A Modified Binomial Tree Method for Currency Lookback Options

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Abstract The binomial tree method is the most popular numerical approach to pricing options. However, for currency lookback options, this method is not consistent with the corresponding continuous models, which leads to slow speed of convergence. On the basis of PDE approach, we develop a consistent numerical scheme called modified binomial tree method. It possesses one order of accuracy and its efficiency is demonstrated by numerical experiments. The convergence proofs are also produced in terms of numerical analysis and the notion of viscosity solution.

Keywords modified binomial tree method, currency lookback options, convergence

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1 Introduction

Lookback options are path dependent options whose payoffs depend on the maximum or the minimum of the underlying asset price during the life of the options (see [1][2][3]). Valuation formulas have been produced for European lookback options by Goldman, Sosin and Gatto (1979) [4]. Just like vanilla options, generally speaking, analytical valuation formulas are not available for American lookback options. As a result, numerical methods should be adopted to price American lookback options.

The binomial tree method, as a discrete model proposed by Cox, Ross and Rubinstein (1979) [5], is the most popular numerical approach to pricing options. Hull and White (1993) and Barraquand and Pudet (1996) propose a binomial tree method for lookback options, respectively [6][7]. Cheuck and Vorst (1997) suggest an equivalent but simple algorithm [8]. Dai (1999) establish the equivalence of the binomial tree method and an explicit difference

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scheme and proved the convergence of the binomial tree method for American lookback call options based on the notion of viscosity solution [9].

In this paper we focus on currency lookback options, since lookback options are mostly structured with a foreign exchange rate as an underlying variable. In the over-the-counter market, such options are traded with continuously monitoring. It has been observed by many researchers that the binomial tree method for currency lookback options converges very slowly. Cheuck and Vorst (1997) point out that this method is for discretely monitoring [8]. Later we will show that, in the viewpoint of PDE, the binomial tree method is not consistent with the continuous models for continuously monitored lookback options. This leads to slow speed of convergence of the binomial tree method.

The purpose of this paper is to develop a more efficient scheme. On the basis of PDE approach, we propose a consistent scheme that we call modified binomial tree method since it can be regarded as a slight modification of the binomial tree method. This method possesses one order of accuracy and its efficiency is demonstrated by numerical experiments. Numerical results show that the modified binomial tree method presented here prevails over the binomial tree method in any cases.

In addition, we show that the values of currency lookback options computed from the modified binomial tree method converge to the true solutions of corresponding continuous model. In fact, due to consistency, the convergence can be directly deduced by Lax theorem for European cases. For American currency lookback call options, the proof presented in [9] also applies to the modified binomial tree method. But, for American currency lookback put options, it would cause some difficulties. The key point is to obtain uniform estimates of bounds of the approximate solutions sequences computed from modified binomial tree method.

This paper is organized as follows. In the next section we recall the continuous models for lookback options. In section 3 we recall the binomial tree method and point out that the binomial tree method for lookback options is not consistent with the continuous models. In section 4 we present the modified binomial tree method. Section 5 is devoted to the convergence proofs. At last numerical experiments are given.

2 Continuous models

A (floating strike) lookback call gives the holder the right to buy at the lowest realized price while a (floating strike) put allows the holder to sell at the highest realized price over the lookback period. Throughout the remaining of this paper, we only consider the currency lookback put option. The case of currency lookback call is similar.

Let $T$ be the time of expiration and $[0, T]$ the lookback period. Suppose the foreign exchange rate process $(S_t)_{0 \leq t \leq T}$ is governed by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$
where \( (B_t)_{0 \leq t \leq T} \) is a standard Brownian motion, defined on some probability space, \( \mu \) is the expected return rate and \( \sigma \) is volatility. Denote the maximum foreign exchange rate from 0 to \( t \) \( (t \in [0, T]) \) by \( M = \max_{0 \leq \tau \leq t} S_\tau \). Let \( P = P(S, M, t) \) stand for option price. The continuous model for the European currency lookback put is given by

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r_d - r_f) S \frac{\partial P}{\partial S} - r_d P = 0, & \quad 0 \leq S \leq M, \quad 0 \leq t < T \\
P(S, M, T) = M - S & \\
\frac{\partial P}{\partial M}(M, M, t) = 0
\end{align*}
\]

