Limit of functions

Definition 4.1.1 Let $A \subset \mathbb{R}$. A point $c \in \mathbb{R}$ is a cluster point of $A$ if for every $\delta > 0$, there exists $x \in A$ such that $0 < |x - c| < \delta$ (or $x \in (V_\delta(c) \cap A) \setminus \{c\}$ the $\delta$-deleted neighborhood of $c$ in $A$.)

Equivalently, there exists a sequence $(x_n), x_n \in A \setminus \{c\}$ for all $n$ (for simplicity, we will say $(x_n)$ is a sequence in $A \setminus \{c\}$) such that $\lim(x_n) = c$. (Theorem 4.1.2).

Examples: $A = (0, 1); \mathbb{Q}; \{\sin n : n \in \mathbb{N}\}; \{1, 0\}; \mathbb{N}.$

Definition 4.1.4 Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ and $c$ is a cluster point of $A$. $f$ is said to have a limit $L$ at $c$ if given any $\varepsilon > 0$, there exists $\varepsilon > 0$ such that

$$|f(x) - L| < \varepsilon, \text{ for all } x \in A \cap V_\delta(c) \setminus \{c\} = (A \setminus \{c\}) \cap (c - \delta, c + \delta)$$

(or $0 < |x - c| < \delta$ and $x \in A$. ) We will write $\lim_{x \to c} f(x) = L$.

Theorem 4.1.8 $\lim_{x \to c} f(x) = L$ if and only if $\lim(f(x_n)) = L$ for all sequences $(x_n)$ in $A \setminus \{c\}$ that converges to $c$.

Some examples

$$\lim_{x \to c} x = c, \quad \lim_{x \to c} x^2 = c^2, \quad \lim_{x \to c} |x|^{1/n} = |c|^{1/n}, \quad n \in \mathbb{N} \quad \lim_{x \to c} 1/x = 1/c \text{ where } c \neq 0.$$
$f(x) = \chi_Q(x) = 1$ if $x \in \mathbb{Q}$ and $0$ otherwise.

$g : \mathbb{Q} \to \mathbb{R}$, $g(x) = 1$ for all $x \in \mathbb{Q}$.

**Example where limit does not exist**

$$\lim_{x \to 0} \sin(1/x), \lim_{x \to 0} 1/x, \lim_{x \to 0} x/|x|.$$ 

**Remark: a convention.** When dealing with an explicit algebraic function when domain is not specified, we always assume its domain to be the largest possible domain.

$$\sin(1/x), 1/x, \frac{(x + 1)}{(x^2 - 1)}, \frac{x^2 - 1}{x - 1}, \ln x, \sec x, \sqrt{x}$$

and rational function $p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials.

**Limit Theorems** Similar to limit theorems for sequences we have

4.2.4 Suppose $\lim_{x \to c} f(x) = l_1$, $\lim_{x \to c} g(x) = l_2$. Then

$$\lim_{x \to c} f(x)g(x) = l_1l_2, \quad \text{and} \quad \lim_{x \to c} f(x) + g(x) = l_1 + l_2.$$ 

Furthermore, if $l_2 \neq 0$, then $\lim_{x \to c} f(x)/g(x) = l_1/l_2$. 

**Example** Let \( p(x) \) and \( q(x) \) be polynomials. Then

\[
\lim_{x \to c} p(x) = p(c) \quad \text{and} \quad \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \text{if} \quad q(c) \neq 0.
\]

We also have comparison of limits and squeeze theorem.

**An extension of 4.2.6** Let \( f, g : A \to \mathbb{R} \) and let \( c \) be a cluster point of \( A \). If \( f(x) \leq g(x) \) on a deleted neighborhood \( V_\delta(c) \cap A \setminus \{c\} \) of \( c \) in \( A \) and \( \lim_{x \to c} f(x) = l_1 \), \( \lim_{x \to c} g(x) = l_2 \), then \( l_1 \leq l_2 \).

**4.2.7 Squeeze theorem**

Let \( f, g, h : A \to \mathbb{R} \) and let \( c \) be a cluster point of \( A \). If \( f(x) \leq g(x) \leq h(x) \) on a deleted neighborhood \( V_\delta(c) \cap A \setminus \{c\} \) of \( c \) in \( A \) and \( \lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l \), then \( \lim_{x \to c} g(x) \) exists and equals to \( l \).

