Question 1

If \( f \) is measurable and \( \lambda \) is any real number, \( f + \lambda \) and \( \lambda f \) are measurable.

Proof. Since \( \{ f > a - \lambda \} \) is measurable, \( \{ f + \lambda > a \} = \{ f > a - \lambda \} \) is measurable, then \( f + \lambda \) is measurable.

If \( \lambda = 0 \), then \( \lambda f = 0 \), of course is measurable. If \( \lambda > 0 \), then \( \{ \lambda f > a \} = \{ f > \frac{a}{\lambda} \} \), if \( \lambda < 0 \), \( \{ \lambda f > a \} = \{ f < \frac{a}{\lambda} \} \), these two sets are both measurable, so \( \lambda f \) is measurable.

Question 3

**Theorem 4.3** can be used to define measurability for vector-valued (e.g, complex-valued) functions. Suppose, for example, that \( f \) and \( g \) are real-valued and defined in \( \mathbb{R}^n \), and let \( F(x) = (f(x), g(x)) \). Then \( F \) is said to be measurable if \( F^{-1}(G) \) is measurable for every open \( G \subseteq \mathbb{R}^2 \). Prove that \( F \) is measurable if and only if both \( f \) and \( g \) are measurable in \( \mathbb{R}^n \).

Proof. \(( \iff )\) Since \( G \) is open, by **Theorem 1.11** we have \( G = \bigcup_{k=1}^{\infty} I_k \), \( I_k = \{ y : a_1^{(k)} \leq y_1 \leq b_1^{(k)}, i = 1, 2 \} \) are nonoverlapping intervals. \( F^{-1}(I_k) = \{ a_1^{(k)} \leq f \leq b_1^{(k)} \} \cap \{ a_2^{(k)} \leq g \leq b_2^{(k)} \} \), since \( f \) and \( g \) are measurable, \( F^{-1}(I_k) \) is measurable. \( F^{-1}(G) = \bigcup_{k=1}^{\infty} F^{-1}(I_k) \) is measurable, thus \( F \) is measurable.

\(( \implies )\) For every finite \( a \), since \( F \) is measurable, \( F^{-1}((a, +\infty) \times \mathbb{R}) = \{ f > a \} \) is measurable, then \( f \) is measurable in \( \mathbb{R}^n \). Similarly we can get \( g \) is measurable in \( \mathbb{R}^n \).

Question 4

Let \( f \) be defined and measurable in \( \mathbb{R}^n \). If \( T \) is a nonsingular linear transformation of \( \mathbb{R}^n \), show that \( f(Tx) \) is measurable. [ If \( E_1 = \{ x : f(x) > a \} \) and \( E_2 = \{ x : f(Tx) > a \} \), show that \( E_2 = T^{-1}E_1 \)].

Proof. \( E_2 = \{ x : f(Tx) > a \} = \{ x : Tx \in f^{-1}((a, +\infty)) \} = \{ x : Tx \in E_1 \} = \{ x : x \in T^{-1}E_1 \} = T^{-1}E_1 \). Since \( T \) is nonsingular, \( T^{-1} \) is a Lipschitz transformation. By **Theorem 3.33**, \( E_2 = T^{-1}E_1 \) is measurable, thus \( f(Tx) \) is measurable.
Question 5

Give an example to show that $\phi(f(x))$ may not be measurable if $\phi$ and $f$ are measurable.

Example: Let $F$ be the Cantor-Lebesgue function. Define $g : [0, 1] \to [0, 1]$ and $g = (F(x) + x)/2$, note that $g$ is continuous and strictly increasing. Let $C$ denote the Cantor set, then $|g(C)| = \frac{1}{2}$. By Corollary 3.39 we know there exists a nonmeasurable set $N \subset g(C)$. Put $f = g^{-1}$, then $f(N) \subset C$ and $|f(N)| = 0$ since $|C| = 0$. Put $Z = f(N)$ and $\phi$ is the characteristic function of $Z$. Consider $\{\phi(f(x)) > 0\}$, when $f(x) \in Z$, then we have $\{\phi(f(x)) > 0\} = N$ which is nonmeasurable, thus $\phi(f(x))$ is not measurable.

