \cup_{\phi_{n-1}} Z_n \cup_{\phi_n} Z_{n+1} \cup_{\phi_{n+1}} \cdots$$

It is homeomorphic to the connected infinite cyclic cover $S^2 \times \mathbb{R}$ of X . (The other infinite cyclic covers have the form $S^2 \times \mathbb{R} \times \{\text{finite set}\}$ or $S^2 \times S^1 \times \mathbb{Z}$, and so are not connected.)

Now, λ intersects \mathcal{F} algebraically in one point and lifts to an open arc κ' in $S^2 \times \mathbb{R}$. As $S^2 \times \mathbb{R}$ is homeomorphic to $S^3 \setminus \{\mathbf{p}, \mathbf{q}\}$, we may embed $S^2 \times \mathbb{R}$ into S^3 so that the ‘ends’ of $S^2 \times \mathbb{R}$ converge to \mathbf{p} and \mathbf{q} respectively. Then, $\kappa = \kappa' \cup \{\mathbf{p}, \mathbf{q}\}$ is a wild arc in S^3 , and $S^3 \setminus \kappa$ is the connected infinite cyclic cover of $X \setminus \lambda$.

Proof of Theorem B. First, we show that $H_1(X \setminus \lambda)$ is infinite cyclic and is generated by the homology class of a pushoff of λ in $X \setminus \lambda$. To do this, let λ intersect \mathcal{F} in points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$. As λ intersects \mathcal{F} algebraically at one

point, n must be odd. We may assume that the points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ are arranged in that order along the loop λ . Modify the surface \mathcal{F} by adding ‘tubes’ joining \mathbf{p}_i to \mathbf{p}_{i+1} , $i = 1, \dots, n-2$. Let the resulting surface be \mathcal{F}' . Then λ intersects \mathcal{F}' only at \mathbf{p}_n . The boundary of a regular neighbourhood $N(\lambda)$ of λ intersects \mathcal{F}' in a loop which is a meridian γ of λ . This means that γ bounds a surface in $X \setminus \lambda$. Hence, $[\gamma] = 0$ in $H_1(X \setminus \lambda)$. If a 2-handle D is attached to $X \setminus N(\lambda)$ along γ , the resulting space is homeomorphic to X minus a 3-ball. Consider the Mayer-Vietoris sequence:

$$\langle [\gamma] \rangle \longrightarrow H_1(X \setminus N(\lambda)) \oplus H_1(D) \longrightarrow H_1(X \setminus \text{ball}) = H_1(X) = \mathbb{Z}.$$

As $[\gamma]$ is mapped to 0 in $H_1(X \setminus N(\lambda))$, we have $H_1(X \setminus \lambda) = H_1(X) = \mathbb{Z}$ and is generated by $[\lambda]$ or precisely the homology class of a pushoff of λ in $X \setminus \lambda$.

Proof of (ii). Consider the cover $S^3 \setminus \kappa$ of $X \setminus \lambda$. By construction,

$$\pi_1(X \setminus \lambda) / \pi_1(S^3 \setminus \kappa) = \mathbb{Z}.$$

Hence the commutator subgroup H of $\pi_1(X \setminus \lambda)$ lies in $\pi_1(S^3 \setminus \kappa)$. Moreover,

$$\begin{aligned} \mathbb{Z} &= \pi_1(X \setminus \lambda) / \pi_1(S^3 \setminus \kappa) \\ &= \frac{\pi_1(X \setminus \lambda) / H}{\pi_1(S^3 \setminus \kappa) / H} \\ &= \frac{\mathbb{Z}}{\pi_1(S^3 \setminus \kappa) / H}. \end{aligned}$$

Hence $\pi_1(S^3 \setminus \kappa) = H$ is the commutator subgroup of $\pi_1(X \setminus \lambda)$.

Proof of (iii). The meridian γ of λ lifts to a meridian $\tilde{\gamma}$ of κ in $S^3 \setminus \kappa$. Also the surface that γ bounds in $X \setminus \lambda$ lifts to a surface in $S^3 \setminus \kappa$. Hence $[\tilde{\gamma}] = 0$ in $H_1(S^3 \setminus \kappa)$. Now add a 2-handle D to $S^3 \setminus \kappa$ along γ . Consider the exact sequence:

$$H_1(S^1) \xrightarrow{j_*} H_1(D) \oplus H_1(S^3 \setminus \kappa) \longrightarrow H_1(S^3 \setminus \text{twopoints}) = 0.$$

As j_* maps a generator of S^1 to $[\tilde{\gamma}]$ and $[\tilde{\gamma}] = 0$ in $H_1(S^3 \setminus \kappa)$, we have j_* is the zero map. Hence, $H_1(S^3 \setminus \kappa) = 0$.

Next consider the continuation of the above exact sequence:

$$0 \longrightarrow H_2(D) \oplus H_2(S^3 \setminus \kappa) \longrightarrow H_2(S^3 \setminus \text{twopoints}) \xrightarrow{\partial} H_1(S^1) \longrightarrow 0.$$

As ∂ is an isomorphism, $H_2(S^3 \setminus \kappa) = 0$.

