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PORTFOLIO SELECTION WITH CAPITAL GAINS TAX, RECURSIVE UTILITY, AND REGIME SWITCHING

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Abstract. Capital gains taxation has important implications for investors' portfolio choice decisions. To explore these implications, we develop a continuous time investment and consumption model with capital gains tax, Epstein-Zin recursive utility, and regime switching. We find that various factors, such as tax rate, risk aversion, interest rate, stock return, and volatility jointly affect optimal portfolio allocation, whereas intertemporal substitution does not. In a regime switching market, investors may trade or stop trading purely because of a change in regime, and there is a distinct cross-regime effect on optimal portfolio allocation. In particular, investors tend to raise stock investment in a bear regime so as to reduce potential tax payments upon regime switching. Given reasonable parameter values, regime switching has a greater impact on optimal portfolio allocation in a bear regime than in a bull regime.

Keywords: portfolio selection; capital gains tax; recursive utility; regime switching.

[1]This work is partially supported by Singapore MOE AcRF grant R-146-000-188/201-112, USPC-SPO-NUS grant R-146-000-203-133, NUS Global Asia Institute - LCF Fund R-146-000-160-646, NSFC grant 11671292, and NSF grant DMS-1008905. The authors thank Neng Wang (the department editor), the associate editor, and an anonymous referee for helpful comments.

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1. Introduction

It is well recognized that transaction costs dramatically alter investors' portfolio choices. The capital gains tax rate, typically ranging from 10% to 40%, is much higher than the rate of transaction costs. Thus, capital gains tax must have significant implications for investors' portfolio choice decisions. However, in contrast to extensive literature on continuous-time portfolio selection with transaction costs,[1] relatively little research exists on capital gains tax.

In this paper, we develop a *continuous-time* investment and consumption model with capital gains tax and the following two important features. First, we choose the recursive utility introduced by Epstein and Zin (1989) and Duffie and Epstein (1992). We aim to examine how various factors, including tax rate, risk aversion, and intertemporal substitution, affect optimal portfolio allocation. Second, we consider a regime-switching market in which there are two regimes ("bull" and "bear") with different fundamental parameters, such as expected return and volatility, and the two regimes may switch from one to the other according to a Markov chain.[2] This allows us to investigate the effect of capital gains tax on optimal portfolio allocation when the investment opportunity set is not constant.

Because no capital gains tax is paid until capital gains are realized, investors tend to defer realization of positive capital gains so as to save interest. This deferral option is constrained by suboptimal risk exposure due to the inability to maintain the optimal portfolio, namely, the Merton line.[3] As such, investors need to achieve a trade-off between the benefit of tax deferral and the cost of suboptimal risk exposure, which leads to a no-trade region enclosed by two trading boundaries: the optimal

[1]See, e.g., Magill and Constantinides (1976), Constantinides (1986), Davis and Norman (1990), Shreve and Soner (1994), Leland (1999), Liu and Loewenstein (2002), Liu (2004), Jang et al. (2007), Dai and Yi (2009), Dai et al. (2009), Kallsen and Muhle-Karbe (2010), Soner and Touzi (2013), and references therein.

[2]The two state Markov chain has been widely used to describe the economic cycle, see, e.g., Jang et al. (2007), Hackbarth, Miao, and Morellec (2006), and Bhamra, Kuehn, and Strebulaev (2010).

[3]Due to the presence of capital gains tax, the definition of the Merton line is slightly different from the standard one, and the portfolio is tax-adjusted. For details, see (2.9).

buy boundary and the sell boundary that define optimal portfolio allocation. Since investors are inclined to take no transaction to defer taxation of capital gains as long as risk exposure is within the no-trade region, the distance between the two trading boundaries determines the expected tax deferral time and then significantly affects the tax deferral option. As buying does not incur capital gains tax, one may expect that the buy boundary would get closer to the Merton line. We find that, on the contrary, the buy boundary generally deviates more from the Merton line than the sell boundary does if the stock risk premium is positive. The financial intuition for this is that a positive stock risk premium will potentially increase risk exposure of investors who defer taxation of capital gains; thus, to keep the average risk exposure close to the Merton line, investors should lower the buy boundary more than raise the sell boundary. However, when the no-trade region is very narrow, the two trading boundaries become symmetric with respect to the Merton line, because in this scenario a transaction is primarily caused by the random walk of stock price reaching the trading boundaries.

Using asymptotic and numerical analysis, we provide qualitative and quantitative insights about the effects of various factors on optimal portfolio allocation. For example, we find that reducing the interest rate (or tax rate, risk premium) or raising the relative risk aversion level (or stock volatility) can increase the frequency of trading. The intuition behind this is clear: if the interest rate or tax rate shrinks, then the incentive for tax deferral declines; and if investors are more risk averse or face larger stock volatility (or smaller risk premium), then they tend to invest less in stock and thus to incur a smaller amount of tax. All of these lead investors to trade more frequently.

It is well known that, in the absence of capital gains tax, the consumption decision depends on both the elasticity of intertemporal substitution (EIS) and risk aversion; whereas, the portfolio allocation decision is independent of the EIS. We find that this property is inherited in the presence of capital gains tax, and on the other hand, the

EIS does affect the tax deferral option through the optimal consumption decision. The impact of the EIS on the consumption decision depends on parameter values, as does the impact of the EIS on the tax deferral option. Assuming that the risk-free rate or stock return is sufficiently high compared to the subjective discount rate for consumption, we find that the tax deferral option becomes more valuable as the EIS increases. Intuitively, when the EIS increases, investors are more willing to substitute consumption intertemporally, and thus consume less for such a relatively high risk-free rate or stock return. Since consuming less implies investing more and then raises the incentive of tax deferral, the deferral option becomes more attractive.

When the market may switch between bull and bear regimes, we find that investors may trade or stop trading purely because of a change in regime, and there is a distinct cross-regime effect on optimal portfolio allocation and the deferral option. In particular, in contrast to the myopic portfolio allocation strategy in the absence of capital gains tax, the optimal portfolio allocation is affected by the investment opportunity in the other regime because of tax costs. Intuitively, when the market switches from a bull regime to a bear regime, investors may have to sell stock and then pay capital gains tax. This motivates investors to raise stock investment in the bear regime so as to reduce potential tax payments upon regime switching. The cross-regime effect is more distinct when there is a large capital gain, because more capital gains tax may be incurred. Further, we find that, given reasonable parameter values, regime switching has a greater impact on optimal portfolio allocation in a bear regime than in a bull regime, because a bull regime usually has a longer duration and then dominates.

Related Literature. Capital gains taxes differ from transaction costs because: 1) investors pay taxes for capital gains, but receive tax returns for capital losses; and 2) the amount of capital gains taxed depends on the purchase price of stock holdings, known as the tax basis, which incurs strong path-dependency. As a consequence, much of the extant literature on portfolio selection with capital gains tax has been restricted to discrete-time models with a very limited number of time steps.[4] The continuous time model developed in the present paper offers the advantage of using analytical techniques. Despite that closed-form solutions are generally unavailable, we utilize asymptotic analysis to provide a quantitative and qualitative characterization of the deferral option and optimal portfolio allocation. Numerical solutions are also employed as a complementary tool.