(2.1)

where \( r_d > 0 \) and \( r_f > 0 \) represent the riskfree domestic rate and foreign rate, respectively. For American case, the continuous model is the following parabolic obstacle problem

\[
\begin{align*}
\min \{ -\frac{\partial P}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} - (r_d - r_f) S \frac{\partial P}{\partial S} + r_d P, P - (M - S) \} = 0 & \quad 0 \leq M \leq S < \infty, \quad 0 \leq t < T \\
P(S, M, T) = M - S & \\
\frac{\partial P}{\partial M}(M, M, t) = 0
\end{align*}
\]

(2.2)

With the transformation

\[
x = \ln \frac{M}{S}, \quad V(x, t) = \frac{P(S, M, t)}{S},
\]

(2.3)

(2.1) can be reduced to

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} + (r_f - r_d - \frac{\sigma^2}{2}) \frac{\partial V}{\partial x} - r_f V = 0, & \quad x > 0, \quad 0 \leq t < T \\
V(x, T) = e^x - 1 & \\
\frac{\partial V}{\partial x}(0, t) = 0
\end{align*}
\]

(2.4)

For American case, (2.2) can be reduced to

\[
\begin{align*}
\min \{ -\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} - (r_f - r_d - \frac{\sigma^2}{2}) \frac{\partial V}{\partial x} + r_f V, V - (e^x - 1) \} = 0 & \quad x > 0, \quad 0 \leq t < T \\
V(x, T) = e^x - 1 & \\
\frac{\partial V}{\partial x}(0, t) = 0
\end{align*}
\]

(2.5)

3 Binomial tree method

The standard binomial tree method for lookback options involves the path function so that the amount of computation is very large \[7][6\]. Here we present the single state variable binomial tree method proposed by Cheuck and Vorst (1997) \[8\].

If \( N \) is the number of discrete time points, we have time points \( n \Delta t, \ n = 0, 1, \ldots, N \) with \( \Delta t = T/N \). Set \( u = e^{\sigma \sqrt{\Delta t}}, \ d = 1/u \) and \( \rho = e^{r_d \Delta t} \). Let \( P^n(S_n, M_n) \) be the price of
European currency lookback put option at time point $n\Delta t$ with foreign exchange rate $S_n$ and $M_n = \max_{0 \leq i \leq n} S_i$. Suppose that $S_n$ will be either $S_nu$ or $S_nd$ after a small time interval $\Delta t$. The transformation (2.3) for continuous models implies the following transformation might be adopted:

$$P^n(S_n, M_n) = S_n V^n_j$$

(3.1)

with $j = \ln(M_n/S_n)/\ln u$. Clearly $j$ is a non-negative integer. Consider $j \geq 1$, in which case $M_n = M_{n+1}$. Following Cox et al. arbitrage argument, one has

$$P^n(S_n, M_n) = \frac{1}{\rho} [p P^{n+1}(S_nu, M_{n+1}) + (1-p) P^{n+1}(S_nd, M_{n+1})],$$

(3.2)

where

$$p = \frac{\rho e^{-r\Delta t} - d}{u - d}.$$

It follows from (3.1) and (3.2)

$$V^n_j = \frac{1}{\rho} [pu V^{n+1}_{j-1} + (1-p) d V^{n+1}_{j+1}], \quad j \geq 1.$$  

(3.3)

For $j = 0$, similarly one derives

$$V^n_0 = \frac{1}{\rho} [pu V^{n+1}_0 + (1-p) d V^{n+1}_1].$$

(3.4)

As at expiration date

$$P^N(S_N, M_N) = M_N - S_N = S_N(u^j - 1),$$

then

$$V^N_j = u^j - 1, \quad j = 0, 1, \cdots, N.$$  

(3.5)

Using the backward induction (3.3)-(3.5), option prices can be calculated. This is the single state variable binomial tree method.

