Ch.5. Qualitative Properties of Solutions
§5.1. Oscillations and Comparison (sec 24,25, lecture 30 (Essentials of ODE))

In this section we discuss the oscillation properties of solutions of second order equations

\[ \ddot{y} + p(t)\dot{y} + q(t)y = 0, \]  

where \( p(t) \) and \( q(t) \) are continuous real functions on \( \mathbb{R} \).

**Definition.** \( t_0 \) is called a **zero** of \( y(t) \) if \( y(t_0) = 0 \). \( t_0 \) is called a **simple zero** of \( y(t) \) if

\[ y(t_0) = 0, \quad \dot{y}(t_0) \neq 0. \]

A zero \( t_0 \) of \( y(t) \) is called **isolated** if there exists \( \delta > 0 \) such that \( t_0 \) is the only zero of \( y(t) \) on the interval \( (t_0 - \delta, t_0 + \delta) \).

A solution of (5.1.1) is **non-trivial** if it is not identically equal to zero. We shall discuss the distribution of zeros of nontrivial solutions of (5.1.1).
Lemma 5.1 Let $y(t)$ be a non-trivial solution of (5.1.1). Then the zeros of $y(t)$ are simple.

Proof Suppose not. There exists $t_0$ such that $y(t_0) = 0$ and $\dot{y}(t_0) = 0$. Then $y(t)$ is a solution of the initial value problem

$$\begin{cases}
\ddot{y} + p(t)\dot{y} + q(t)y = 0, \\
y(t_0) = 0, \quad \dot{y}(t_0) = 0.
\end{cases}$$

Since $y = 0$ is also a solution of the initial value problem. By the uniqueness theorem we have $y(t) \equiv 0$, which contradicts the assumption $y(t) \not\equiv 0$. \qed

Example. The equation

$$\ddot{y} + y = 0$$

has two linearly independent solutions $y_1 = \sin t$ and $y_2 = \cos t$. Their zeros are distinct and occur alternately, namely, $\sin(t)$ vanishes exactly once between any two successive zeros of $\cos(t)$, and conversely.

Any non-trivial solution of the equation

$$\ddot{y} - y = 0$$

has at most one zero.
What about 

\[ \ddot{y} + \alpha^2 y = 0? \]
Lemma 5.2 The zeros of a non-trivial solution of (5.1.1) are isolated. Hence on any bounded interval \( y(t) \) has at most a finite number of zeros.

Proof Suppose \( y(t) \) has a non-isolated zero \( t_0 \). Thus there exists a sequence of zeros of \( y(t) \), say, \( \{ t_j \} \), such that \( t_j \to t_0 \) as \( j \to \infty \). Without loss of generality we may assume that \( \{ t_j \} \) is an increasing sequence. Applying the mean value theorem, there exists \( x_j \in (t_j, t_{j+1}) \) such that \( \dot{y}(x_j) = 0 \). Note that \( x_j \to t_0 \) as \( j \to \infty \). Since \( \dot{y}(t) \) is continuous,

\[
\dot{y}(t_0) = \lim_{j \to \infty} \dot{y}(x_j) = 0.
\]

Thus \( t_0 \) is non-simple zero. This contradicts the conclusion of Lemma 5.1.

Suppose \( y(t) \) has infinitely many zeros on a bounded interval \([a, b]\). By Bolzano Theorem, we can find a sequence of zeros of \( y(t) \), say \( \{ t_j \} \), which converges to \( t_0 \in [a, b] \). By the continuity of \( y(t) \) we have

\[
y(t_0) = \lim_{j \to \infty} y(t_j) = 0.
\]

Thus \( y(t) \) has a non-isolated zero \( t_0 \). This contradicts the conclusion we just proved. \( \square \)
Let $y(t) \neq 0$ be a solution of (5.1.1) and let $t_0$ be a zero of $y(t)$. From Lemma 5.2, $t_0$ is an isolated zero. If $y(t)$ has other zeros located in the left of $t_0$, we can consider the largest zero $t_1$ in the left of $t_0$. There is no other zeros between $t_1$ and $t_0$. We call $t_1$ and $t_0$ a pair of successive zeros. Similarly we can consider the least zero $t_2$ in the right of $t_0$, and call $t_0$ and $t_2$ a pair of successive zeros.

**Theorem 5.3 (Sturm Separation Theorem)**

Let $y_1(t)$ and $y_2(t)$ be two non-trivial solutions of (5.1.1). Then

(i) $y_1(t)$ and $y_2(t)$ are linearly dependent if and only if their zeros are coincide.