Using the average tax basis as an approximation to the exact tax basis,[5] Dammon, Spatt, and Zhang (2001, 2004) make a significant contribution by developing a binomial tree model that is capable of effectively working with multi-step portfolio choice decisions. Gallmeyer, Kaniel, and Tompaidis (2006) further extend to the multiple stocks case. These models assume that investors can receive tax returns unlimitedly for capital losses. However, the U.S. tax code stipulates that the amount of capital losses that investors can claim on their tax returns is limited to $3,000 per year, and the rest is carried forward indefinitely. Both Ehling et al. (2010) and Marekwica (2012) incorporate such limited tax returns into a discrete-time model with the average tax basis. Ehling et al. (2010) find that the restriction does not significantly affect the investors' portfolio choice decision provided that there are large embedded capital gains. Marekwica (2012) shows that immediately realizing capital losses is still optimal.

The discrete time model developed by Dammon, Spatt, and Zhang (2001, 2004) is extended by Ben Tahar, Soner, and Touzi (2007, 2010) to the continuous time Merton problem with capital gains tax, where the constant relative risk aversion (CRRA) utility is chosen and the constant investment opportunity set is assumed. In the U.S., to encourage long-term investment, the tax rate for long-term capital gains is significantly lower than the rate for short-term capital gains. Using the average holding period with the average tax basis as an approximation, Dai et al.

[4]See, e.g., Constantinides (1983, 1984), Dybvig and Koo (1996), DeMiguel and Uppal (2005), and Garlappi and Huang (2006).

[5]While the exact tax basis is used in the U.S., the average tax basis is indeed adopted in Canada.

(2015) propose a continuous-time model to examine how the asymmetric tax structure together with limited tax returns affects the behavior of investors.

As in most of the extant literature, we restrict our attention to the single tax rate case with the average tax basis and unlimited tax returns. Hence, our model can be regarded as an extension of the model in Ben Tahar, Soner, and Touzi (2007, 2010) by incorporating recursive utility and regime switching. However, our analysis provides a better understanding of optimal portfolio allocation and the deferral option, even with the CRRA utility and the constant investment opportunity set. For example, besides the results that we have presented above, we find that the initial *tax-adjusted* optimal fraction of wealth in stocks is slightly higher than the Merton line, because the deferral option reduces risk and motivates investors to invest more in stocks.[6] We also find that the optimal consumption strategy is close to the suboptimal consumption strategy without deferring capital gains taxation. Moreover, our paper contributes to the literature on asymptotic analysis for portfolio selection. A large body of literature exists on asymptotic analysis for portfolio selection with transaction costs.[7] To the best of our knowledge, the present paper is the first to adopt asymptotic analysis to study portfolio selection with capital gains tax.

The remainder of the paper is organized as follows. In the next section, we formulate the problem as a stochastic control problem and present some basic properties of the associated value function and the optimal strategy. In Section 3, we focus on the single regime market and investigate the impact of various factors on optimal portfolio choice decision. The regime-switching case is studied in Section 4. We conclude in Section 5. All technical proofs and complementary results are found in the Appendix.

[6]The tax return for capital losses also motivates investors to invest more in stocks. This effect has been absorbed by the tax-adjusted fraction.

[7]See, e.g., Shreve and Soner (1994), Roger (2004), Soner and Touzi (2013), and Bichuch and Shreve (2013).

2. Model Formulation

In this section, we present a mathematical formulation for the continuous-time investment and consumption problem with the average tax basis, extending the model of Ben Tahar, Soner, and Touzi (2010) to incorporate the Epstein-Zin recursive utility and a regime-switching market.

2.1. **The Model.**

We work on a filtered probability space $(\mathcal{D}, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$, where $\{\mathscr{F}_t\}_{t\geq 0}$ is the filtration generated by a standard Wiener process $\{W_t, t \geq 0\}$ with $W_0 = 0$ and an independent Markov chain $\{\iota_t, t \geq 0\}$ with values in $\mathcal{I} = \{1, 2, \cdots\}$ and generator matrix $Q = (q_{ij})_{i,j\in\mathcal{I}}$, satisfying $\sum_{j\in\mathcal{I}} q_{ij} = 0, \forall i \in \mathcal{I}$. Without loss of generality, we focus on the single regime case $\mathcal{I} = \{1\}$ and the two regimes case $\mathcal{I} = \{1, 2\}$.

Assume that there are two assets that an investor can trade without any transaction costs. The first asset ("the bond") is a money market account growing at a continuously compounded, after-tax interest rate $r(\iota_t)$. The second asset ("the stock") is a risky investment and its price S_t follows

$$dS_t = \mu(\iota_t)S_t dt + \sigma(\iota_t)S_t dW_t,$$

where the market parameters $r(\cdot)$, $\mu(\cdot)$, and $\sigma(\cdot)$ are regime-dependent.

The investor is subject to capital gains tax. As in Dammon, Spatt, and Zhang (2001) and Ben Tahar, Soner, and Touzi (2010), we assume that: (i) capital gains tax is realized immediately after the sale; (ii) there is no wash sale restriction;[8] (iii) short selling is prohibited; and (iv) the tax basis used to evaluate capital gains is defined as the weighted average of past purchase prices.

Let x_t, y_t, and k_t be the amount invested in the bond, the current dollar value of, and the total cost basis of the stock holding, respectively. We introduce two right-continuous (with left limits), nonnegative, and nondecreasing $\{\mathscr{F}_t\}_{t\geqslant 0}$-adapted

[8] Jensen and Marekwica (2011) examine the effect of wash sale constraints using a two-period model with one risky asset.

processes L_t and M_t with $L_0 = M_0 = 0$, where $dM_t \leqslant 1$ represents the *fraction of the current stock holding* that is sold at time t, while dL_t represents the *dollar amount* purchased at time t. It is worth noting that the average tax basis is used to evaluate capital gains. Therefore, when one sells stock at time t, the purchase price k_t declines by the same proportion dM_t as the dollar value of the stock holding y_t does, and the realized capital gain is $(y_t - k_t)dM_t$. Then, the evolution processes of x_t, y_t, and k_t are:

$$\begin{aligned} dx_t &= [r(\iota_t)x_t - C_t]\,dt - dL_t + [y_t - \alpha\,(y_t - k_t)]\,dM_t, \\ dy_t &= \mu(\iota_t)y_t dt + \sigma(\iota_t)y_t dW_t + dL_t - y_t dM_t, \\ dk_t &= dL_t - k_t dM_t, \end{aligned}$$

where α and $\{C_t, t \geq 0\}$ are the tax rate and the consumption streams, respectively.