For American lookback puts, the binomial tree method is given by

$$\begin{cases}
V^n_j = \max\left\{ \frac{1}{\rho} [pu V^{n+1}_{j-1} + (1-p) d V^{n+1}_{j+1}], u^j - 1 \right\}, & j > 0 \\
V^n_0 = \frac{1}{\rho} [pu V^{n+1}_0 + (1-p) d V^{n+1}_1] \\
V^N_j = u^j - 1, & j \geq 0
\end{cases}.$$  

(3.6)

In the following we show that binomial tree method is not consistent with continuous model. To illustrate method, we take European case for example. First, let us present an explicit difference scheme for (2.4). Given mesh size $\Delta x, \Delta t > 0$, $N\Delta t = T$, let $Q = \{(x_j, t_n) : x_j = j\Delta x, t_n = n\Delta t, -1 \leq j \in Z, 0 \leq n \leq N\}$ stand for the lattice. $V^n_j = V(x_j, t_n)$
represents the value of numerical approximation at \((j\Delta x, n\Delta t)\). Taking the explicit difference discretization for time and the conventional difference discretization for space, we have
\[
\frac{V_j^{n+1} - V_j^n}{\Delta t} + \frac{\sigma^2 V_{j+1}^{n+1} + V_j^{n+1} - 2V_j^{n+1}}{\Delta x^2} + (r_f - r_d - \frac{\sigma^2}{2}) \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta x} - r_f V_j^n = 0, \quad j \geq 0.
\]
Set \(\Delta x = \sigma\sqrt{\Delta t}\) and denote
\[
a = \frac{1}{2} + \frac{\Delta x}{2} \left( \frac{1}{2} + \frac{r_d - r_f}{\sigma^2} \right),
\]
then we get
\[
V_j^n = \frac{1}{1 + r_f \Delta t} \left[ aV_{j-1}^{n+1} + (1 - a)V_{j+1}^{n+1} \right], \quad j \geq 0.
\] (3.7)

In order to approximate the Neumann boundary condition, we let
\[
V_{-1}^{n+1} = V_{0}^{n+1}.
\] (3.9)

Substituting (3.9) into (3.8) for \(j = 0\), we obtain an explicit difference scheme
\[
\begin{cases}
V_j^n = \frac{1}{1 + r_f \Delta t} \left[ aV_{j-1}^{n+1} + (1 - a)V_{j+1}^{n+1} \right], \quad j \geq 1 \\
V_0^n = \frac{1}{1 + r_f \Delta t} \left[ aV_{0}^{n+1} + (1 - a)V_{1}^{n+1} \right]
\end{cases}
\] (3.10)

with the final condition
\[
V_j^N = e^{j\Delta x} - 1, \quad j \geq 0
\] (3.11)

We claim that scheme (3.10) is not consistent with continuous model (2.4) at \(x = 0\) \((j = 0)\). Indeed, let \(v\) be a solution to equation (2.4) and write
\[
\Delta t v_0^n = \frac{1}{1 + r_f \Delta t} \left( av_0^{n+1} + (1 - a)v_1^{n+1} \right) - v_0^n
\]
\[
= \frac{1}{1 + r_f \Delta t} \left( av_0^{n+1} - v_0^n + (1 - a)v_1^{n+1} - r_f v_0^n \Delta t \right).
\]

It follows from Taylor expansions and the boundary condition \((v_{xx})_0^{n+1} = 0\)
\[
\Delta t v_0^n = - \frac{\Delta t}{1 + q \Delta t} \frac{\sigma^2}{4} v_{xx} + O(\Delta t \Delta x),
\]
which implies the inconsistency (3.10) and (2.4).

On the other hand, it is not hard to check that
\[
\frac{1}{\rho} p u = \frac{a}{1 + r_f \Delta t} + (\Delta t \frac{1}{2}), \quad \frac{1}{\rho} (1 - p) d = \frac{1 - a}{1 + r_f \Delta t} + (\Delta t \frac{1}{2}).
\] (3.12)

Then we conclude that the binomial tree method (3.3)-(3.5) is equivalent to explicit difference scheme (3.10)-(3.11) in the sense of neglecting a high order of \(\Delta t\). So, the binomial tree method (3.3)-(3.5) is also not consistent with continuous model (2.4).
Remark 1 In fact, (3.9) is a $O(\Delta x)$ approximation to the Neumann boundary condition. Due to the way that (3.8) and (3.9) fit together, one order of accuracy is lost. This leads to slow convergence speed of difference scheme (3.10)-(3.11) and binomial tree method (3.3)-(3.5).

Remark 2 For American case, similar result can be obtained.