Since $S^3 \setminus \kappa$ is a non-compact 3-manifold, $H_i(S^3 \setminus \kappa) = 0$ for all $i \geq 3$.

Proof of (i). This follows from Corollary 26.2 of [21], the version of the Sphere Theorem that applies to the present setting. \square

In the event that $\pi_1(X \setminus \lambda)$ has a presentation consisting of two generators and one relation, then, by [4], one has an alternative proof from (ii) that $\pi_1(S^3 \setminus \kappa)$ is acyclic. This frequently occurs in the examples that we give in the next section.

The complement of a wild arc in S^3 is an example of a noncompact manifold that fits into the framework of Freedman and Freedman [14].

Corollary 3.2. *If a knot in $S^2 \times S^1$ gives rise to a wild arc κ in S^3 that is not isotopic to the trivial arc, then $\pi_1(S^3 \setminus \kappa)$ contains a closed surface group as a subgroup. In particular, κ is isotopic to the trivial arc if and only if $\pi_1(S^3 \setminus \kappa) = 1$.*

Proof. To see that $\pi_1(S^3 \setminus \kappa)$ must be nontrivial, recall that $S^3 \setminus \kappa = S^2 \times \mathbb{R} \setminus \kappa'$, where κ' is the inverse image of a smooth knot λ in $S^2 \times S^1$ and $[\lambda]$ generates $H_1(S^2 \times S^1)$. Cut $S^2 \times S^1$ open along $S^2 \times \{\text{point}\}$ to obtain $S^2 \times I$, where I is the interval $[-1, 1]$. Since λ is smooth, it follows for dimension reasons that the cut open arc λ' corresponding to λ does not map onto $S^2 \times \{1\}$ under the projection from $S^2 \times I$ to $S^2 \times \{1\}$. Hence, there exists a closed 2-disk E in $S^2 \times \{1\}$ such that $(E \times I) \cap \lambda' = \emptyset$. Consequently, $(E \times \mathbb{R}) \cap \kappa' = \emptyset$. In other words, κ' lies inside $(S^2 \setminus E^\circ) \times \mathbb{R}$ which is just $D \times \mathbb{R}$, where D is a 2-disk. Now if $\pi_1(S^3 \setminus \kappa) = \pi_1(S^2 \times \mathbb{R} \setminus \kappa') = 1$, then $\pi_1(D \times \mathbb{R} \setminus \kappa') = \mathbb{Z}$. By Corollary 2 in [14], κ' is isotopic to a core of $D \times \mathbb{R}$, so that κ is just the trivial arc in S^3 , contradicting our assumption.

Therefore, by Theorem B, $\pi_1(S^3 \setminus \kappa)$ is a nontrivial perfect group. Hence it cannot be a nontrivial free group. By Theorem 3 in [14], it is not locally free. It follows that the equivalent assertions of Theorem 2 in [14] all hold, namely that (i) $S^3 \setminus \kappa$ always contains a closed incompressible surface, (ii) $S^3 \setminus \kappa$ cannot be exhausted by handlebodies and (iii) $\pi_1(S^3 \setminus \kappa)$ is not locally free. In other words, the group of a nontrivial wild arc always contains a surface group. \square

By comparison, in classical knot theory, a knot in S^3 is a fibred knot if and only if its commutator subgroup is finitely generated and free. The results in [14] imply that the commutator subgroup of a non-fibred knot in S^3 always contains a surface group. However, the infinite cyclic cover of a non-fibred knot in S^3 is a complicated open 3-manifold not necessarily embeddable in S^3 . In this context, one recalls the construction in [3] of an acyclic group that contains all closed surface groups (of genus ≥ 2).

It is interesting to observe that $X = S^2 \times S^1$ can be built by identifying the two boundary components of a connected sum of two handlebodies of genus g . More precisely, let T be a handlebody of genus g , with $\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \beta_2, \dots, \beta_g$ a set of oriented simple closed loops representing a base of $H_1(\partial T)$ such that each α_i bounds a disk in T . Let Z be the connected sum of two copies T' and T'' of T . Then $\partial X = \partial T' \cup \partial T''$. Write $\phi : \partial T' \rightarrow \partial T''$ for the homeomorphism that maps α'_i to β''_i and β'_i to α''_i , and let X be the quotient space of Z obtained by identifying $\partial T'$ with $\partial T''$ via ϕ . Denote the image of the surface $\partial T'$ or $\partial T''$ in X by \mathcal{F} . Then X is $S^2 \times S^1$. Now the infinite cyclic cover $S^2 \times \mathbb{R}$ is the union of a countable number of copies of Z . Two cases receive special attention. First, Epstein's construction, discussed above, has $\partial T = S^1 \times S^1$.

We next analyze the case where $\partial T = S^2$. In fact, this is the construction introduced by Fox and Artin in [13].