Lecturer’s REMARK It may not be clear that $|g(C)| > 0$. Thus I will prefer defining $F^{-1}(x) = \inf\{y : F(y) = x\}$ for $x \in [0, 1]$ and let $f = F^{-1}$.

Question 6

Let $f$ and $g$ be measurable functions on $E$.

(a) If $f$ and $g$ are finite a.e. in $E$, show that $f + g$ is measurable no matter how we define it at the points when it has the form $+\infty + (-\infty)$ or $-\infty + \infty$.

(b) Show that $fg$ is measurable without restriction on the finiteness of $f$ and $g$. Show that $f + g$ is measurable if it is defined to have the same value at every point where it has the form $+\infty + (-\infty)$ or $-\infty + \infty$.

Proof. (a) Put $Z_1 = \{x \in E : f = \pm\infty\}$ and $Z_2 = \{x \in E : g = \pm\infty\}$, $|Z_1| = 0$, $|Z_2| = 0$, then $|Z_1 \cap Z_2| = 0$. Put $Z = \{x \in E : (f(x) = +\infty$ and $g(x) = -\infty$) or $(f(x) = -\infty$ and $g(x) = +\infty)\} \subset Z_1 \cup Z_2$, then $|Z| = 0$. Let

$$
\overline{f}(x) = \begin{cases} 
  f(x) & x \in E - Z \\
  0 & x \in Z
\end{cases}
$$

$$
\overline{g}(x) = \begin{cases} 
  g(x) & x \in E - Z \\
  0 & x \in Z
\end{cases}
$$

Since $\overline{f} = f$ a.e and $\overline{g} = g$ a.e., $\overline{f}, \overline{g}$ are measurable. By Theorem 4.9, $\overline{f} + \overline{g}$ is measurable on $E \setminus Z$ and hence measurable on $E$. Since $\overline{f} + \overline{g} = f + g$ a.e. no matter how we define the value of $f + g$ on $Z$, we conclude that $f + g$ is measurable on $E$. 

2
(b) First, for any measurable set $D$, we will let $\mathcal{M}_{\mathcal{F}}(D)$ be the set of all measurable functions on $D$.

Define $E_i, i = 1, \ldots, 6$:

<table>
<thead>
<tr>
<th></th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
<th>$E_5$</th>
<th>$E_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
<td>0</td>
<td>0</td>
<td>$\pm \infty$</td>
<td>$\neq 0$</td>
</tr>
<tr>
<td>$g$</td>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
<td>$-\infty$</td>
<td>$\neq 0$</td>
<td>$\pm \infty$</td>
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<td>$fg$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
<td>$\beta_3$</td>
<td>$\beta_4$</td>
<td>$\pm \infty$</td>
<td>$\pm \infty$</td>
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</table>

Since $f, g \in \mathcal{M}_{\mathcal{F}}(E)$, we have $f, g \in \mathcal{M}_{\mathcal{F}}(E_i), i = 1, \ldots, 6$ and $f, g \in \mathcal{M}_{\mathcal{F}}(E \cup E_i)$, then $fg \in \mathcal{M}_{\mathcal{F}}(E_i), i = 1, \ldots, 6$ and $fg \in \mathcal{M}_{\mathcal{F}}(E \cup E_i)$. Thus, $fg \in \mathcal{M}_{\mathcal{F}}(E)$ without restriction on the finiteness of $f$ and $g$.