By the squeeze theorem, we have,

\[
\lim_{x \to 0} x \sin \frac{1}{x} = 0.
\]

Besides all the trigonometric identities, we will assume the following is known.

\[
0 \leq \sin x \leq x \leq \tan x \quad \text{for all} \quad 0 \leq x \leq \pi/2.
\]

\[
\lim_{x \to c} \sin x = \sin c, \lim_{x \to c} \cos x = \cos c \quad \text{for all} \quad x.
\]

Then \( \lim_{x \to c} \tan x = \tan c \) for all \( c \in \mathbb{R} \) such that \( \cos c \neq 0 \) and

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]
One-sided limits

Right hand limit: \( \lim_{x \to c^+} f(x) = L \) if given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
|f(x) - L| < \varepsilon \text{ for all } 0 < x - c < \delta, x \in A
\]
(or for all \( x \in A \cap (c, c + \delta) \)).

Alternatively, one has
\[
\lim_{x \to c^+} f(x) = \lim_{x \to c} f(x) \quad \text{in} \quad A \cap (c, \infty)
\]
where \( \tilde{f} = f \bigg|_{A \cap (c, \infty)} \).

Similarly, we can define left hand limits: \( \lim_{x \to c^-} f(x) \).

4.3.2 \( \lim_{x \to c^+} f(x) = L \) if and only if \( \lim (f(x_n)) = L \) for all sequences \( (x_n) \) in \( A \cap (c, \infty) \) that converges to \( c \).

4.3.3 \( \lim_{x \to c} f(x) = L \) if and only if \( \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L \).

Of course, one also has comparison theorem and squeeze theorem for one-sided limits.

Exercise: 4.1: 1-6, 9-12,14, \quad (4th ed: 11-16 becomes 12-17) 4.2: 2, 4-6,11,13 \quad (4th ed: 13-14 becomes 14-15)

Homework7 due 28/3 4.1: 7,8,9c,10b,13 4.2: 3,12, 14 (take note of the difference if you are using 4th ed.)
**Infinite limits**

One can also define infinite limits.

For example, we write \( \lim_{x \to c^+} f(x) = \infty \) if given any \( \alpha \in \mathbb{R} \), there exists \( \delta > 0 \) such that

\[
  f(x) > \alpha \quad \text{for all } x \in A \cap (c, c + \delta).
\]

Moreover, one can define \( \lim_{x \to \infty} f(x) = L \) if given any \( \varepsilon > 0 \), there exists \( N > 0 \) such that

\[
  |f(x) - L| < \varepsilon \quad \text{for all } x \in A \cap (N, \infty).
\]

Similarly, one can define \( \lim_{x \to -\infty} f(x) = \infty \).

**Continuous functions**

Let \( f : A \to \mathbb{R} \) and \( c \in A \). Then we say \( f \) is continuous at \( c \) if given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
  |f(x) - f(c)| < \varepsilon \quad \text{for all } x \in A \cap (c - \delta, c + \delta) \quad \text{i.e., } |x - c| < \varepsilon \quad \text{and } x \in A.
\]

**Remark**  (1) In most cases, \( c \) is a cluster point of \( A \). Then \( f \) is continuous at \( c \) if and only if \( \lim_{x \to c} f(x) = f(c) \), that is, the limit \( \lim_{x \to c} f(x) \) exists and equals to \( f(c) \).

(2) Usually we are interested at the case where \( A \) is an interval, then every point in \( A \) is a cluster point.

**5.1.3** Let \( f : A \to \mathbb{R} \) and \( c \in A \). Then \( f \) is continuous at \( c \) if and only if \( \lim(f(x_n)) = f(c) \) for all sequences \( (x_n) \) in \( A \) that converges to \( c \). Note that here we allow \( x_n = c \). In particular \( (f(x_n)) \) can be the constant sequence \( (f(c)) \).
**Definition 5.1.5** Let \( f : A \to \mathbb{R} \) and \( B \subset A \). We say that \( f \) is continuous on the set \( B \) if \( f \) is continuous at at every point of \( B \).