We use the continuous time Epstein-Zin recursive utility $\{V_t, t \geq 0\}$ that is implicitly determined by (cf. Duffie and Epstein 1992)

$$V_t = \mathbb{E}\Big[\int_t^\infty F(C_s, V_s)ds\Big|\mathcal{F}_t\Big], \quad \forall t \geq 0, \tag{2.1}$$

where the normalized aggregator $F(\cdot,\cdot)$ is

$$F(C,V) = \frac{\beta}{1 - 1/\kappa}\Big[C^{1-1/\kappa}\big((1-\gamma)V\big)^{(1/\kappa-\gamma)/(1-\gamma)} - (1-\gamma)V\Big],$$

$\beta \in (0,\infty)$ is the subjective discount rate, $\gamma \in (0,1) \cup (1,\infty)$ is the relative risk aversion level, and $\kappa \in (0,1) \cup (1,\infty)$ is the EIS. Note that (2.1) reduces to the standard CRRA utility case when $\gamma = 1/\kappa$.

Define the solvency region as:

$$\Omega := \{(x_t, y_t, k_t) \in \mathbb{R}^3 : x_t + y_t - \alpha(y_t - k_t) \geqslant 0, y_t \geqslant 0, k_t \geqslant 0\}.$$

Then, a consumption and investment strategy $\{C_s, L_s, M_s\}_{s\geq 0}$ is said to be admissible if: (i) $(x_t, y_t, k_t) \in \Omega$ for all $t \geq 0$; and (ii) the recursive utility V_t is uniquely determined by (2.1). Let $\mathscr{A}_i(x,y,k)$ be the set of all admissible strategies under initial condition $x_0 = x$, $y_0 = y$, $k_0 = k$, and $\iota_0 = i$. The investor aims to maximize his or her utility over all admissible strategies, namely:

$$\varphi^{(i)}(x,y,k) := \sup_{\{C_t,L_t,M_t\}_{t\geq 0}\in\mathscr{A}_i(x,y,k)} V_0, \tag{2.2}$$

where $\varphi^{(i)}(x,y,k)$ is known as the value function under regime $i \in \mathcal{I}$.

For notational simplicity, we denote the constant market parameters under regime $i \in \mathcal{I}$ by $r_i = r(i)$, $\mu_i = \mu(i)$, and $\sigma_i = \sigma(i)$. Later, we will see that the value function in a tax market is bounded from above by the value function in a tax-free market. Hence, we shall assume that the parameters r_i, μ_i, σ_i, γ, κ, and β satisfy a certain condition such that the value function in a tax-free market is finite.[9] For $\mathcal{I} = \{1\}$, the condition is

$$\beta > (1-1/\kappa)\Big[r_1 + \frac{(\mu_1-r_1)^2}{2\gamma\sigma_1^2}\Big]. \tag{2.3}$$

For $\mathcal{I} = \{1,2\}$, we refer readers to Theorem 6.2 in Xiao (2012).

2.2. **Properties of the Value Function.**

The value function has the following properties:

Proposition 1. *Let $\varphi^{(i)}(x,y,k)$ be the value function under regime $i \in \mathcal{I}$ defined in (2.2). Denote wealth after liquidation by $z = x + (1-\alpha)y + \alpha k$. Define*

$$\begin{aligned}\overline{K}_i &:= \kappa\Big[\beta - (1-1/\kappa)\Big(r_i + \frac{(\mu_i-r_i)^2}{2\gamma\sigma_i^2}\Big)\Big],\\ K_i &:= \kappa\Big[\beta - (1-1/\kappa)\Big(r_i + \frac{(\mu_i-r_i/(1-\alpha))^2}{2\gamma\sigma_i^2}\Big)\Big].\end{aligned}$$

(i) For any $k \geqslant y$ (i.e., under capital losses),

$$\varphi^{(i)}(x,y,k) = \varphi^{(i)}(z,0,0) = \varphi^{(i)}((1-\pi_i^*)z, \pi_i^* z, \pi_i^* z), \tag{2.4}$$

where π_i^* is the optimal fraction of wealth in stock after realizing capital losses under regime i.

(ii) $\varphi^{(i)}(x,y,k)$ has lower and upper bounds:

$$\frac{G_i^{1-\gamma}}{1-\gamma} z^{1-\gamma} \leqslant \varphi^{(i)}(x,y,k) \leqslant \frac{\overline{G}_i^{1-\gamma}}{1-\gamma} z^{1-\gamma}, \tag{2.5}$$

[9] We always assume $\mu_i \geq r/(1-\alpha)$, which ensures that a short sale is never optimal for the benchmark suboptimal strategy to be introduced soon [see (2.9)].

where G_i and $\overline{G}_i$ are the unique solution to:

$$\beta^\kappa G_i^{1-\kappa} - K_i + \frac{\kappa-1}{1-\gamma}\sum_{j\in\mathcal{I}} q_{ij}\frac{G_j^{1-\gamma}}{G_i^{1-\gamma}} = 0, \quad i\in\mathcal{I},$$

$$\beta^\kappa \overline{G}_i^{1-\kappa} - \overline{K}_i + \frac{\kappa-1}{1-\gamma}\sum_{j\in\mathcal{I}} q_{ij}\frac{\overline{G}_j^{1-\gamma}}{\overline{G}_i^{1-\gamma}} = 0, \quad i\in\mathcal{I},$$

respectively.

The proof is in Appendix A.

The proposition extends the results obtained by Ben Tahar, Soner, and Touzi (2010) to the regime-switching market with recursive utility. Part (i) implies that immediately realizing capital losses is optimal, because the investor would like to receive tax returns as early as possible to earn interest. Hence, when $k_t > y_t$, the wash sale

$$(x_t, y_t, k_t) \longrightarrow (z_t, 0, 0) \longrightarrow ((1-\pi_i^*)z_t, \pi_i^* z_t, \pi_i^* z_t) \tag{2.6}$$

is an optimal strategy.

2.2.1. *Certainty Equivalent Wealth Loss.*

Note that the upper bound in part (ii) of Proposition 1 is nothing but the value function in a tax-free market with the following optimal strategy (c.f. Xiao 2012):

$$\frac{y_t}{x_t+y_t} = \bar{\xi}_i^* =: \frac{\mu_i - r_i}{\gamma\sigma_i^2}, \qquad \frac{C_t}{x_t+y_t} = \beta^\kappa \overline{G}_i^{1-\kappa}, \tag{2.7}$$

where $\bar{\xi}_i^*$ is the well-known Merton line. This implies that the investor cannot take advantage of tax returns to do better than in a tax-free market. Then, we can define the *certainty equivalent wealth loss* (CEWL) $\overline{\Delta}^{(i)}$ incurred by capital gains tax in terms of the following equation:

$$\varphi^{(i)}(x,y,k) = \frac{\overline{G}_i^{1-\gamma}}{1-\gamma}[(1-\overline{\Delta}^{(i)})z]^{1-\gamma},$$

that is,

$$\overline{\Delta}^{(i)} \approx -\log(1-\overline{\Delta}^{(i)}) = -\frac{1}{1-\gamma}\log\left[\,(1-\gamma)\,z^{-(1-\gamma)}\varphi^{(i)}\right] + \log\overline{G}_i. \tag{2.8}$$

In Appendix G, we can see that the capital gains tax may significantly affect the CEWL.