We now define $E_i, i = 1, \ldots, 4$:

<table>
<thead>
<tr>
<th></th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
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</thead>
<tbody>
<tr>
<td>$f$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>$g$</td>
<td>$-\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>$f+g$</td>
<td>$\beta_1$</td>
<td>$\beta_2$</td>
<td>$\infty$</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

Since $f, g \in \mathcal{M}_{\mathcal{F}}(E)$, we have $f, g \in \mathcal{M}_{\mathcal{F}}(E_i), i = 1, \ldots, 4$ and $f, g \in \mathcal{M}_{\mathcal{F}}(E \cup E_i)$, then $f + g \in \mathcal{M}_{\mathcal{F}}(E_i), i = 1, \ldots, 4$ and $f + g \in \mathcal{M}_{\mathcal{F}}(E \cup E_i)$. Hence, $f + g \in \mathcal{M}_{\mathcal{F}}(E)$.

**Question 7**

Let $f$ be usc and less than $+\infty$ on a compact set $E$. Show that $f$ is bounded above on $E$. Show also that $f$ assumes its maximum on $E$, i.e., that there exists $x_0 \in E$ such that $f(x_0) \geq f(x)$ for all $x \in E$.

Proof. Since $f$ is usc on $E$, then $\{x \in E : f(x) < k\}$ is open for $k = 1, 2, \ldots$. Since $f$ is less than $+\infty$, $\bigcup_{k=1}^{\infty} \{x \in E : f(x) < k\}$ is an open cover of $E$. Since $E$ is compact, then there exist a finite subcover, that is $\exists N \in \mathbb{N}$ such that $\bigcup_{k=1}^{N} \{x \in E : f(x) < k\} \supset E$, this means $f(x) < N$, so $f$ is bounded above on $E$. Put $M = \sup_{x \in E} f(x)$, then $M < N$, for each $k = 1, 2, \ldots$, there exists $x_k \in E$ such that $f(x_k) > M - \frac{1}{k}$. Since $E$ is compact, there exists a subsequence $x_{n_k}$ of $x_k$ which converges to a point $x_0$ of $E$. $f(x_{n_k}) > M - \frac{1}{n_k}$, let $k \to \infty$, we have $f(x_0) \geq M$, since $f(x_0) \leq M$, then $f(x_0) = M$. Thus, $f$ assumes its maximum on $E$. 

3
Question 8

(a) Let $f$ and $g$ be two functions which are usc at $x_0$. Show that $f + g$ is usc at $x_0$. Is $f - g$ usc at $x_0$? When is $fg$ usc at $x_0$?

Proof. If $f(x_0) = +\infty$ or $g(x_0) = +\infty$, then $f(x_0) + g(x_0) = +\infty$, so $f + g$ is usc at $x_0$. If $f(x_0) < +\infty$ and $g(x_0) < +\infty$, since $f$ is usc at $x_0$, given $M_1 > f(x_0)$, there exists $\delta_1 > 0$ such that $f(x) < M_1$ for all $x \in E$ which lie in the ball $|x - x_0| < \delta_1$. Since $g$ is also usc at $x_0$, given $M_2 > f(x_0)$, there exists $\delta_2 > 0$ such that $g(x) < M_2$ for all $x \in E$ which lie in the ball $|x - x_0| < \delta_2$. Put $\delta = \min(\delta_1, \delta_2)$, then $f(x) + g(x) < M_1 + M_2$ for all $x \in E$ which lie in the ball $|x - x_0| < \delta$, thus $f + g$ is usc at $x_0$.

$f - g$ is not usc at $x_0$. Since $g$ is usc at $x_0$, $-g$ is lsc at $x_0$, put $f = 0$, then $f - g = -g$ is lsc at $x_0$.

When $f \geq 0$ and $g \geq 0$, $fg$ is usc at $x_0$. If $f(x_0) = +\infty$ or $g(x_0) = +\infty$, obviously $fg$ is usc at $x_0$. Otherwise, by the above proof, put $\delta = \min(\delta_1, \delta_2)$, then $f(x)g(x) < M_1M_2$ for all $x \in E$ which lie in the ball $|x - x_0| < \delta$, thus $fg$ is usc at $x_0$.