4. ACYCLIC GROUPS OF FOX-ARTIN TYPE (STITCH GROUPS)

In this section, we give some simple examples of wild arcs constructed according to the above recipe. By Theorem B, their groups are all acyclic. We

find that all our examples admit nontrivial representations into the simple group $\mathfrak{A}_5 = PSL(2, 5)$ of order 60. We begin with a formal definition of an n -stitch, and describe how it leads to a wild arc.

Let n be an odd positive integer and $K = [-1, 1]^3$ the cube. Let $y_i = -1 + \frac{2i}{n+1}$, so that y_1, y_2, \dots, y_n are n equally spaced points in $(-1, 1)$, and for $i \in \{-1, \dots, -n, 1, \dots, n\}$ write $Y_i = (\text{sgn}(i), y_{|i|}, 0)$.

Definition 4.1. An n -stitch \mathbf{s} is a set $\{s_1, \dots, s_n\}$ of mutually disjoint arcs (called *strands*) properly and tamely embedded in K such that for some ordering s_1, \dots, s_n there is a permutation

$$\sigma = \begin{bmatrix} 1 & 2 & \cdots & n \\ \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{bmatrix}$$

of $(1, 2, \dots, n)$ whereby

- (i) s_1 joins $Y_{-\sigma_1}$ to Y_{σ_2} ,
- (ii) s_i joins $Y_{(-1)^{i+1}\sigma_i}$ to $Y_{(-1)^{i+1}\sigma_{i+1}}$, $i = 2, \dots, n-1$, and
- (iii) s_n joins Y_{σ_n} to Y_{σ_1} .

Observe that s_1 is the unique strand that joins $\{-1\} \times [-1, 1] \times \{0\}$ to $\{1\} \times [-1, 1] \times \{0\}$. The ordering of strands, and thereby the permutation σ , is therefore uniquely determined and so, in a sense, redundant. However, the information that the self-map σ is a bijection is crucial.

We represent a stitch by drawing its projection on the xy -plane. Below is an example of a 5-stitch associated with the permutation

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{bmatrix}.$$



Figure 4. A 5-stitch

Given a stitch \mathbf{s} , take a countable number of copies \mathbf{s}_i of \mathbf{s} , $i \in \mathbb{Z}$. By identifying the right yz -face of \mathbf{s}_i with the left yz -face of \mathbf{s}_{i+1} , $i \in \mathbb{Z}$, we obtain $\mathbb{R} \times [-1, 1]^2$ together with an embedding of an open arc κ' in it. We may take a trivial embedding of this pair $(\mathbb{R} \times [-1, 1]^2, \kappa')$ into S^3 so that the ends of $\mathbb{R} \times [-1, 1]^2$ converge respectively to two distinct points \mathbf{p} and \mathbf{q} . This gives rise to a wild arc $\kappa = \kappa' \cup \{\mathbf{p}, \mathbf{q}\}$ in S^3 . The fact that κ' is an arc is guaranteed by the permutation σ associated with the stitch. Evidently, an n -stitch has penetration index n , the index being defined as the least integer m such that every point of the arc is the centre of arbitrarily small copies of S^2 that meet the arc in at most m points [2]. Our interest is in the *stitch group* of \mathbf{s} , which is the group of the wild arc κ , namely $\pi_1(S^3 - \kappa)$.

Example 4.2. Fox's stitch. This example is obtained by taking a section from the simple closed curve of Fox [12]. It forms a 3-stitch as shown in Figure 5. It has also been reborn as Figure 4 of [9], Figure 2 of [20], and the X shown in Figure 2 of [14]. (To pass from the original stitch to its reincarnation, reflect about a horizontal plane perpendicular to the page and disjoint from the stitch.)

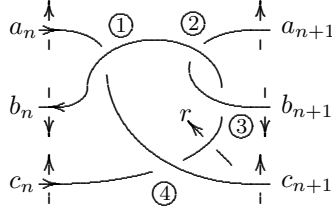


Figure 5. Fox's Stitch

Let a_n, b_n and c_n be generators of $\pi_1(S^3 \setminus \kappa)$ as shown in the diagram. We also introduce a generator r between the crossings 3 and 4. The group's Wirtinger relations, (1) - (4) below, are obtained by following a loop going around once underneath each crossing. Further, the loop $c_n b_n^{-1} a_n$ can be pulled off the stitch completely, giving the relation (0).

$$\begin{aligned} c_n b_n^{-1} a_n &= 1 & (0) \\ a_n b_n &= b_n c_{n+1} & (1) \\ b_n b_{n+1} &= a_{n+1} b_n & (2) \\ b_n b_{n+1} &= b_{n+1} r & (3) \\ c_{n+1} r &= c_n c_{n+1} & (4) \end{aligned}$$