(2.7) indicates that, in the absence of capital gains tax, the optimal investment strategy is myopic in the sense that the Merton line only depends on the parameters of the current regime. This follows from the fact that the risk of regime switching is unhedgeable, and portfolio rebalancing does not incur any costs at regime-switching time. Later, we will see that the presence of capital gains tax makes the optimal investment strategy no longer myopic.

2.2.2. *The Value of Deferring Capital Gains Realization.*

The lower bound in part (ii) of Proposition 1 is the value function associated with a sub-optimal strategy described below: for any $t > 0$ and total liquidated wealth z_t, one keeps liquidating the portfolio and maintains[10]:

$$\frac{(1-\alpha)y_t}{z_t} = \xi_i^* =: \frac{\mu_i - r_i/(1-\alpha)}{\gamma\sigma_i^2}, \qquad \frac{C_t}{z_t} = c_i^* =: \beta^\kappa G_i^{1-\kappa}, \tag{2.9}$$

where ξ_i^* is called the tax-adjusted Merton line. Hereafter, the Merton line is always referred to as ξ_i^*.

The sub-optimal strategy is optimal in a restricted set of admissible strategies in which the investor cannot defer realizing any capital gains. Because tax is paid only when capital gains are realized, the investor has an option and an incentive to defer capital gains taxation. We then define the value of this deferral option, denoted by $\Delta^{(i)}$, as follows:

$$\varphi^{(i)}(x,y,k) \equiv \frac{G_i^{1-\gamma}}{1-\gamma}[(1+\Delta^{(i)})z]^{1-\gamma},$$

that is,

$$\Delta^{(i)} \approx \log(1+\Delta^{(i)}) = \frac{1}{1-\gamma}\log\left[(1-\gamma)z^{-(1-\gamma)}\varphi^{(i)}\right] - \log G_i. \tag{2.10}$$

Later, we will see that (2.10) implies a transformation which plays a critical role in our asymptotic analysis.

[10]Strictly speaking, the strategy is not in the set of admissible strategies. However, we can always use a sequence of admissible strategies to approach the strategy.

The only reason for investors to defer capital gains taxation is to save interest. Hence, the deferral option is worthless as the interest rate vanishes.[11] This is consistent with our result that $\overline{K}_i = K_i$ and $\overline{G}_i = G_i$ when $r_1 = r_2 = 0$, from which one infers that the sub-optimal strategy (2.9) is indeed optimal.

2.3. **Optimal Strategy.**

2.3.1. *Consumption Strategy.* Using the dynamic programming principle, the value function $\varphi^{(i)}$ turns out to satisfy an HJB equation (see Appendix B), by which the optimal consumption proves to be

$$C_i^* = \Big(\frac{1}{\beta}\frac{\varphi_x^{(i)}}{((1-\gamma)\varphi^{(i)})^{(\kappa^{-1}-\gamma)/(1-\gamma)}}\Big)^{-\kappa}. \tag{2.11}$$

2.3.2. *Investment Strategy.* The recursive utility (2.1) is homogeneous, so the corresponding value function has a scaling invariance property: $\varphi^{(i)}(\rho x, \rho y, \rho k) = \rho^{1-\gamma}\varphi^{(i)}(x, y, k)$ for any $\rho > 0$. This allows us to investigate the optimal investment strategy in the $\xi - b$ plane, where

$$\xi = \frac{(1-\alpha)\, y}{x + (1-\alpha)\, y + \alpha k}, \ b = \frac{k}{y}. \tag{2.12}$$

Note that ξ is the tax-adjusted fraction of wealth in stock, and b is referred to as the basis-price ratio which represents the average tax basis per dollar of share price. Thanks to part (i) of Proposition 1, the wash sale is optimal when there are capital losses ($b > 1$). As a consequence, we are able to restrict our attention to the domain $0 \leqslant b \leqslant 1$. The combination of the risk aversion utility function and the homogeneity property implies that the sell region ($\mathbf{SR}_i$), buy region ($\mathbf{BR}_i$), and no-trade region ($\mathbf{NT}_i$) at regime i can be described as follows using two functions $\xi_i^+(b)$ and $\xi_i^-(b)$ for $b \in [0, 1]$:[12]

$$\begin{aligned} \mathbf{SR}_i &= \{(b,\xi) \mid 0 \leqslant b < 1, \xi \geqslant \xi_i^+(b)\}, \\ \mathbf{BR}_i &= \{(b,\xi) \mid 0 \leqslant b < 1, \xi \leqslant \xi_i^-(b)\}, \\ \mathbf{NT}_i &= \{(b,\xi) \mid 0 \leqslant b < 1, \xi_i^-(b) < \xi < \xi_i^+(b)\}. \end{aligned}$$

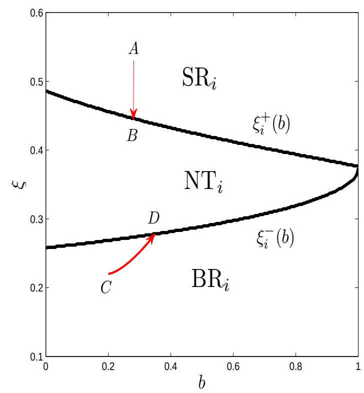
[11]The insight was first documented by Domar and Musgrave (1944).

[12]For a rigorous definition of these regions, see Appendix B.

These regions are depicted in Figure 1. The financial intuition is that, for a given basis-price ratio b, a relatively risk-averse investor should sell (buy) stock when the fraction of wealth in stock is sufficiently high (low). The existence of the $\mathbf{NT}_i$ implies that it may be optimal for the investor to defer capital gains taxation.

It is worth pointing out that a sale of stock does not change the basis-price ratio b (see the vertical line from A to B in Figure 1), but a purchase of stock does (see the curve from C to D in Figure 1).[13] Intuitively, when one sells stock, the average purchase price per remaining share is unchanged. However, when one buys stock and there are capital gains, the average purchase price per share increases.

Figure 1. Trading and No-trade Regions



We call $\xi_i^{\pm}(b))$ the optimal *sell boundary/buy boundary*. It is interesting to note[14]

$$\xi_i^+(1) = \xi_i^-(1).$$

This indicates that, after the wash sale, the fraction of wealth in stock should be re-balanced to[15] $\pi_i^* = \xi_i^{\pm}(1)/(1-\alpha)$, where π_i^*, being the same as the one in (2.6), can also be regarded as the *initial* optimal fraction of wealth in stock if the initial endowment is all in the money account. Our analysis shows that $\xi_i^{\pm}(1)$ is higher than

[13]The curve from C to D is determined by the characteristic line of the first-order equation $\xi w_\xi^{(i)} + (1-b)w_b^{(i)} = 0$ in $\mathbf{BR}_i$, as given in the Appendix.