(b) If $\{f_k\}$ is a sequence of functions which are usc at $x_0$. Show that $\inf_k f_k(x)$ is usc at $x_0$.

Proof. Since $f_k$ is usc at $x_0$, given $\varepsilon > 0$, $f_k(x_0) + \varepsilon > f_k(x_0)$, then $\exists \delta_k > 0$ such that $f_k(x) < f_k(x_0) + \varepsilon$ for $x \in E_k = (\{x : |x - x_0| < \delta_k\} \cap E)$. When $x \in \cap E_k$, $\inf_k f_k(x) < \inf_k f_k(x_0) + \varepsilon$, so $\inf_k f_k(x)$ is usc at $x_0$.

(c) If $\{f_k\}$ is a sequence of functions which are usc at $x_0$ and which converge uniformly near $x_0$, show that $\lim f_k$ is usc at $x_0$.

Proof. Since $f_k$ is usc at $x_0$, given $\varepsilon > 0$, there exists $\delta_k > 0$ such that $f_k(x) < f_k(x_0) + \frac{\varepsilon}{2}$ for all $x \in E$ which lie in the ball $|x - x_0| < \delta_k$. Since $f_k$ converge uniformly near $x_0$, there exists $K$, when $k > K$ and $x \in \{x : |x - x_0| < \delta_k\} \cap E$, $|\lim f_k(x) - f_k(x)| < \frac{\varepsilon}{2}$, then $\lim f_k(x) < f_k(x) + \frac{\varepsilon}{2} < f_k(x_0) + \varepsilon$, $\lim f_k(x) < \lim f_k(x_0) + \varepsilon$, thus $\lim f_k$ is usc at $x_0$.

Question 9

Show that the limit of a decreasing (increasing) sequence of functions usc (lsc) at $x_0$ is usc (lsc) at $x_0$. In particular, the limit of a decreasing (increasing) sequence of functions continuous at $x_0$ is usc (lsc) at $x_0$.

Proof. Let $\{f_k\}$ be the decreasing sequences usc at $x_0$, since $f_k$ is decreasing, we have $f = \lim_{k \to \infty} f_k = \inf_k f_k$, by Question 8(b) we know that $\inf_k f_k$ is usc at $x_0$, then
\[ f = \lim_{x \to \infty} f_k \] is usc at \( x_0 \).

**Question 10**

(a) If \( f \) is defined and continuous on \( E \), show that \( \{a < f < b\} \) is relatively open, and that \( \{a \leq f \leq b\} \) and \( \{f = a\} \) are relatively closed.

(b) If \( f \) is a finite function on \( \mathbb{R}^n \). Show that \( f \) is continuous on \( \mathbb{R}^n \) if and only if \( f^{-1}(G) \) is open for every open \( G \) in \( \mathbb{R}^1 \), or if and only if \( f^{-1}(F) \) is closed for every closed \( F \) in \( \mathbb{R}^1 \).

Proof. (a) For each \( x_k \in \{a < f < b\} \), since \( a < f(x_k) < b \) and \( f \) is continuous on \( E \), there exists \( \delta_1, \delta_2 > 0 \) such that \( f(x) < b \) when \( x \in B(x_k, \delta_1) \cap E \) and \( f(x) > a \) when \( x \in B(x_k, \delta_2) \cap E \), put \( \delta_k = \min(\delta_1, \delta_2) \), we have \( a < f(x) < b \) when \( x \in B(x_k, \delta_k) \cap E \).

Then \( \cup B(x_k, \delta_k) \cap E = \{a < f < b\} \), since \( \cup B(x_k, \delta_k) \) is open, \( \{a < f < b\} \) is relatively open.