(ii) $y_1(t)$ and $y_2(t)$ are linearly independent if and only if their zeros are distinct and occur alternately in the sense that $y_1(t)$ vanishes exactly once between any two successive zeros of $y_2(t)$, and conversely.
Proof (i) If $y_1(t)$ and $y_2(t)$ are linearly dependent, since $y_1(t)$, $y_2(t) \neq 0$, there is a constant $c \neq 0$ such that $y_2(t) = cy_1(t)$. So they have same zeros.

If $y_1(t)$ and $y_2(t)$ have a coincide zero $t_0$, then their Wronskian $W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ vanishes at $t_0$. Thus $y_1(t)$ and $y_2(t)$ are linearly dependent.

(ii) If the zeros of $y_1(t)$ and $y_2(t)$ are distinct and occur alternately, then from (i) they are linearly independent.

Now we assume $y_1(t)$ and $y_2(t)$ are linearly independent. From the proof of (i) we see that they do not have any coincide zero. Let $t_1 < t_2$ be two successive zeros of $y_2(t)$. Then $t_1$ and $t_2$ are not zeros of $y_1(t)$. We shall show that $y_1(t)$ has exactly one zero in the interval $(t_1, t_2)$.

$y_2(t)$ does not change its sign in the interval $(t_1, t_2)$. Without loss of generality we may assume that $y_2(t) > 0$ for $t_1 < t < t_2$. Since $y_2(t_1) = y_2(t_2) = 0$, we find $\dot{y}_2(t_1) \geq 0$ and $\dot{y}_2(t_2) \leq 0$. Combining these with Lemma 5.1 we see that

$$\dot{y}_2(t_1) > 0, \quad \dot{y}_2(t_2) < 0.$$  \hfill 5.1.2
The Wronskian $W(t) = y_1(t)\dot{y}_2(t) - \dot{y}_1(t)y_2(t)$ does not vanish, hence has constant sign since it is continuous. So

$$W(t_1)W(t_2) = y_1(t_1)\dot{y}_2(t_1) \cdot y_1(t_2)\dot{y}_2(t_2) > 0.$$  \hspace{1cm} 5.1.3

Combining (5.1.2) and (5.1.3) we see that $y_1(t_1)y_1(t_2) < 0$. Using the Intermediate Value Theorem we conclude that $y_1(t)$ has at least one zero in the interval $(t_1, t_2)$.

$y_1(t)$ can not have more than one zeros in $(t_1, t_2)$. In fact, if $y_1(t)$ has zeros $t_3$ and $t_4$ between $t_1$ and $t_2$, then we repeat the above argument and show that $y_2(t)$ must vanish at a point between $t_3$ and $t_4$. But this is impossible as $y_2(t)$ does not vanish between $t_1$ and $t_2$.

So we have proved that, $y_1(t)$ has exactly one zero between any pair of successive zeros of $y_2(t)$. Similarly, $y_2(t)$ has exactly one zero between any pair of successive zeros of $y_1(t)$. \hspace{1cm} \square

**Example** Consider $\ddot{y} + \alpha^2 y = 0$ and $\ddot{y} + \beta^2 y = 0$. 
Theorem 5.4 (Sturm Comparison Theorem)

Assume that $y(t)$ and $z(t)$ are non-trivial solutions of the following equations respectively

$$\ddot{y} + p(t)\dot{y} + R(t)y = 0 \quad \text{5.1.4}$$

and

$$\ddot{z} + p(t)\dot{z} + r(t)z = 0, \quad \text{5.1.5}$$

where $p(t)$, $R(t)$ and $r(t)$ are continuous functions in $\mathbb{R}$.

(i) Assume that

$$r(t) < R(t) \quad \text{for all } t \in \mathbb{R}. \quad \text{5.1.6}$$

Then $y(t)$ vanishes at least once between any pair of successive zeros of $z(t)$, namely, for any pair of successive zeros $t_1, t_2$ of $z(t)$, $y(t)$ has at least one zero in the open interval $(t_1, t_2)$.

(ii) Assume that

$$r(t) \leq R(t) \quad \text{for all } t. \quad \text{5.1.7}$$

Then for any pair of successive zeros $t_1, t_2$ of $z(t)$, $y(t)$ has at least one zero in the closed interval $[t_1, t_2]$. 
Proof First prove (i). Let $t_1$ and $t_2$ be a pair of successive zeros of $z(t)$. Hence $z(t) \neq 0$ on $(t_1, t_2)$. We shall show that $y(t)$ has at least one zero in the open interval $(t_1, t_2)$. Suppose not. Then $y(t) \neq 0$ for all $t \in (t_1, t_2)$. Without loss of generality we may assume that $y(t) > 0$ and $z(t) > 0$ on $(t_1, t_2)$. Then we have, from Lemma 5.1, $\dot{z}(t_1) > 0$ and $\dot{z}(t_2) < 0$.