4 Modified binomial tree method

In this section we present a consistent scheme. Let us still take the European option for example. We now choose the grid $Q = \{(\bar{x}_j, t_n) : \bar{x}_j = (j + \frac{1}{2})\Delta x, t_n = n\Delta t, 0 \leq n \leq N, -1 \leq j \in \mathbb{Z}\}$. Let $V_j = V(\bar{x}_j, t_n)$ be the value of numerical approximation at $(\bar{x}_j, t_n)$. We still use

$$V_{n+1}^j = V_n^{n+1}$$

(4.1)

to approximate the boundary condition. By similar arguments as in last section and (4.1), again we get an explicit difference scheme

\[
\begin{cases}
V_j^n = \frac{1}{1+r_f\Delta t}[aV_{j-1}^n + (1-a)V_{j+1}^n], & j \geq 1 \\
V_0^n = \frac{1}{1+r_f\Delta t}[aV_0^{n+1} + (1-a)V_1^{n+1}]
\end{cases}
\]

where $a$ is defined by (3.7). It should be noted that the final condition becomes

$$V_N^j = e^{(j+1/2)\Delta x} - 1, \quad j \geq 0$$

In virtue of (3.12), neglecting a high order of $\Delta t$, the above scheme is equivalent to

\[
\begin{cases}
V_j^n = \frac{1}{p}[puV_{j-1}^n + (1-p)dV_{j+1}^n], & j \geq 1 \\
V_0^n = \frac{1}{p}[puV_0^n + (1-p)dV_1^n] \\
V_j^N = u^{j+1/2} - 1, \quad j \geq 0
\end{cases}
\]

(4.2)

The scheme (4.2) can be regarded as a slight modification of the binomial tree method. Since the $x = 0$ boundary occurs at the middle of $\bar{x}_-1$ and $\bar{x}_0$, (4.1) is a $O(\Delta x^2)$ approximation to the Neumann boundary condition. It is not hard to verify that (4.2) is consistent with equation (2.4) and is accurate of order $O(\Delta x)$.

For American lookback put, similarly, we can derive the modified binomial tree method as follows:

\[
\begin{cases}
V_j^n = \max\{\frac{1}{p}[puV_{j-1}^n + (1-p)dV_{j+1}^n], u^{j+1/2} - 1\}, & j \geq 1 \\
V_0^n = \max\{\frac{1}{p}[puV_0^n + (1-p)dV_1^n], u^{1/2} - 1\} \\
V_j^N = u^{j+1/2} - 1, \quad j \geq 0
\end{cases}
\]

(4.3)
5 Convergence

This section is devoted to the convergence proof of the modified binomial tree method. First let us consider the European case. It is not hard to verify that the modified tree method (4.2) is stable. Since (4.2) is consistent with equation (2.4) and is accurate of order $O(\Delta x)$, we infer by Lax theorem (see [10])

**Theorem 1** The modified binomial tree method (4.2) is convergent with $O(\Delta x)$ for European currency lookback put option.

Next let us consider modified binomial tree method (4.3) for American lookback put option. The convergence proof relies on the notion of viscosity solution introduced by Crandal, Ishii and Lions (1992) [11]. The basic idea stems from Barles and Souganidis (1991) and Jiang and Dai (1998) [12][13].

In what follows we always denote by $V_j^n(x)$ the solution to (4.3). In addition, we suppose $0 < p < 1$,

which keeps valid for sufficiently small $\Delta t$. Under this assumption, (4.3) is a monotone scheme. We now introduce several lemmata.

**Lemma 2** (1) $V_j^n \leq V_j^{n-1}$ for all $j \geq 0$ and $n \leq N$.
(2) $V_j^n \leq V_{j+1}^n$ for all $j \geq 0$ and $n \leq N$.

Proof. The proof is obvious.

**Lemma 3** Let $W_j^n$ be the solution to the following scheme

\[
\begin{aligned}
W_j^n &= \max\left\{ \phi_j^n + \frac{1}{2}(1-p)dW_{j+1}^{n+1}, u_j^{n+1/2} \right\}, \quad j \geq 1 \\
W_0^n &= \max\left\{ \phi_0^n + \frac{1}{2}(1-p)dW_1^{n+1}, u_0^{1/2} \right\} \\
W_j^N &= u_j^{n+1/2}
\end{aligned}
\]  

(5.4)

Then for each $n \leq N$, we have
(1) $V_j^n \leq W_j^n$ for all $j \geq 0$.
(2) $W_j^n \leq W_j^{n-1}$ for all $j \geq 0$.
(3) $W_j^n = u_j^{n+1/2}$ for $j \geq N - n$.