[14]This is supported by numerical results.

[15]Recall that there is a factor $1-\alpha$ in the transformation (2.12).

the Merton line ξ_i^* (see Appendix C), which implies that the investor is inclined to be more aggressive when there is a benefit of tax deferral. The reason behind this is that in contrast to the immediate realization of capital losses, the investor has an option to defer the taxation of capital gains. This deferral option reduces risk and then motivates the investor to invest more in stock.

3. EIS, Risk Aversion, and Capital Gains Tax

This section is devoted to the impact of EIS, risk aversion, and other factors on optimal portfolio allocation and consumption decision in the presence of capital gains tax. Without loss of generality, we focus on the single-regime case, namely, $\mathcal{I} = \{1\}$. To simplify notation, we suppress the regime index i in this section.

Closed-form solutions are unavailable, in general, when capital gains tax is incurred, so we will employ asymptotic analysis. Since asymptotic analysis is rather technical, we will focus on the economic implications of analysis results and move technical arguments to the Appendix. We also carry out numerical analysis where the penalty method with finite difference discretization is utilized (cf. Forsyth and Vetzal 2002, Dai and Zhong 2010).[16] The following default parameter values are adopted: $\beta = 0.05$, $r = 0.03$, $\mu = 0.08$, $\sigma = 0.20$, $\alpha = 0.24$, $\gamma = 3$, and $\kappa = 1.43$.[17]

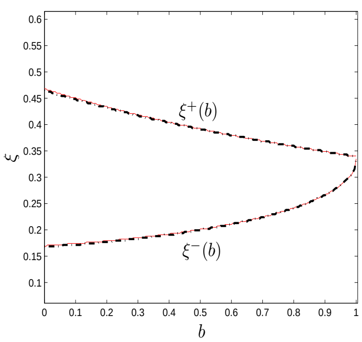
3.1. **EIS.**

It is well known that, in the absence of capital gains tax, the EIS does not affect optimal portfolio allocation. We find that this property is inherited in the presence of capital gains tax, as shown in Figure 2, where the optimal trading boundaries with different values of κ coincide with each other. This is also supported by our asymptotic analysis (D.2), where the approximating optimal trading boundaries are independent of the EIS.

[16] It should be emphasized that implementing the penalty method with finite difference discretization is absolutely non-trivial for the present problem because the resulting matrix is, in general, not an M-matrix. One needs to carefully choose the penalty parameter to guarantee convergence.

[17] There is no consensus on the magnitude of κ (see, e.g., Hall 2009, Wang, Wang, and Yang 2015). We always choose $\kappa \in (1, 2)$ which is consistent with the macro-finance literature.

Figure 2. The Independence of Trading Boundaries on EIS



The dashed-dotted lines and the solid lines correspond to the trading boundaries with $\kappa = 1.25$ and $\kappa = 1.43$, respectively. Default parameter values: $\beta = 0.05$, $r = 0.03$, $\mu = 0.08$, $\sigma = 0.2$, $\alpha = 0.24$, and $\gamma = 3$.

Asymptotic analysis (D.3) indicates that the optimal consumption decision is close to the suboptimal consumption decision $C_t/z_t = c^* \equiv K$, as given in (2.9), where the investor does not defer capital gains taxation.[18] In terms of the explicit expression of K, it is easy to verify

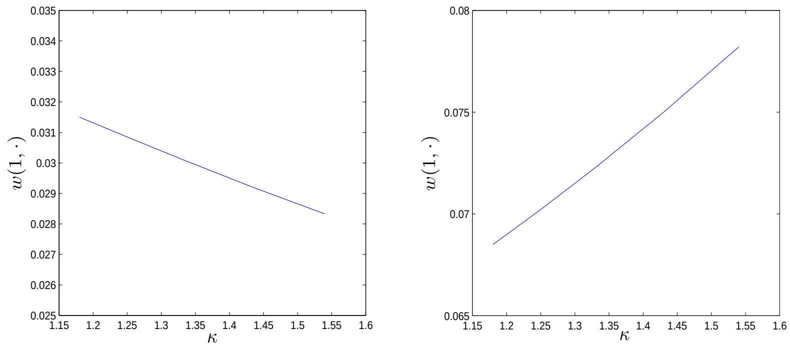
$$\frac{\partial K}{\partial \kappa} = \beta - r - \frac{(\mu - r/(1-\alpha))^2}{2\gamma\sigma^2}, \tag{3.1}$$

which indicates that the impact of the EIS on consumption depends on the relative magnitude of the subjective discount rate β and the risk-free rate r, as well as other parameters. In particular, when the risk-free rate or the stock return rate is sufficiently high compared with the subjective discount rate, optimal consumption decreases with the EIS. Intuitively, an investor with a higher EIS is more willing to substitute consumption intertemporally, and thus consumes less for such a relatively large risk-free rate or stock return.

Our asymptotic analysis together with numerical results shows that the value of deferral is inversely proportional to K [see, e.g., (D.1)]. We then infer that ceteris paribus, the lower the consumption, the larger the value of deferral. Intuitively, this is because consuming less results in investing more and then increases the incentive

[18]In contrast, Roger (2004) finds that, for the transaction cost problem, the optimal consumption strategy is close to the one without transaction costs.

FIGURE 3. The Value of Deferral against the EIS



(a) $\beta = 0.05$ (b) $\beta = 0.03$

The value of deferral $w(1,\cdot)$ is obtained by the penalty method. Default parameter values: $\mu = 0.08$, $\alpha = 0.24$, $r = 0.03$, $\sigma = 0.2$, and $\gamma = 3$.

of tax deferral. Combining with (3.1), we deduce that the EIS influences the value of deferral through consumption K. In particular:

(i) if $\beta - r - \frac{(\mu - r/(1-\alpha))^2}{2\gamma\sigma^2} > 0$, then the value of deferral declines as the EIS grows;

(ii) if $\beta - r - \frac{(\mu - r/(1-\alpha))^2}{2\gamma\sigma^2} < 0$, then the value of deferral increases as the EIS grows.

Figure 3 reports different monotonicity of the value of the deferral with respect to the EIS, corresponding to case (i) (Figure 3a) and case (ii) (Figure 3b), respectively. Observe that, in Figure 3b where $r = \beta$, the value of deferral is increased up to 15% as the EIS changes from 1.2 to 1.5. This is not surprising because we have argued that an investor with a higher EIS consumes less for such a relatively large r, which leads to more investment in stock and then raises the incentive of tax deferral.