Substituting (1) and (2) into (0), we get $b_{n-1} = [b_{n-2}, b_{n-1}^{-1}][b_{n-1}, b_n^{-1}]$. The relation obtained by eliminating r in (3) and (4) turns out to be a consequence of this relation. Hence, $\pi_1(S^3 \setminus \kappa)$ has the presentation:

$$\langle b_n \mid b_n = [b_{n-1}, b_n^{-1}][b_n, b_{n+1}^{-1}] \rangle_{n \in \mathbb{Z}}.$$

A different reduction leads to the alternative presentation:

$$\langle c_n \mid c_n = [c_{n-1}^{-1}, c_n c_{n+1} c_n^{-1}] \rangle_{n \in \mathbb{Z}}.$$

It admits a representation into \mathfrak{A}_5 by setting

$$b_n = \begin{cases} (21435) & \text{if } n \equiv 0 \pmod{3} \\ (13425) & \text{if } n \equiv 1 \pmod{3} \\ (15324) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

If we identify the generators b_n with b_{n+3} for all n , we obtain a group with a presentation similar to Higman's group in Example 1.1. However, as shown above, this group, unlike the ones in Example 1.1, does admit a nontrivial finite quotient.

Note that Relation (0) (or its n -ary analogue in the case of an n -stitch) always occurs in the group of any wild arc obtained from a stitch. Since all the generators in the Wirtinger presentation of the group of a wild arc are conjugate to each other, Relation (0), which contains $\frac{n+1}{2}$ generators with exponent $+1$ and $\frac{n-1}{2}$ with exponent -1 , has the effect of killing the

abelianization of the group, thus making the group perfect. In the case $n = 1$, it kills the group itself of any wild arc obtained from a 1-stitch.



Figure 6. The wild arc of Fox's stitch

Example 4.3. The first Fox-Artin stitch. This yields the wild arc κ_1 of Example 1.1 in [13]. Following [13], let c_n, a_n and b_n be oriented loops going round the upper, middle and lower arcs respectively in each section of the wild arc.

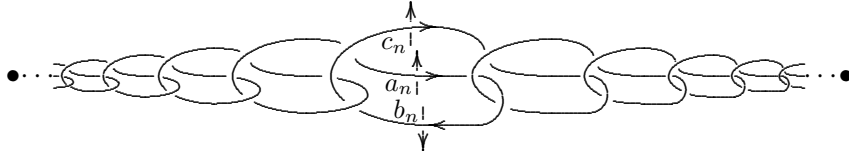


Figure 7. The wild arc κ_1 of Example 1.2 in [13]

$\pi_1(S^3 \setminus \kappa_1)$ has a presentation given by

$$\langle c_n \mid c_{n-1}c_n c_{n+1} = c_n c_{n+1} c_{n-1} c_n \rangle_{n \in \mathbb{Z}}.$$

The relation may be rewritten in the form $c_n = [c_{n-1}^{-1}, c_{n+1}^{-1} c_n^{-1}]$. The group is perfect and is the commutator subgroup of the group

$$\langle x, y \mid x = [y^{-1}xy, yx^{-1}y^{-1}x^{-1}] \rangle$$

with abelianization \mathbb{Z} . Hence, $\pi_1(S^3 \setminus \kappa_1)$ is acyclic, by either Theorem A or Theorem B. As shown in [13], it also admits a nontrivial homomorphism into \mathfrak{A}_5 , by sending c_n to (12345) for n odd and to (14235) for n even.

Example 4.4. The second Fox-Artin stitch. By switching each of the upper crossings in κ_1 of Example 4.3, the wild arc κ_2 in Figure 6 of [13] is obtained. See Figure 8.

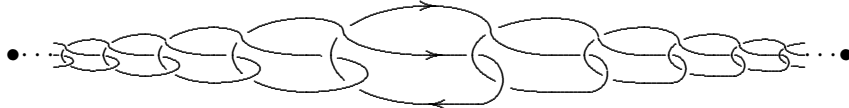


Figure 8. The wild arc κ_2

As noted by Fox, this wild arc is isotopic to that of Figure 6. The isotopy is achieved by flipping the lower clasps of κ_2 , and then rotating by π in the plane of the page.

Example 4.5. The first Fox-Artin stitch with an extra twist. A slightly more complicated-looking stitch appears in Figure 9. It generates a wild arc as shown in Figure 10.

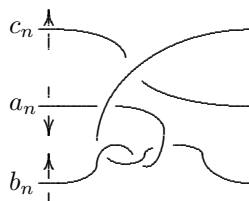


Figure 9.



Figure 10.

The group of this wild arc has a presentation given by

$$\langle c_n \mid c_n = [(c_{n-1}c_n c_{n+1})^2, c_{n-1}] \rangle_{n \in \mathbb{Z}}.$$

This group again is acyclic, being the commutator subgroup of the group $\langle x, y \mid x = [(yxy^{-1}xyxy^{-1})^2, y^{-1}xy] \rangle$. It admits a representation in \mathfrak{A}_5 given by

$$b_n = \begin{cases} (15243) & \text{if } n \equiv 0 \pmod{5} \\ (15432) & \text{if } n \equiv 1 \pmod{5} \\ (13254) & \text{if } n \equiv 2 \pmod{5} \\ (12534) & \text{if } n \equiv 3 \pmod{5} \\ (12453) & \text{if } n \equiv 4 \pmod{5}. \end{cases}$$

However, a computer search shows that there are no representations in \mathfrak{A}_5 in terms of 3-cycles or products of two disjoint cycles.