(5.5)
Proof. (1) and (2) are obvious. In order to prove (3), we use induction. Suppose (5.5) holds for \( n = k + 1 \), namely \( V_{k+1}^{j+1} \leq u_{j+1/2} \) for \( j \geq N - k - 1 \). Then for \( j \geq N - k \)

\[
W_j^k = \max\{\frac{1}{\rho}[puW_{j-1}^{k+1} + (1 - p)dW_{j+1}^{k+1}], u_{j+1/2}\} = \max\{\frac{1}{\rho}[puu_{j-1}^{j+1/2} + (1 - p)du_{j+1/2}], u_{j+1/2}\} = \max\{\frac{1}{\rho}u_{j+1/2}, u_{j+1/2}\} = u_{j+1/2}.
\]

This is the desired result.

The following lemma plays a critical role in the convergence proof.

**Lemma 4** (1) For \( \Delta t \) given, there exists a unique element \( \{W_j^{\Delta t}\}_{j \geq 0} \) satisfying \( \{W_j^{\Delta t} - u_{j+1/2}\}_{j \geq 0} \in l^\infty(\mathbb{Z}^+) \) such that

\[
\begin{cases}
W_j^{\Delta t} = \max\{\frac{1}{\rho}[puW_{j-1}^{\Delta t} + (1 - p)dW_{j+1}^{\Delta t}], u_{j+1/2}\}, j \geq 1 \\
W_0^{\Delta t} = \max\{\frac{1}{\rho}[puW_0^{\Delta t} + (1 - p)dW_1^{\Delta t}], u_{j+1/2}\}
\end{cases}
\]  

(5.6)

and

\[ W_j^n \leq W_j^{\Delta t} \text{ for all } n \text{ and } j \geq 0, \quad W_j^{\Delta t} \leq W_{j+1}^{\Delta t} \text{ for } j \geq 0. \]  

(5.7)

(5.8)

(2)

\[ W_j^{\Delta t} = \begin{cases}
C_1\xi_1^j + C_2\xi_2^j, & 0 \leq j < j_\infty \\
u_{j+1/2}, & j \geq j_\infty
\end{cases} \]  

(5.9)

where

\[
\xi_{1,2} = \frac{\rho \pm \sqrt{\rho^2 - 4\rho(1 - p)}}{2(1 - p)d}, \quad C_1 = \frac{\xi_2u_{j_\infty+1/2} - u_{j_\infty+3/2}}{\xi_1^{j_\infty} - \xi_1}, \quad C_2 = \frac{\xi_2^{j_\infty} - (\xi_2 - \xi_1)}{\xi_2^{j_\infty} - \xi_1}
\]  

(5.10)

(5.11)

and

\[
j_\infty = \frac{1}{\ln\xi_2 - \ln\xi_1} \ln\left(\frac{(\rho - pu) - (1 - p)d\xi_2}{(1 - p)d}\right). \]  

(5.12)

(3)

\[
\lim_{\Delta t \to 0} j_\infty \Delta x = \frac{1}{\lambda_- - \lambda_+} \ln\frac{\lambda_- (\lambda_+ - 1)}{\lambda_+ (\lambda_- - 1)} < \infty.
\]  

(5.13)
where
\[
\lambda_\pm = \frac{r_d - r_f}{\sigma^2} + \frac{1}{2} \pm \sqrt{\left(\frac{r_d - r_f}{\sigma^2}\right)^2 + \frac{1}{2} + \frac{2r_f}{\sigma^2}}. 
\]