Example of continuous functions:

1. All polynomials are continuous on \( \mathbb{R} \).
2. All rational functions are continuous on their (largest possible) domains.
3. Algebraic functions (such as \( x^r, r \in \mathbb{Q} \)) are also continuous on their (largest possible) domains.
4. Trigonometric functions are continuous on their domains.
5. Exponential functions are continuous on \( \mathbb{R} \), logarithm functions are continuous on their domains.
6. Absolute value function \( (f(x) = |x|) \) is continuous on \( \mathbb{R} \).
7. Compositions of continuous functions are continuous. That is, if \( f : A \to B \), \( g : B \to \mathbb{R} \) are continuous functions, then \( g \circ f : A \to \mathbb{R} \) is also continuous. In particular if \( f \) is continuous at \( c \) (\( c \in A \)) and \( g \) is continuous at \( f(c) \) (\( f(c) \in B \)), then \( g \circ f \) is continuous at \( c \). (See Theorems 5.2.6 & 5.2.7).

**Remark 5.1.7** Suppose \( f : A \to \mathbb{R} \) and \( c \notin A \) but \( c \) is a cluster point of \( A \). Since \( f \) is not defined at \( c \), \( f \) cannot be continuous at \( c \). However, if \( \lim_{x \to c} f(x) = L \) exists, one can extend the function to \( A \cup \{c\} \) such that the new function is continuous at \( c \).

**Example 5.1.8** The function \( x \sin(1/x) \) is only defined for \( x \neq 0 \). However, if we define \( f(0) = 0 \) and \( f(x) = x \sin(1/x) \) otherwise, then \( f \) is an extension of \( x \sin(1/x) \) such that \( f \) is continuous at 0.
**Trigonometric functions**  \( \sin x, \cos x \) are continuous on \( \mathbb{R} \). Other trigonometric functions such as \( \tan x, \cot x, \sec x, \csc x \), are continuous on their largest possible domains. We will only show that \( \sin x \) is continuous. Indeed it suffices to note that

\[
|\sin x - \sin x_0| = 2|\sin \frac{x - x_0}{2} \cos \frac{x + x_0}{2}| \leq |x - x_0|.
\]

Exercise: 4.3: 3-9, 5.1: 3-7,9-13 (4.3 & 5.1: 4th ed is the same as 3rd ed).

Homework8: 4/4 4.3: 6,10,13, 5.1:1,2,8,14,15

**5.2.1 & 5.2.2** Suppose \( f, g : A \to \mathbb{R} \). If both \( f \) and \( g \) are continuous at \( c \in A \), then the functions \( f + g, fg \) are both continuous at \( c \). In addition, if \( g(c) \neq 0 \), then \( f/g \) is continuous at \( c \). Consequently, if both \( f \) and \( g \) are continuous on a subset \( B \) of \( A \), then both functions \( f + g \) and \( fg \) are continuous on \( B \). Moreover, \( f/g \) is continuous on \( \{x \in B : g(x) \neq 0 \} \).

**Theorems 5.2.6 & 5.2.7** Compositions of continuous functions are continuous. That is, if \( f : A \to B \subset \mathbb{R}, g : B \to \mathbb{R} \) are continuous functions, then \( g \circ f : A \to \mathbb{R} \) is also continuous. In particular if \( f \) is continuous at \( c \ (c \in A) \) and \( g \) is continuous at \( f(c) \ (f(c) \in B) \), then \( g \circ f \) is continuous at \( c \).

It follows from the above theorems that if \( f : A \to \mathbb{R} \) is continuous on \( A \), then \( |f(x)| \) and \( \sqrt{|f(x)|} \) are continuous (recall that \( \sqrt{x} \) is continuous on its domain).
Unfortunately, there are many instances we need to deal with composite of functions that may not be continuous. Let us state a easy fact here.

**Fact:** If \( g \) is a continuous function at \( c \) and \( \lim_{x \to a} f(x) = c \), then \( \lim_{x \to a} g(f(x)) = g(c) = g(\lim_{x \to a} f(x)) \).

For example, \( \lim_{x \to 0} \frac{\sin^2 x}{x^2} = 1 = \lim_{x \to 0} \cos(\sin x/x) \).

Moreover, note that in the above, \( a \) could be infinite. For example, \( \lim_{x \to \infty} \cos(1/x) = 1 \).