3.2. **Risk Aversion and Other Parameter Values.**

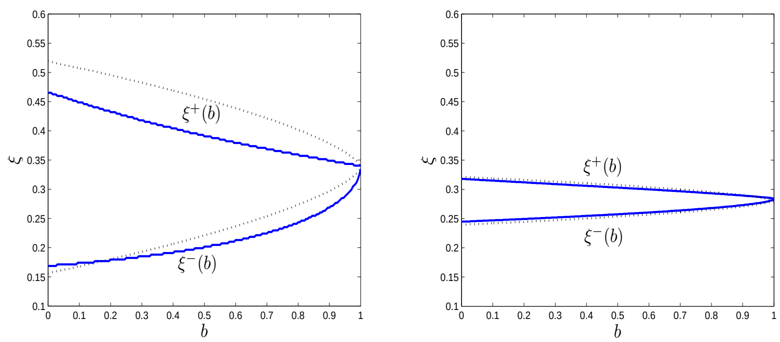
Using asymptotic analysis, we introduce a perturbation parameter

$$\varepsilon := \sqrt{\frac{2\alpha r\xi^*}{(1-\alpha)\gamma\sigma^2}} = \frac{\sqrt{2\alpha r(\mu - r/(1-\alpha))/(1-\alpha)}}{\gamma\sigma^2} \tag{3.2}$$

to integrate the effects of various factors on portfolio decisions. We find that the **NT** shrinks as ε declines [see (D.2)]. This is also verified by Figure 4, where the **NT** in Figure 4a with larger ε is apparently wider than that in Figure 4b with smaller

ε.[19] By the definition of ε, we then infer that, reducing the interest rate (or tax rate, risk premium) or raising the relatively risk aversion level of the investor (or stock volatility) can increase the frequency of trading. The intuition behind this is clear: if r or α shrinks, then the incentive of tax deferral declines; and if γ or σ is higher, or $\mu - r$ is lower, then the investor tends to invest less in stock and thus to incur a smaller amount of tax. All of these lead the investor to trade more frequently.

FIGURE 4. The Optimal Trading Boundaries



(a) $r = 0.03$, $\sigma = 0.2$, $\varepsilon = 0.23$ (b) $r = 0.0025$, $\sigma = 0.3$, $\varepsilon = 0.04$

The solid lines and the dotted lines represent the trading boundaries computed from the penalty method and approximation (D.2), respectively. Default parameter values: $\beta = 0.05$, $\mu = 0.08$, $\alpha = 0.24$, $\gamma = 3$, and $\kappa = 1.43$.

The approximating buy and sell boundaries, as given in (D.2), are symmetric with respect to the Merton line and are quite close to the numerical results for Figure 4b. For Figure 4a, the approximation is acceptable, but not so ideal. This is because (D.2) derived from asymptotic expansions is accurate only for small ε.

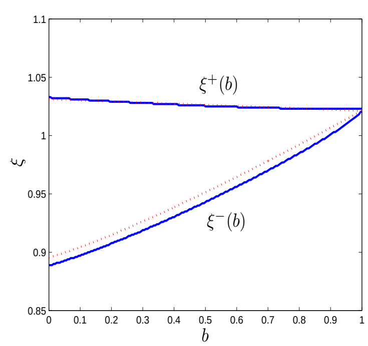
Figure 4a apparently displays the asymmetry of the trading boundaries with respect to the Merton line, which is typical when ε is large. This can be explained as follows. The investor is inclined to take no transaction to defer taxation of capital gains as long as risk exposure is within a certain risk tolerance, namely, within the no-trading region. Hence, the distance between the two trading boundaries determines the expected tax deferral time and then significantly affects the tax deferral option.

[19]In Figure 4, we only vary the values of r and σ. The result remains valid when other parameter values in ε change.

Under the assumption of positive stock risk premium, deferring realization of capital gains will potentially increase the investor's risk exposure. As such, to keep the average risk exposure close to the Merton line, the investor should lower the buy boundary more than raise the sell boundary, which gives rise to the asymmetry of the trading boundaries, as shown in Figure 4a. However, for small ε, the benefit of tax deferral is tiny and the distance between the buy and sell boundaries becomes very narrow. In this case, the trading boundaries are reached primarily owing to the random walk of stock price, which leads to symmetric trading boundaries with respect to the Merton line, as exhibited in Figure 4b.

It is worth noting that, when $\xi^* \approx 1$, the asymmetry becomes more distinct. In fact, in this scenario, risk exposure around the Merton line is insensitive to the random walk of stock price,[20] which motivates the investor to sell stock when risk exposure is very close to the Merton line. Hence, the sell boundary almost matches the Merton line, whereas the buy boundary still deviates substantially from the Merton line in order to defer realization of capital gains. This phenomena is numerically verified by Figure 5 and supported by an asymptotic analysis at $\xi^* \approx 1$ (see Appendix F).

FIGURE 5. The Optimal Trading Boundaries when $\xi_* \approx 1$



The solid line and the dotted line represent the optimal trading boundaries computed from the penalty method and approximation (F.2), respectively. Parameter values: $\beta = 0.05$, $r = 0.0025$, $\mu = 0.085$, $\sigma = 0.2$, $\alpha = 0.24$, $\gamma = 2$, $\kappa = 1.43$, and $\xi_* = 1.021$.

[20]Using Ito's lemma, it is easy to see that the standard deviation of $d\xi_t/\xi_t$ is $\sigma(1 - \xi_t)$.

4. Regime Switching and Capital Gains Tax

Now, we turn to the regime-switching case where $\mathcal{I} = \{1, 2\}$. We rewrite the generator as

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix},$$

where $\lambda = (\lambda_1, \lambda_2)$ are the transition intensities between two regimes. In particular, if $\lambda = \mathbf{0} := (0, 0)$, then the two regimes are independent of each other.

4.1. **The Value of Deferral.**

To emphasize the dependence on λ, we denote by $\Delta_\lambda^{(i)}$ the value of deferral at regime i with $\lambda = (\lambda_1, \lambda_2)$. The following proposition presents a simple linear rule to evaluate $(\Delta_\lambda^{(1)}, \Delta_\lambda^{(2)})$ in terms of $(\Delta_{\mathbf{0}}^{(1)}, \Delta_{\mathbf{0}}^{(2)})$.