Proof: Denote \(\overline{W}^n = \{\overline{W}_j^n\}_{j \geq 0}\). To simplify notation, (5.4) will also be written as
\[
\overline{W}_n = F(\Delta t)\overline{W}_n^{n+1}(x) 
\]
with \(\overline{W}^N = \{u_{j+1/2}\}_{j \geq 0}\). Let \(\overline{W}_n = \{\overline{W}_j^n - u_{j+1/2}\}_{j \geq 0}\). Then \(\overline{W}_n(x)\) satisfies
\[
\overline{W}_n(x) = F(\Delta t)(\overline{W}_n^{n+1} + \{u_{j+1/2}\}_{j \geq 0} - \{u_{j+1/2}\}_{j \geq 0} = G(\Delta t)\overline{W}_n^{n+1}. 
\]
Owing to (5.5), \(\overline{W}_n \in l^\infty(Z^+)\). Hence \(G(\Delta t)\) can be regarded as a mapping from \(l^\infty(Z^+)\) to \(l^\infty(Z^+)\). We claim \(G(\Delta t)\) is a contraction mapping. Indeed,
\[
\|G(\Delta t)U(x) - G(\Delta t)V(x)\|_\infty 
= \|F(\Delta t)(U(x) + \{u_{j+1/2}\}_{j \geq 0}) - F(\Delta t)(V(x) + \{u_{j+1/2}\}_{j \geq 0})\|_\infty 
\leq \frac{1}{\rho}[pu + (1 - p)d]\|U(x) - V(x)\|_\infty = e^{-r_f \Delta t}\|U(x) - V(x)\|_\infty 
\]
with \(r_f > 0\) for \(U^n, V^n \in l^\infty(Z^+)\), where \(\|\cdot\|_\infty\) stands for the norm of \(l^\infty(Z^+)\) and the last equality is due to
\[
\frac{1}{\rho}[pu + (1 - p)d] = e^{-r_f \Delta t}. 
\]
Therefore, there exists a unique element \(\overline{W}^{\Delta t} \in l^\infty(Z^+)\) such that \(\overline{W}^{\Delta t} = G(\Delta t)\overline{W}^{\Delta t}\). Due to Lemma 2, one has \(\overline{W}^n_j \leq \overline{W}^{\Delta t}_j\) for all \(n\) and \(j \geq 0\) and \(\overline{W}^{\Delta t}_j \leq \overline{W}^{\Delta t}_{j+1}\) for \(j \geq 0\). This completes the proof of (1) by denoting \(\overline{W}^{\Delta t}_j(x) = \overline{W}^{\Delta t}_j(x) + \{u_{j+1/2}\}_{j \geq 0}\). Solving the system (5.6) yields (5.9). By symbol operation, it is not hard to check that (5.13) is valid. The proof is complete.

The following lemma is the well-known comparison principle (see [11]).

**Lemma 5** Suppose \(u\) and \(v\) are viscosity subsolution and supersolution of problem (2.5) respectively, then \(u \leq v\).

Define the extension function \(\overline{V}_\Delta t(x, t), (x, t) \in [0, \infty) \times [0, T]\) of modified binomial approximate solution as follows: for \(x \in [(j - \frac{1}{2})\Delta x, (j + \frac{1}{2})\Delta x), t \in [(n - \frac{1}{2})\Delta t, (n + \frac{1}{2})\Delta t), \overline{V}_\Delta t(x, t) = \overline{V}_j^n.\)
Theorem 6 Suppose that $V(x,t)$ is the viscosity solution to the problem (2.5). Then, as $\Delta t \to 0$, $V_{\Delta t}(x,t)$ converges locally uniformly to $V(x,t)$ in $[0, \infty) \times [0,T]$. That is, the modified binomial tree method for American lookback options is convergent.

Proof. Set

$$
V^*(x,t) = \limsup_{\Delta t \to 0, (y,s) \to (x,t)} V_{\Delta t}(y,s),
$$
$$
V_*(x,t) = \liminf_{\Delta t \to 0, (y,s) \to (x,t)} V_{\Delta t}(y,s).
$$

Combing (5.8), (5.9) and (5.13) yields

$$
W_j^{\Delta t} \leq \max\{e^{\tau_j}, \left(\frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)}\right)^{1/(\lambda_--\lambda_+)} + 1 \}
$$

for sufficiently small $\Delta t$. By (5.7), (5.16) and Lemma 3 (1), one has

$$
V^*_j \leq W_j^0 \leq W_j^{\Delta t} \leq \max\{e^{\tau_j}, \left(\frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)}\right)^{1/(\lambda_--\lambda_+)} + 1 \}.
$$

This implies that $V^*$ and $V_*$ are well defined. It is obvious that $V^* \in USC$ and $V_* \in LSC$, and $V_*(x,t) \leq V^*(x,t)$. Since the modified binomial tree scheme (4.3) is monotone and consistent with (2.5), we can use the so-called “half-relaxed” technique to show that $V^*$ and $V_*$ are viscosity subsolution and supersolution of (2.5) respectively (see [12] and [13]). Then in terms of comparison principle (Lemma 5) we deduce $V^*(x,t) \leq V_*(x,t)$ and thus $V^*(x,t) = V_*(x,t) = V(x,t)$, which is the desired results.