Example 4.6. The Borromean stitch. Consider the stitch in Figure 11, with wild arc as in Figure 12.

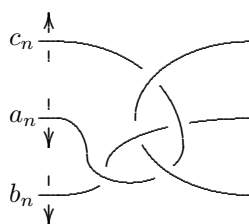


Figure 11.



Figure 12.

The group of this wild arc has a presentation given by

$$\langle a_n, b_n \mid a_{n+1}b_{n+1}a_nb_n = (a_nb_na_n^{-1})(b_{n+1}a_n) = (b_{n+1}a_n)(b_{n+1}^{-1}a_{n+1}b_{n+1}) \rangle_{n \in \mathbb{Z}}.$$

Thus it does not appear to be a group of Baumslag-Gruenberg type. By the identification $a_n = b_{n+1}$, this group maps onto the group

$$\begin{aligned} & \langle a_n \mid a_{n+1} a_n^2 a_{n-1} = a_n a_{n-1} a_n = a_n a_{n+1} a_n \rangle_{n \in \mathbb{Z}} \\ & = \langle x, y \mid yx^2y = xyx, xy^2x = yxy \rangle, \end{aligned}$$

which is just the double cover $SL(2, 5)$ of \mathfrak{A}_5 .

Example 4.7. The stitch in Figure 13 produces a wild arc whose group maps onto $PSL(2, 7)$.

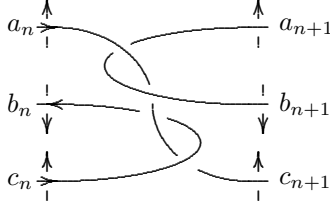


Figure 13.



Figure 14.

The group of this wild arc has a presentation with generators a_n and relations

$$a_n = [[a_{n-1}, a_n^{-1}], [a_{n+1}^{-1}, a_n]], \quad n \in \mathbb{Z}.$$

This group admits a representation into \mathfrak{A}_5 by setting a_n equal to (12345) for n odd, and to (14235) for n even. It also maps onto the simple group $PSL(2, 7)$ of order 168, by sending a_n to the image in $PSL(2, 7)$ of the integral matrix

$$\begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{if } n \equiv 0 \pmod{3} \\ \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} & \text{if } n \equiv 1 \pmod{3} \\ \begin{pmatrix} 4 & 4 \\ 1 & 3 \end{pmatrix} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Example 4.8. The 5-stitch in Figure 4 above gives a wild arc whose group has the presentation with generators a_n , $n \in \mathbb{Z}$ and relations

$$a_n = x_n [a_{n-1}^{-1}, a_n] x_n^{-1}$$

where

$$x_n = a_{n-1} a_n^{-1} [a_{n+1}, a_n^{-1}] [a_n, a_{n-1}^{-1}] a_n^{-1} [a_{n+1}, a_n^{-1}].$$

This group also admits a representation into \mathfrak{A}_5 , given by:

$$a_n = \begin{cases} (12345) & \text{if } n \equiv 0 \pmod{3} \\ (23145) & \text{if } n \equiv 1 \pmod{3} \\ (34125) & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

The above examples (and others) suggest that it may be of interest to investigate finite quotients of stitch groups.

Proposition 4.9. [8] *All stitch groups are residually finite.*

Our examples lead us to ask how common it is for a stitch group to map onto \mathfrak{A}_5 . Observe that, for an acyclic space W , there is a natural bijection between homomorphisms from $\pi_1(W)$ to \mathfrak{A}_5 and pointed homotopy classes of maps from W to the Poincaré homology 3-sphere S^3/I^* . This is because, since $G = \pi_1(W)$ is superperfect, any homomorphism from G to \mathfrak{A}_5 lifts uniquely to its universal central extension, the binary icosahedral group $I^* = SL(2, 5)$. Thus there is a natural (with respect to W) one-to-one correspondence between $\text{Hom}(G, \mathfrak{A}_5)$ and $\text{Hom}(G, I^*)$, and so also $[W, BI^*]$. The inclusion of I^* as a subgroup of S^3 induces a fibration

$$S^3/I^* \rightarrow BI^* \rightarrow BS^3$$

and so a principal fibration

$$S^3 \rightarrow S^3/I^* \rightarrow BI^*.$$

Then the obstructions to existence and uniqueness of liftings of maps $W \rightarrow BI^*$ to maps $W \rightarrow S^3/I^*$ lie in cohomology groups of W with trivial coefficients. By acyclicity, such groups are zero.