Proposition 2. *Let $\Delta_\lambda^{(i)}$ be the value of deferral at regime $i = 1, 2$ with $\lambda = (\lambda_1, \lambda_2)$. Then*

$$\begin{pmatrix} \Delta_\lambda^{(1)} \\ \Delta_\lambda^{(2)} \end{pmatrix} \approx \frac{1}{M_{11} + M_{22} - 1} \begin{pmatrix} M_{22} & M_{11} - 1 \\ M_{22} - 1 & M_{11} \end{pmatrix} \begin{pmatrix} \frac{c^*_{1,\mathbf{0}}}{c^*_{1,\lambda}} \Delta_{\mathbf{0}}^{(1)} \\ \frac{c^*_{2,\mathbf{0}}}{c^*_{2,\lambda}} \Delta_{\mathbf{0}}^{(2)} \end{pmatrix}, \tag{4.1}$$

where $c^*_{i,\lambda}$ is as given in (2.9) with $\lambda = (\lambda_1, \lambda_2)$, and

$$M_{11} = 1 + \frac{1-\gamma}{\kappa - 1}\Big(\frac{K_1}{c^*_{1,\lambda}} - 1\Big) + \frac{\lambda_1}{c^*_{1,\lambda}}, \tag{4.2}$$

$$M_{22} = 1 + \frac{1-\gamma}{\kappa - 1}\Big(\frac{K_2}{c^*_{2,\lambda}} - 1\Big) + \frac{\lambda_2}{c^*_{2,\lambda}}. \tag{4.3}$$

The proof is in Appendix H.

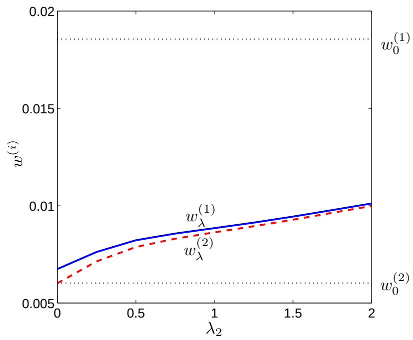
Numerical results show that the optimal consumption rates $c^*_{i,\lambda}, i = 1, 2$ are insensitive to the value of λ. Hence (4.1) can be rewritten approximately as

$$\Delta_\lambda^{(1)} \approx \frac{M_{22}}{M_{11} + M_{22} - 1} \Delta_{\mathbf{0}}^{(1)} + \frac{M_{11} - 1}{M_{11} + M_{22} - 1} \Delta_{\mathbf{0}}^{(2)}, \tag{4.4}$$

$$\Delta_\lambda^{(2)} \approx \frac{M_{22} - 1}{M_{11} + M_{22} - 1} \Delta_{\mathbf{0}}^{(1)} + \frac{M_{11}}{M_{11} + M_{22} - 1} \Delta_{\mathbf{0}}^{(2)}. \tag{4.5}$$

For reasonable parameter values, we have $M_{11} > 1$ and $M_{22} > 1$. As such, (4.4) and (4.5) imply that $\Delta_\lambda^{(i)}$ is approximately a convex combination of $\Delta_{\mathbf{0}}^{(1)}$ and $\Delta_{\mathbf{0}}^{(2)}$, indicating that a smoothing effect exists for the value of deferral across regimes.

Figure 6. The Value of Deferral with Regime Switching against λ_2



The solid line, dashed line, and two dotted lines are, respectively, the numerical values of $\Delta_\lambda^{(1)}(1,\cdot)$, $\Delta_\lambda^{(2)}(1,\cdot)$, $\Delta_{\mathbf{0}}^{(1)}(1,\cdot)$, and $\Delta_{\mathbf{0}}^{(2)}(1,\cdot)$ obtained from the penalty method. Parameter values: $\lambda_1 = 0.2353$, $\alpha = 0.2$, $r_1 = r_2 = 0.02$, $\mu_1 = \mu_2 = 0.06$, $\kappa = 1.43$, $\beta = 0.05$, $\gamma = 3$, $\sigma_1 = 0.15$, and $\sigma_2 = 0.25$.

To verify the cross-regime smoothing effect predicted by Proposition 2, we consider a regime-switching market, where the bull regime ($i = 1$) is less volatile ($\sigma_1 = 0.15$) while the bear regime ($i = 2$) is more volatile ($\sigma_2 = 0.25$). Fixing $\lambda_1 = 0.2353$, Figure 6 reports the value of deferral $\Delta_\lambda^{(i)}$ against λ_2, the transition intensity from the bear regime to the bull regime. As λ_2 grows, M_{22} becomes larger and the weight of $\Delta_{\mathbf{0}}^{(1)}$ in (4.4) and (4.5) increases, therefore both $\Delta_\lambda^{(1)}$ and $\Delta_\lambda^{(2)}$ move towards $\Delta_{\mathbf{0}}^{(1)}$. Since $\Delta_{\mathbf{0}}^{(1)}$ is greater than $\Delta_{\mathbf{0}}^{(2)}$, we then infer that $\Delta_\lambda^{(1)}$ and $\Delta_\lambda^{(2)}$ are increasing in λ_2, which is verified by Figure 6. Moreover, we observe that the cross-regime difference between $\Delta_\lambda^{(1)}$ and $\Delta_\lambda^{(2)}$ declines as λ_2 grows, which is also consistent with the prediction of (4.4) and (4.5):

$$\Delta_\lambda^{(1)} - \Delta_\lambda^{(2)} \approx \frac{1}{M_{11} + M_{22} - 1}(\Delta_{\mathbf{0}}^{(1)} - \Delta_{\mathbf{0}}^{(2)}),$$

where the denominator increases with M_{22} as λ_2 grows.

4.2. **Trading Boundaries.**

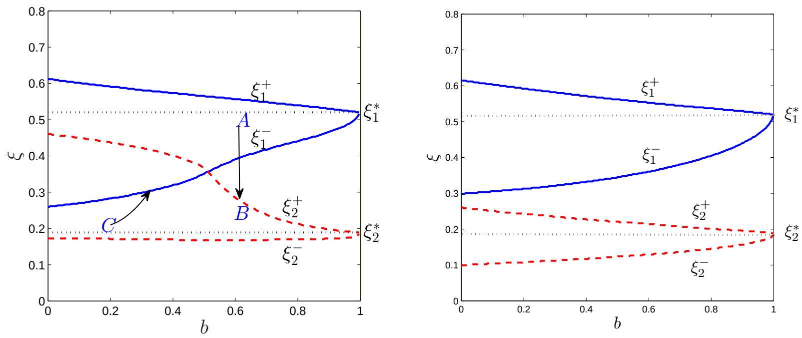
Now, we investigate how the presence of regime switching affects optimal portfolio allocation. In Figure 7a, we plot the optimal trading boundaries when the market may switch between a bull regime and a bear regime (Figure 7a with $\lambda_1 = 0.2353$

and $\lambda_2 = 1.7391$).[21] We find that, in the regime-switching market, an investor may trade or stop trading purely because of a change in regime. For example, at Point A in Figure 7a, it is optimal for the investor to take no action provided that the current market regime is bull, because Point A is inside of the no-trade region of the bull regime. However, if the market switches from a bull regime to a bear regime at the moment, the investor should immediately sell to reach the sell boundary of the bear regime (i.e., Point B), because Point A is in the sell region of the bear regime. Similarly, at Point C in Figure 7a, the investor would buy to reach the buy boundary of the bull regime provided that the current regime is bull, because Point C is in the buy region of the bull regime. If the market switches from bull to bear at the moment, the investor will stop trading because Point C is inside of the no-trade region of the bear regime.