We prove the complement of \( \{a \leq f \leq b\} \) is relatively open, the complement of \( \{a \leq f \leq b\} \) is \( \{f > b\} \cup \{f < a\} \). From the above proof we know \( \{f > b\} \) and \( \{f < a\} \) are relatively open, then \( \{f > b\} \cup \{f < a\} \) is relatively open, so \( \{a \leq f \leq b\} \) is relatively closed. Put \( b = a \), \( \{f = a\} \) is also relatively closed.

(b) \( \Rightarrow \) If \( f \) is continuous, put \( f^{-1}(G) \neq \emptyset \). For \( x_0 \in f^{-1}(G) \), then \( f(x_0) \in G \). Since \( G \) is open, there exists \( \varepsilon > 0 \) such that \( (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \subset G \). Since \( f \) is continuous, for the above \( \varepsilon \), \( \exists \delta > 0 \) such that \( -\varepsilon < f(x) - f(x_0) < \varepsilon \) when \( x \in (x_0 - \delta, x_0 + \delta) \), so when \( x \in (x_0 - \delta, x_0 + \delta) \), \( f(x) \in G \), then \( (x_0 - \delta, x_0 + \delta) \subset f^{-1}(G) \). This means \( x_0 \) is the interior point of \( f^{-1}(G) \), so \( f^{-1}(G) \) is open.

( \( \Leftarrow \) ) Since \( f^{-1}(G) \) is open for every open \( G \) in \( \mathbb{R}^1 \), for each \( x_0 \) and \( \varepsilon > 0 \), the pre-image \( U \) of \( (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \) is open. Since \( x_0 \in U \), then \( \exists \delta > 0 \) such that \( (x_0 - \delta, x_0 + \delta) \subset U \). So when \( x \in (x_0 - \delta, x_0 + \delta) \), we have \( f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \), this means \( f \) is continuous at \( x_0 \). Thus \( f \) is continuous.

**Question 11**

Let \( f \) be defined on \( \mathbb{R}^n \) and let \( B(x) \) denote the open ball \( \{y : |x - y| < r\} \) with center \( x \) and fixed radius \( r \). Show that the function \( g(x) = \sup\{f(y) : y \in B(x)\} \) is lsc and that the function \( h(x) = \inf\{f(y) : y \in B(x)\} \) is usc on \( \mathbb{R}^n \). Is the same true for the closed ball \( \{y : |x - y| \leq r\} \)?
Proof. For any \( x_0 \in \mathbb{R}^n \), \( g(x_0) = \sup\{ f(y) : y \in B(x_0) \} \), then \( \forall y \in B(x_0), f(y) \leq g(x_0) \). Moreover, \( \forall \varepsilon > 0 \), there exists \( x_\varepsilon \in B(x_0, 1) \) such that \( f(x_\varepsilon) > g(x_0) - \varepsilon \). Choose \( r_\varepsilon < 1 \) such that \( |x_0 - x_\varepsilon| < r_\varepsilon \). Since \( x_\varepsilon \in B(z, r) \) if \( z \in B(x_0, r - r_\varepsilon) \) and \( g(x) = \sup\{ f(y) : y \in B(z) \} \), we have \( g(x) \geq f(x_\varepsilon) > g(x_0) - \varepsilon \) when \( x \in B(x_0, r - r_\varepsilon) \), so \( g(x) \) is lsc at \( x_0 \). Thus, \( g(x) \) is lsc on \( \mathbb{R}^n \). We can use the similar way to prove \( h(x) = \inf\{ f(y) : y \in B(x) \} \) is usc on \( \mathbb{R}^n \).

It is not the same for the closed ball \( \{ y : |x - y| \leq r \} \), for example, let \( f(0) = 1 \) and \( f(x) = 0 \) for \( x \neq 0 \), it is clear that \( h(x) = \sup\{ f(y) : y \in B(y, 1) \} \) is not lsc.

**Question 12**

If \( f(x), x \in \mathbb{R}^1 \), is continuous at almost every point of an interval \([a, b]\), show that \( f \) is measurable on \([a, b]\).