Remark 3 The above proofs can also be applied to show the convergence of the binomial tree method for American currency lookback put options.

Remark 4 In next section numerical results show that the modified binomial tree (4.3) has also one order of convergence. Unfortunately, we cannot prove it. This problem is open.

Remark 5 Relying on finite volume method, we are able to construct another consistent scheme for American currency lookback put option as follows:

$$
\begin{align*}
V^n_j &= \max\left\{\frac{1}{\rho}[puV_{j-1}^{n+1} + (1-p)dV_{j+1}^{n+1}], w_j - 1\right\}, \ j \geq 1, \\
V^n_0 &= \frac{1}{\rho}[(pu - (1-p)d)V_0^{n+1} + 2(1-p)dV_1^{n+1}] \\
V^n_N &= w^j - 1, \ j \geq 0.
\end{align*}
$$

It can be shown by numerical experiments that this scheme also performs well. However, this scheme is not monotone and its convergence is under the investigation.
6 Numerical experiments

We will compare the modified binomial tree method (MBTM) ((4.2) for European case and (4.3) for American case) with the binomial tree method (BTM) ((3.3)-(3.5) for European case and (3.6) for American case). Table 1 contains computations for European and American currency lookback options with $S = 100$, $r_d = 0.07$, $r_f = 0.04$, $\sigma = 0.2$, $T = 1$ year and several values of $M$ and $N = T/\Delta t$. Analytical solutions are also given for European case. Clearly the modified binomial tree method is superior to the binomial tree method. Even with the large time step, the outstanding performance of the modified binomial tree method can be recognized. Note that the nearer to $M = S$, the larger the error. This fact shows that the main errors are from the approximation at boundary.

In table 2 we compare the speeds of convergence of two methods. Since the main errors appear at the boundary, values of European and American currency lookback put options are given at $S = M = 100$ with $r_d = 0.07$, $r_f = 0.04$, $\sigma = 0.2$ and several values of $T$. For the modified binomial tree method, one order of convergence is observed both for European and American currency lookback options. However, there is no order of convergence for the binomial tree method. In any case the modified binomial tree method prevails over the binomial tree method.

<table>
<thead>
<tr>
<th>Table 1 Numerical results for different $M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(S = 100$, $r_d = 0.07$, $r_f = 0.04$, $\sigma = 0.2$, $T = 1$ year, $N = T/\Delta t)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>M</th>
<th>100</th>
<th>102</th>
<th>106</th>
<th>112</th>
<th>120</th>
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<td>European</td>
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<td></td>
<td></td>
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<tr>
<td>American</td>
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<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>50</td>
<td>13.894</td>
<td>14.115</td>
<td>14.920</td>
<td>17.219</td>
<td>22.062</td>
<td>40.005</td>
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<tr>
<td>50</td>
<td>15.310</td>
<td>15.535</td>
<td>16.007</td>
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<tr>
<td>500</td>
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<td>15.445</td>
<td>16.066</td>
<td>18.076</td>
<td>22.526</td>
<td>40.001</td>
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</tbody>
</table>
A modified BTM to currency lookback options

Table 2 Numerical results for different $T$
($S = 100, M = 100, r_d = 0.07, r_f = 0.04, \sigma = 0.2, N = T/\Delta t$)

<table>
<thead>
<tr>
<th>$T$ (year)</th>
<th>0.25</th>
<th>0.5</th>
<th>0.75</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
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<tbody>
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<tr>
<td>$N = 5$</td>
<td>5.046</td>
<td>7.470</td>
<td>11.688</td>
<td>13.240</td>
<td>15.637</td>
<td>17.443</td>
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<tr>
<td>$N = 5$</td>
<td>7.754</td>
<td>10.759</td>
<td>12.950</td>
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<td>17.448</td>
<td>19.537</td>
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<tr>
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<td>American:</td>
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</table>

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References

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