In these considerations, the class of stitch groups can also be replaced by other classes, such as the class of acyclic groups of Baumslag-Gruenberg type, or the intersection of these two classes, or the class of all groups of wild arcs derived as in the previous section.

5. STITCH GROUPS OF BAUMSLAG-GRUENBERG TYPE

Here is a class of stitches whose groups we show to be of Baumslag-Gruenberg type. We define the notion of a tunnel number 1 knot in $S^2 \times S^1$ by analogy with the familiar definition in S^3 . Denote the longitude $\{0\} \times S^1$ of $S^2 \times S^1$ by σ_1 . An arc σ_2 in $S^2 \times S^1$ is said to be *trivial* if $\sigma_1 \cap \sigma_2 = \partial\sigma_2$ and if σ_2 together with an arc in σ_1 joining the two boundary points of σ_2 bounds a disk properly embedded in $S^2 \times S^1 \setminus (\sigma_1 \cup \sigma_2)$. In this case, the 1-complex $\sigma = \sigma_1 \cup \sigma_2$ is said to be *trivial* in $S^2 \times S^1$.

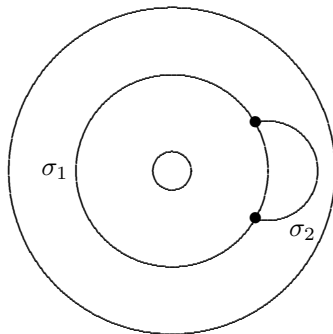


Figure 15. A trivial 1-complex $\sigma = \sigma_1 \cup \sigma_2$ in $S^2 \times S^1$

Note that $\pi_1(S^2 \times S^1 \setminus \sigma)$ is free of rank 2. To identify its generators, observe that deleting a regular neighbourhood of σ_1 leaves a solid torus T_1 . One generator (called a longitude of σ_1) corresponds to a longitude of T_1 , and the other (called a meridian of σ_2) to the boundary of a disc in T_1 that intersects σ_2 in a single point. Also, the complement of a regular neighbourhood of σ in $S^2 \times S^1$ is homeomorphic to a genus 2 handlebody.

Definition 5.1. A knot λ in $S^2 \times S^1$ is called a *tunnel number 1 knot* if there exists an arc μ in $S^2 \times S^1$ with $\lambda \cap \mu = \partial\mu$ and with $\lambda \cup \mu$ isotopic to a trivial 1-complex in $S^2 \times S^1$. The arc μ is called a *tunnel* of $S^2 \times S^1 \setminus \lambda$.

Lemma 5.2. Let λ be a tunnel number 1 knot in $S^2 \times S^1$. Then the group $\pi_1(S^2 \times S^1 \setminus \lambda)$ has a presentation having two generators and one relation.

Proof. Let μ be a tunnel of $S^2 \times S^1 \setminus \lambda$ such that the 1-complex $\lambda \cup \mu$ is isotopic to a trivial 1-complex $\sigma = \sigma_1 \cup \sigma_2$ in $S^2 \times S^1$. Let x be a meridian of σ_2 and y a longitude of σ_1 . Also let r be a meridian of the tunnel μ such that r bounds a disk in $S^2 \times S^1 \setminus \lambda$. Then $S^2 \times S^1 \setminus \lambda$ can be obtained by attaching a 2-disk to $S^2 \times S^1 \setminus \sigma$ along r . Thus, r regarded as an element of $\pi_1(S^2 \times S^1 \setminus \sigma)$ can be expressed as a relator $r(x, y)$ in terms of the generators x, y . Then

$$\pi_1(S^2 \times S^1 \setminus \lambda) = \langle x, y \mid r(x, y) = 1 \rangle. \quad \square$$

Let \mathbf{s} be a stitch. By identifying the two ends of \mathbf{s} (that is, $\{-1\} \times I^2 \sim \{1\} \times I^2$), we get a copy of $D^2 \times S^1$. Since $S^2 \times S^1$ is the union of two copies of $D^2 \times S^1$ whose intersection is a torus, we proceed by adding a 2-handle along a meridian and capping off with a 3-ball. In this way we obtain a pair homeomorphic to $(S^2 \times S^1, \lambda)$, where λ is the knot obtained by identifying the horizontal pairs of endpoints $Y_{\pm i}$ of \mathbf{s} . The definition of a stitch ensures that λ is a knot rather than a link with more than one component.

Definition 5.3. The stitch \mathbf{s} is called a *tunnel number 1 stitch* if, in the resulting pair $(S^2 \times S^1, \lambda)$, λ is a tunnel number 1 knot in $S^2 \times S^1$.

It is now easy to check that such stitches give rise to acyclic groups of Baumslag-Gruenberg type.