Proof. Put \( Z = \{ f \text{ is not continuous} \} \), then \(|Z| = 0\). Put \( E = [a, b] - Z \), then \( f \) is continuous relative to \( E \). \( \{ f > \alpha \} = \{ x \in E : f > \alpha \} \cup \{ x \in Z : f > \alpha \} \). Since \( f \) is continuous on \( E \), \( \{ x \in E : f > \alpha \} \) is measurable. \( |\{ x \in Z : f > \alpha \}| \leq |z| = 0 \), then \( \{ x \in Z : f > \alpha \} \) is measurable. Thus, \( f \) is measurable on \([a, b]\).

**Question 15**

Let \( \{ f_k \} \) be a sequence of measurable functions defined on a measurable \( E \) with \( |E| < +\infty \). If \( |f_k(x)| \leq M_x < +\infty \) for all \( k \) for each \( x \in E \), show that given \( \varepsilon > 0 \), there is a closed \( F \subset E \) and a finite \( M \) such that \(|E - F| < \varepsilon \) and \( |f_k(x)| \leq M \) for all \( k \) and all \( x \in F \).

Proof. Fix \( \varepsilon > 0 \), for each \( n \), let \( E_n = \{ x \in E : M_x \leq n \} \). Then \( E_n \) is measurable. Clearly \( E_n \subset E_{n+1} \) and \( E_n \rightarrow E \). Hence, \(|E_n| \rightarrow |E|\). Since \(|E| < +\infty\), it follows that \(|E - E_n| \rightarrow 0\). Choose \( n_0 \) such that \(|E - E_{n_0}| < \frac{\varepsilon}{2}\), and let \( F \) be a closed subset of \( E_{n_0} \) with \(|E_{n_0} - F| < \frac{\varepsilon}{2}\). Then \(|E - F| < \varepsilon\) and \( |f_k(x)| \leq n_0 \) for all \( k \) and all \( x \in F \).

**Remark** Unfortunately, we do not know why \( E_n \) in the above is measurable (unless we are given that the function \( g(x) = M_x \) is measurable. Thus, it is better to consider

\[
F_n = \{ x \in E : \sup_k |f_k(x)| \leq n \}.
\]
Question 17

Suppose that \( f_k \xrightarrow{m} f \) and \( g_k \xrightarrow{m} g \) on \( E \). Show that \( f_k + g_k \xrightarrow{m} f + g \) on \( E \) and, if \( |E| < +\infty \), that \( f_kg_k \xrightarrow{m} fg \) on \( E \). If, in addition, \( g_k \xrightarrow{m} g \) on \( E \), \( g \neq 0 \) a.e., and \( |E| < +\infty \), show that \( f_k/g_k \xrightarrow{m} f/g \) on \( E \).

Proof. (1) \( \{x \in E : |f + g - (f_k + g_k)| > \varepsilon\} \subset \{x \in E : |f - f_k| > \frac{\varepsilon}{2}\} \cup \{x \in E : |g - g_k| > \frac{\varepsilon}{2}\} \), since \( \lim_{k \to \infty} \{x \in E : |f - f_k| > \frac{\varepsilon}{2}\} = 0 \) and \( \lim_{k \to \infty} \{x \in E : |g - g_k| > \frac{\varepsilon}{2}\} = 0 \), then \( \lim_{k \to \infty} \{x \in E : |f + g - (f_k + g_k)| > \varepsilon\} = 0 \), \( f_k + g_k \xrightarrow{m} f + g \) on \( E \).

(2) Since \( f_kg_k - fg = (f_k-f)(g_k-g)+f(g_k-g)+g(f_k-f), \{x \in E : |f_kg_k-fg| > \varepsilon\} \subset \{x \in E : |(f_k-f)(g_k-g)| > \frac{\varepsilon}{3}\} \cup \{x \in E : |f(g_k-g)| > \frac{\varepsilon}{3}\} \cup \{x \in E : |g(f_k-f)| > \frac{\varepsilon}{3}\} \).