Consider the Wronskian $W(t)$ of $y(t)$ and $z(t)$, $W(t) = y(t)\dot{z}(t) - y(t)\dot{z}(t)$. We have

$$W(t_1) = y(t_1)\dot{z}(t_1) \geq 0, \quad W(t_2) = y(t_2)\dot{z}(t_2) \leq 0.$$  \hspace{1cm} \text{5.1.8}

On the other hand,

$$\dot{W} = y\dddot{z} - \dddot{y}z = y(-p\dot{z} - rz) - z(-p\dot{y} - Ry) = -pW + (R - r)yz,$$

namely

$$\dot{W} + pW = (R - r)yz.$$

Multiplying the equality by $e^{\int_{t_1}^{t} p(s) ds}$ we get

$$\frac{d}{dt} \left[ e^{\int_{t_1}^{t} p(s) ds} W(t) \right] = e^{\int_{t_1}^{t} p(s) ds} [R(t) - r(t)]y(t)z(t) > 0$$
for all $t > t_1$. Thus

$$e^{\int_{t_1}^{t_2} p(s) ds} W(t_2) > W(t_1) \geq 0.$$ 

Thus $W(t_2) > 0$, which contradicts (5.1.8). \qed

Next, we prove (ii).
Next we use Theorem 5.4 to study the oscillation behavior of solutions of (5.1.1).

**Theorem 5.5** Assume \( q(t) \leq 0 \) for all \( t \). Then any non-trivial solution of (5.1.1) has at most one zero.

**Proof** Let \( z(t) \) be a non-trivial solution of (5.1.1). We shall show that \( z(t) \) has at most one zero. Suppose not, and let \( t_1 < t_2 \) be two successive zeros of \( z(t) \). We shall compare \( z(t) \) with a non-trivial solution of the equation

\[
\ddot{y} + p(t)\dot{y} = 0.
\]

This equation has a non-trivial solution \( y(t) \equiv 1 \). Applying Theorem 5.4 (with \( r(t) = q(t) \) and \( R(t) = 0 \) we conclude that \( y(t) \) has a zero in the interval \([t_1, t_2]\). This contradiction shows that \( z(t) \) can not have more than one zeros. \( \square \)
Theorem 5.6 Let \( y(t) \) be a non-trivial solution of the equation
\[
\ddot{y} + Q(t)y = 0, \tag{5.1.9}
\]
where \( Q(t) \) is a continuous function in \( \mathbb{R} \) and there exists a constant \( m > 0 \) such that
\[
Q(t) \geq m^2 \quad \text{for all } t. \tag{5.1.10}
\]
Then \( y(t) \) has infinitely many zeros, and the distance between two successive zeros is at most \( \pi/m \).

Proof We shall show that, for any constant \( a \), \( y(t) \) has at least one zero on the interval \([a, b]\), where \( b = a + \pi/m \). Suppose \( y(t) \) has no zeros on \([a, b]\). We compare \( y(t) \) with a non-trivial solution of
\[
\ddot{z} + m^2z = 0.
\]
This equation has a non-trivial solution \( z(t) = \sin(mt - ma) \).
\( z(a) = z(b) = 0 \). Applying Theorem 5.4 (with \( r(t) = m^2 \) and \( R(t) = Q(t) \)) we conclude that \( y(t) \) has at least one zero on the interval \([a, b]\). This contradiction verifies Theorem 5.6. \( \Box \)
An equation in the form of (5.1.9) is called an equation of normal form. The equation (5.1.1) can always be transformed to an equation in normal form. In fact, let $y(t)$ be a solution of (5.1.1) and set $y(t) = u(t)v(t)$. From (5.1.1),

$$v\ddot{u} + (2\dot{v} + pv)\dot{u} + (\ddot{v} + p\dot{v} + qv)u = 0.$$ 

Let

$$v = e^{-\frac{1}{2} \int p(t)dt}, \quad Q(t) = q(t) - \frac{1}{4}p(t)^2 - \frac{1}{2}\dot{p}(t).$$

Then $u$ satisfies the following equation in normal form

$$\ddot{u} + Q(t)u = 0.$$
Example.