Proof of Theorem C. The tunnel number 1 stitch gives rise to a tunnel number 1 knot λ in $S^2 \times S^1$. $S^3 \setminus \kappa$ is the infinite cyclic cover of $S^2 \times S^1 \setminus \lambda$. By Lemma 5.2, $\pi_1(S^2 \times S^1 \setminus \lambda)$ has a presentation having two generators x, y and one relator $r(x, y)$. Since the generator y corresponds to the covering translation, we may let $x_n = y^n x y^{-n}$, $n \in \mathbb{Z}$ and express $r(x, y)$ in terms of x_n to obtain $r(x_n, x_{n+1}, \dots, x_{n+k})$ for some positive k . \square

Note that the number k occurring in the above proof can be made arbitrarily large, even for a 3-stitch. This can be seen by reference to 2-bridge (4-plait) knots. It is easy to check that the examples in 4.2, 4.3, 4.4, 4.5 4.7 and 4.8 above are tunnel number 1 stitches.

Example 5.4. McPherson's stitch. Consider the following pattern of $2k + 1$ arcs, which is taken from Figure 2 of [19].

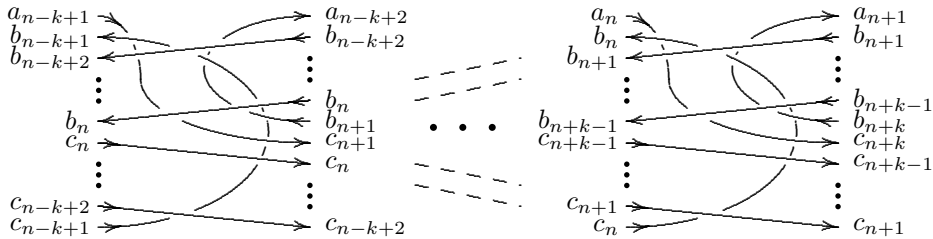


Figure 16. McPherson's stitch

When k of these are juxtaposed, we obtain an n -stitch whose group has the presentation

$$\langle c_n \mid c_n = x_n c_{n+1} x_n^{-1} \cdots x_{n+k-1} c_{n+k} x_{n+k-1}^{-1} c_{n+k}^{-1} \cdots c_{n+1}^{-1} \rangle_{n \in \mathbb{Z}}$$

where

$$x_j = c_j^{-1} \cdots c_{j-k}^{-1} c_{j-k+1} \cdots c_j \quad j \in \mathbb{Z},$$

and so is of Baumslag-Gruenberg type.



Figure 17.

The wild arc corresponding to $k = 2$ is shown in Figure 17, whose group has a presentation that simplifies to

$$\langle b_n \mid b_n = [b_{n-1}, b_n b_{n+1}^{-1} b_n^{-1}] [b_{n-1}^{-1} b_{n-2} b_{n-1}, b_n^{-1}] [b_n, b_{n+1} b_{n+2}^{-1} b_{n+1}^{-1}] \rangle_{n \in \mathbb{Z}}.$$

This group admits a representation into \mathfrak{A}_5 , by setting b_n equal to (12345) for n even and to (21435) for n odd.

APPENDIX

The appendix is devoted to establishing Proposition 2.1.

The asphericity of L is an application of the Sphere Theorem [21] as in [11]. To obtain a presentation of $\pi_1(L)$, observe that, by repeated use of the van Kampen theorem,

$$\pi_1(L) = \langle a_i, b_i, z_i \mid a_i z_i a_i^{-1} = [z_i, b_i], a_i = z_{i+1}^{-1}, z_i = b_{i+1} \rangle_{i \in \mathbb{Z}}.$$

Eliminating a_i and b_i , we have

$$\pi_1(L) = \langle z_i \mid z_i = [z_i, z_{i+1}^{-1}] [z_i, z_{i-1}] \rangle_{i \in \mathbb{Z}}.$$

This presentation clearly satisfies the condition of Theorem A, and so $\pi_1(L)$ is acyclic, giving (iii). Part (iii) of this result is the principal content of [11]. Here is a further proof by direct algebraic homological calculation.

We write $L = \bigcup_{i \geq 0} L_i$, the union of the nested sequence of spaces L_i defined by: $L_0 = M_0$, $L_{2i+1} = L_{2i} \cup_{f_i} M_{i+1}$, $L_{2i} = M_{-i} \cup_{f_{-i}} L_{2i-1}$. Then, by the van Kampen theorem, each successive $\pi_1(L_i)$ is the amalgamated free product of its predecessor with a copy of $\pi_1(M)$.

To begin the calculation of homology, we write $\pi_1(L_0)$ as the free product of the rank-two free groups $\text{Fr}(a_0, z_0)$ and $\text{Fr}(b_0, y_0)$, with amalgamation $z_0 = y_0$ and $a_0 z_0 a_0^{-1} = [y_0, b_0]$.

Then, from the Mayer-Vietoris sequence, $H_j(\pi_1(L_0)) = 0$ for $j \geq 2$. On the other hand, $H_1(\pi_1(L_0))$ is evidently the free abelian group generated by a_0, b_0 , where for convenience we use the same symbols for group elements and their images in the abelianization.