\( \{x \in E : |(f_k-f)(g_k-g)| > \frac{\varepsilon}{3}\} \subset \{x \in E : |f_k-f| > \frac{\varepsilon}{3\sqrt{3}}\} \cup \{x \in E : |g_k-g| > \frac{\varepsilon}{3\sqrt{3}}\} \).

Since \( f \) is the limit (in measure) of a sequence of functions, it must be finite on \( E \).

Since \( |E| < +\infty \), for any \( \eta > 0 \), there exists \( Z_1, Z_2 \subset E, |Z_1|, |Z_2| < \eta \) such that \( f \) is bounded on \( E - Z_1 \) and \( g \) is bounded on \( E - Z_2 \), and . Put \( |f| \leq M_1 \) on \( E - Z_1 \) and \( |g| \leq M_2 \) on \( E - Z_2 \). So \( \{x \in E : |g_k-g| > \frac{\varepsilon}{3}\} \subset \{x \in E : |g_k-g| > \frac{\varepsilon}{3M_2}\} \cup Z_1, \{x \in E : |g(f_k-f)| > \frac{\varepsilon}{3}\} \subset \{x \in E : |f_k-f| > \frac{\varepsilon}{3M_2}\} \cup Z_2, \) since \( f_k \xrightarrow{m} f \) and \( g_k \xrightarrow{m} g \) on \( E \) and \( |Z_1|, |Z_2| < \eta \), we have \( |\{x \in E : |f_kg_k-fg| > \varepsilon\}| < 5\eta \) if \( k \) is sufficiently large.

Hence \( f_kg_k \xrightarrow{m} fg \) on \( E \).

(3) First, since \( g \neq 0 \) a.e., we have \( 1/g \) is finite a.e.. Hence given any \( \eta > 0 \), since \( |E| < \infty \), there exists \( Z \subset E \) such that \( |Z| < \eta \) and \( |1/g| \leq M \) on \( E \setminus Z \) for some constant \( M > 0 \).

Since \( g_k \xrightarrow{m} g, \exists K \) such that \( |E_k| = |\{|g_k-g| > 1/(2M)\}| < \eta \) when \( k \geq K \). Note that \( |g_k| \geq |g| - |g_k - g| \geq 1/(2M) \) on \( E \setminus (Z \cup E_k) \).

Now observe that

\[
\frac{1}{g_k} - \frac{1}{g} = \frac{g_k - g}{g_k g} = \frac{1}{|g_k g|} |g_k - g|.
\]

Note that \( |g_k g| \geq 1/(2M^2) \) on \( E \setminus (Z \cup E_k) \). Since \( g_k \xrightarrow{m} g \), for any \( \varepsilon > 0 \), there exists \( K' \) such that \( |\{|g_k-g| > \varepsilon/(2M^2)\}| < \eta \) for \( k \geq K' \). It is now clear that \( |\{|1/g_k - 1/g| > \varepsilon\}| < 2\eta \) when \( k \geq \max\{K, K'\} \).

Thus \( \frac{1}{g_k} \xrightarrow{m} \frac{1}{g} \) on \( E \). By the conclusion of (2), we have \( f_k/g_k \xrightarrow{m} f/g \) on \( E \).

Question 18

If \( f \) is measurable on \( E \), define \( \omega_f(a) = |\{f > a\}| \) for \( -\infty < a < +\infty \). If \( f_k \searrow f \), show that \( \omega_{f_k} \searrow \omega_f \). If \( f_k \xrightarrow{m} f \), show that \( \omega_{f_k} \xrightarrow{m} \omega_f \) at each point of continuity of \( \omega_f \).
Proof. Put \( E_k = \{ f_k > a \} \), since \( f_k \not\nearrow f \), we have \( E_k \subset E_{k+1} \) and \( E_k \to E = \{ f > a \} \), by Theorem 3.26 we have \( |E_k| \to |E| \), then \( \omega_{f_k} \not\nearrow \omega_f \).