Question: what about \( \lim_{x \to \infty} x \sin(1/x) \)?

**Continuous functions on intervals**

There are special properties of a continuous function on an interval \( I = [c, d], (c, d), [c, d], (c, d] \). (Note that they are all "connected").

1. **Intermediate value theorem (5.3.7) (IVT)** If \( f(a) > 0 \) and \( f(b) < 0 \) and \( f \) is continuous on \( [a, b] \), then there exists \( x_0 \in (a, b) \) such that \( f(x_0) = 0 \). Consequently, if \( f \) is continuous on \( [a, b] \) and \( \alpha \) is a value between \( f(a) \) and \( f(b) \), then there exists \( x^* \) between \( a \) and \( b \) such that \( f(x^*) = \alpha \). (If \( \alpha \) is strictly between \( f(a) \) and \( f(b) \), then \( x_\alpha \) will also strictly between \( a \) and \( b \).)
Some examples: (i) for all \( c \geq 0 \) and \( k \in \mathbb{N} \), there exists \( a \in \mathbb{R} \) such that \( a^k = c \).
(ii) There exists \( c \in [0, 2] \) such that \( c^3 - c = 1 \).
(iii) Let \( a, b, c, d > 0 \). Then there exists \( x \in (0, \infty) \) such that \( a + bx = c + d/x \).

2. **Extreme value theorem (5.3.4) (EVT)** If \( f \) is continuous on \([a, b]\) (a close and bounded interval), then there exist \( x^*, x_* \in [a, b] \) such that \( f(x^*) \leq f(x) \leq f(x_*) \) for all \( x \in [a, b] \).

Some consequences of the above theorems.

a. If \( f \) is a continuous function on an interval, then its image is also an interval. In addition, if that interval is closed and bounded, then its image will be also a closed and bounded interval.

b. If \( f : I \to I \) is continuous and \( I \) is a closed and bounded interval, then \( f \) has a fixed point, i.e., a point \( x_0 \in [a, b] \) such that \( f(x_0) = x_0 \).

Indeed, continuous function on a closed and bounded interval also has one more interesting property:

(3) If \( f \) is continuous on \([a, b]\), then it is also uniformly continuous on \([a, b]\), that is, given any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in [a, b], |x - y| < \delta.
\]
A function $f$ is said to be uniformly continuous on a set $A$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{for all } x, y \in A, |x - y| < \delta.$$ 

**Fact** A uniformly continuous function on a bounded set is bounded.

**5.4.2 (nonuniform continuity criteria)** A function $f$ is not uniformly continuous on $A$ if and only there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n, y_n \in A$, $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$.

**5.4.3** If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.

**5.4.8 Continuous extension theorem** A function $f$ is uniformly continuous on an interval $(a, b)$ if and only if it can be defined on both endpoints $a$ and $b$ such that the extended function is continuous on $[a, b]$.

Exercise: 5.2: 1, 4, 5, 6, 10, 11, 5.3: 2-6, 17, 5.4: 4-8, 12. (all the same as 4th ed).

Homework 9 11/4, 5.2: 8, 12, 13, 5.3: 1, 13, 5.4: 2, 9, 14.
(A) If \( f : (0, \infty) \to (0, \infty) \) is continuous and such that \( f(\sqrt{x}) = f(x) \) for all \( x \in (0, \infty) \), what can you say about the function \( f \)?

**Monotone and inverse functions**

If a function is either increasing or decreasing on a set \( A \), we say that it is monotone on \( A \). If it is strictly increasing or strictly decreasing on \( A \), we say that \( f \) is strictly monotone on \( A \).

5.6.1 Let \( f \) be a monotone function on an interval \( I \). Let \( c \in I \) and \( c \) is not an end point of \( I \). Then both \( \lim_{x \to c^+} f(x) \) and \( \lim_{x \to c^-} f(x) \) exist.

5.6.2 Let \( f \) be a monotone function on an interval \( I \). Then the set of points at which \( f \) is discontinuous is a countable set.

5.6.5 **Continuous Inverse Theorem** Let \( f \) be a strictly monotone and continuous function on an interval. Then it has an inverse \( g : f(I) \to I \) which is strictly monotone and continuous on the interval \( f(I) \).

Exercise: 5.6:4-13. (the same as 4th ed)