For the iteration, we write u_j for the relator $a_j z_{j+1}$, and v_j for the relator $b_{j+1} z_j^{-1}$ ($j \in \mathbb{Z}$). We claim that for all $n \geq 0$

$$\begin{aligned} H_j(\pi_1(L_n)) &= 0 \text{ for } j \geq 2, \\ H_1(\pi_1(L_n)) &= \text{Fr}(a_{[\frac{n+1}{2}], b_{-[\frac{n}{2}]}])_{\text{ab}}. \end{aligned}$$

This is easily checked from the Mayer-Vietoris sequences, since in the abelianizations each z_n becomes zero, so that u_n maps to a_n and v_n to b_{n+1} . Accordingly, inclusion induces the following homomorphisms of H_1 -groups.

$$\begin{array}{ccccc} H_1(\pi_1(L_{2i})) & \longrightarrow & H_1(\pi_1(L_{2i+1})) & \longrightarrow & H_1(\pi_1(L_{2i+2})) \\ a_i & \longmapsto & 0 & \longmapsto & a_{i+1} \\ & \longmapsto & b_{-i} & \longmapsto & 0 \end{array}$$

Thus inclusion induces the zero homomorphism $\tilde{H}_*(L_{2i}) \rightarrow \tilde{H}_*(L_{2i+2})$. Since homology commutes with direct limits, the result follows.

To obtain (ii), we identify ∂N_- and ∂N_+ in M via a homeomorphism that sends a to z^{-1} and b to z . Then we have a 3-manifold X' whose fundamental group has a presentation:

$$\pi_1(X') = \langle a, b, z, t \mid aza^{-1} = [z, b], t^{-1}at = z^{-1}, t^{-1}zt = b \rangle.$$

By eliminating a and b , we obtain the presentation:

$$\pi_1(X') = \langle z, t \mid z = [z, tz^{-1}t^{-1}][z, t^{-1}zt] \rangle.$$

Now observe from Theorem A that L is an infinite cyclic cover of X' , with $\pi_1(L)$ the commutator subgroup of $\pi_1(X')$.

REFERENCES

- [1] R C Alperin & A J Berrick: Linear representations of binate groups, *J Pure Appl Algebra* **94** (1994), 17-23.
- [2] B J Ball: Penetration indices and their applications, in *Topology of 3-Manifolds*, ed M K Fort Jr, Prentice-Hall (New York, 1962), 37-39.
- [3] G Baumslag, E Dyer & A Heller: The topology of discrete groups, *J Pure Appl Algebra* **16** (1980), 1-47.
- [4] G Baumslag & K W Gruenberg: Some reflections on cohomological dimension and freeness, *J Algebra* **6** (1967), 394-409.
- [5] A J Berrick: Universal groups, binate groups and acyclicity, in: *Group Theory, Procs Singapore Conf*, de Gruyter (Berlin, 1989), 253-266.
- [6] A J Berrick: A topologist's view of perfect and acyclic groups, Chapter 1 in book *Invitations to Geometry and Topology*, ed M Bridson, S Salamon, Oxford Univ Press (Oxford, 2002), 1-28.
- [7] A J Berrick: The acyclic group dichotomy, in preparation.
- [8] A J Berrick & J A Hillman: Perfect and acyclic subgroups of finitely presentable groups, *J London Math Soc*, to appear.
- [9] P Churchard & D Spring: Proper knot theory in open 3-manifolds, *Trans Amer Math Soc* **308** (1988), 133-142.
- [10] E Dyer & A T Vasquez: Some small aspherical spaces, *J Australian Math Soc* **16** (1973), 332-352.
- [11] D B A Epstein: A group with zero homology, *Proc Camb Phil Soc* **64** (1968), 599-601.
- [12] R H Fox: A remarkable simple closed curve, *Annals of Math* **50** (1949), 264-265.
- [13] R H Fox & E Artin: Some wild cells and spheres in three-dimensional space, *Annals of Math* **49** (1948), 979-990.
- [14] B Freedman & M H Freedman: Kneser-Haken finiteness for bounded 3-manifolds, locally free groups, and cyclic covers, *Topology* **37** (1998), 133-147.

- [15] P de la Harpe and D McDuff: Acyclic groups of automorphisms, *Comment Math Helvetici* **58** (1983), 48-71.
- [16] J Hempel: 3-manifolds, *Annals of Mathematics Studies* **86**, Princeton University Press (1976).
- [17] G Higman: A finitely generated infinite simple group, *J London Math Soc* **26** (1951), 61-64.
- [18] D L Johnson & R W K Odoni: Some results on symmetrically-presented groups, *Proc Edinburgh Math Soc* **37** (1994), 227-237.
- [19] J M McPherson: On the nullity and enclosure genus of wild knots, *Trans Amer Math Soc* **144** (1969), 545-555.
- [20] O Nanyes: Proper knots in open 3-manifolds have locally unknotted representatives, *Proc Amer Math Soc* **113** (1991), 563-571.
- [21] C D Papakyriakopoulos: On Dehn's lemma and the asphericity of knots, *Annals of Mathematics* **66** (1957), 1-26.

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