Let \( a \) be a continuous point of \( \omega_f \), then \( \forall \eta > 0, \exists \delta > 0 \), when \( |b-a| \leq \delta \), \( |\omega_f(b) - \omega_f(a)| < \eta \). Let \( E_k = \{ x : |f - f_k| > \delta \} \). Then there exists \( N \in \mathbb{N} \) such that \( |E_k| < \eta \) for \( k \geq N \). Since \( f_k = (f_k - f) + f \), \( \{ f_k > a \} \subset E_k \cup \{ f > a - \delta \} \), then \( \omega_{f_k} \leq \eta + \omega_f(a - \delta) < \eta + \omega_f(a) + \eta \) for \( k \geq N \). Next, since \( f = (f - f_k) + f_k \), we have \( \{ f > a + \delta \} \subset (E_k \cup \{ f_k > a \}) \), then \( \omega_{f_k} \geq \omega_f(a + \delta) - \eta \), so \( |\omega_{f_k}(a) - \omega_f(a)| < 2\eta \). Thus \( \omega_{f_k}(a) \to \omega_f(a) \).

**Question 19**

Let \( f(x,y) \) be a function defined on the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) which is continuous in each variable separately. Show that \( f \) is a measurable function of \((x,y)\).

Proof. Divide \([0,1]\) into \( n \) equal subintervals, put \( f_n(x,y) = f(x, \frac{k}{n}) \) when \( \frac{k}{n} \leq y < \frac{k+1}{n} \). Clearly for any fixed \((x,y)\), \( f_n(x,y) \to f(x,y) \). \( \{ f_n(x,y) > a \} = \bigcup_{k=1}^{n-1} \{ x \in [0,1] : f(x, \frac{k}{n}) > a \} \times [\frac{k}{n}, \frac{k+1}{n}) \}, \) since \( f \) is continuous in each variable separately, so \( \{ f_n(x,y) > a \} \) is measurable. Since \( f_n(x,y) \to f(x,y) \), \( f \) is a measurable function of \((x,y)\).

**Question 20**

If \( f \) is measurable and finite a.e. on \([a,b]\), show that given \( \varepsilon > 0 \), there is a continuous \( g \) on \([a,b]\) such that \( |\{x : f(x) \neq g(x)\}| < \varepsilon \).

Proof. By Lusin’s Theorem we have \( \forall \varepsilon > 0 \), there exists a closed set \( F \subset E \) such that \( |E - F| < \varepsilon \) and \( f \) is continuous relative to \( F \). Put \( G = \overline{F} F \), then \( G \) is open, then \( G = \bigcup_{k=1}^{\infty} (a_k, b_k), (a_k, b_k) \) are disjoint. When \( x \in F \), Let \( g(x) = f(x) \). When \( x \notin F \), define \( g(x) \) is a linear on each open interval \((a_k, b_k)\) and let \( g(x) \) be continuous at the endpoint of the interval. When \( x \in (a_k, b_k) \) which is a finite interval, \( g(x) = f(a_k) \frac{x-a_k}{b_k-a_k} + f(b_k) \frac{x-b_k}{b_k-a_k} \). When some open interval is infinite, if \( x \in (c, +\infty) \), \( g(x) = f(c) \), when \( x \in (-\infty, d) \), \( g(x) = f(d) \). So \( g \) is well-defined on \( \mathbb{R} \), and obviously \(|\{ f \neq g \}| \leq |E - F| < \varepsilon \). From the above construction of \( g \), we have \( g \) is continuous at the endpoint of the open interval, so \( g \) is continuous on \( R \), of course \( g \) is continuous on \([a,b]\).