The Merton line in Figure 7a is $\xi_1^* = 0.519$ in the bull regime and $\xi_2^* = 0.187$ in the bear regime, respectively. It can be seen that $\xi_i^{\pm}(1)$ is very close to ξ_i^* and seems irrelevant to the transition intensities. This *myopic* strategy at $b = 1$ follows from the fact that portfolio rebalancing does not incur any taxes when $b = 1$. However, when b is away from 1, the optimal investment strategy with regime switching is no longer myopic and does exhibit a cross-regime smoothing effect. In fact, for the purpose of comparison, we plot in Figure 7b the optimal trading boundaries when the market does not switch (namely, $\lambda_1 = \lambda_2 = 0$). As we can observe from Figure 7a-b, the two trading boundaries in the bear market with regime switching are significantly higher than the corresponding trading boundaries without regime switching. The reasoning is the following. When the market switches from a bull regime to a bear regime, the investor would have to sell stock and then pay capital gains tax. In response, the investor tends to raise stock investment in the bear regime so as to reduce potential tax payments upon regime switching. Moreover, the larger the capital gains (i.e., the

[21] The duration of a bull regime is usually significantly longer than that of a bear regime, so $\lambda_1 < \lambda_2$. The parameter values $(\lambda_1, \lambda_2) = (0.2353, 1.7391)$ are used in Jang et al. (2007).

Figure 7. Trading Boundaries with Two Regimes



(a) $\lambda_1 = 0.2353$, $\lambda_2 = 1.7391$ (b) $\lambda_1 = \lambda_2 = 0$

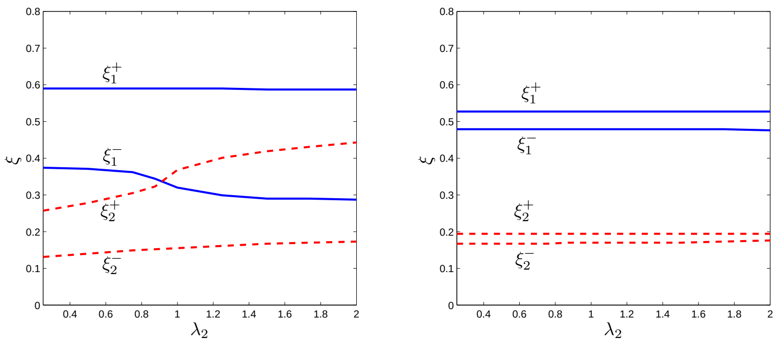
The solid lines and dashed lines represent the trading boundaries against b in the bull regime ($i = 1$) and in the bear regime ($i = 2$), respectively. The dotted lines are the Merton lines. Other parameters: $r_1 = r_2 = 0.02$, $\mu_1 = \mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $\gamma = 3$, $\alpha = 0.2$, $\beta = 0.05$, and $\kappa = 1.43$.

smaller b), the more distinct the cross-regime smoothing effect, because more capital gains tax may be incurred.

We also observe that, in the bull regime, the presence of regime switching only slightly alters the buy boundary and has little impact on the sell boundary, which implies that regime switching has a greater impact on the optimal strategy in the bear regime than in the bull regime. This may be due to two reasons: i) the bull regime has a longer duration and thus dominates; and ii) when the market switches from the bear regime to the bull regime, the investor would have to rebalance the portfolio to the buy boundary in the bull regime, which motivates the investor to slightly lower the buy boundary in the bull regime so as to mitigate the impact of potential sale caused by future regime switching, but this is less relevant to the sell boundary in the bull regime.

Figure 8 provides more information about how the optimal trading boundaries at different b depend on transition intensities, where λ_1 is fixed and λ_2 varies. It can be seen that the trading boundaries at $b = 0.95$ seem almost irrelevant to the change of λ_2, which is consistent with the scenario of $b = 1$ observed in Figure 7. However, for $b = 0.25$, the trading boundaries (excluding the sell boundary in the bull regime)

FIGURE 8. Trading Boundaries against λ_2 for Different b



(a) $b = 0.25$ (b) $b = 0.95$

The solid lines and dashed lines represent the trading boundaries against λ_2 in the bull regime ($i = 1$) and in the bear regime ($i = 2$), respectively. Parameters: $r_1 = r_2 = 0.02$, $\mu_1 = \mu_2 = 0.06$, $\sigma_1 = 0.15$, $\sigma_2 = 0.25$, $\lambda_1 = 0.2353$, $\alpha = 0.2$, $\gamma = 3$, $\beta = 0.05$, and $\kappa = 1.43$.

alter dramatically, and the cross-regime smoothing effect becomes distinct as λ_2 grows. This is not surprising because regime switching may occur more frequently for large λ_2, and with smaller b, the potential tax payment is larger, which makes the trading boundaries more sensitive to the change of λ_2.

5. CONCLUSION

Capital gains taxation has important implications for investors' portfolio choice decisions. In the presence of capital gains tax, investors have an option and an incentive to defer capital gains taxation. To study this deferral option and the corresponding optimal portfolio choice decisions, we develop a continuous-time portfolio selection model with capital gains tax, Epstein-Zin recursive utility, and regime switching. The optimal investment strategy turns out to be determined by the optimal buy and sell boundaries. Although closed form solutions are unavailable in general, the model allows us to employ asymptotic analysis to investigate the effect of various parameters on the deferral option and the optimal strategy. We also conduct an extensive numerical analysis to justify our results.

We have many interesting findings. First, various factors, such as risk aversion, risk-free rate, tax rate, and volatility, dramatically affect the optimal investment strategy. In contrast, the EIS has little impact on the optimal investment strategy, even though it does affect the tax deferral option through consumption. Second, the initial tax-adjusted fraction of wealth in stock is higher than the Merton line, because the deferral option reduces risk and then motivates investors to invest more in stock. Third, the buy boundary generally deviates more from the Merton line than the sell boundary does, provided that the stock risk premium is positive. Fourth, the optimal consumption strategy can be approximated by an explicit suboptimal consumption strategy without deferring capital gains taxation. Last, in a bull-bear switching market, investors may trade or stop trading purely because of a change in regime, and there is a distinct cross-regime effect on the deferral option and the optimal strategy. In particular, the optimal investment strategy in one regime is affected by the investment opportunity in the other regime, and the cross-regime effect becomes more distinct when there are large capital gains. Moreover, given reasonable parameter values, regime switching has a greater impact on the optimal strategy in a bear regime than in a bull regime, because a bull regime usually has a significantly longer duration.

Appendix A. Proof of Proposition 1

Part (i) is a direct extension of Proposition 3.5 in Ben Tahar, Soner, and Touzi (2010) to the regime-switching market with recursive utility. Indeed, for any admissible strategy (C, L, M) of (2.2) where $k_\tau > y_\tau$ for some finite stopping time τ, we can construct a new strategy $(\tilde{C}, \tilde{L}, \tilde{M})$ such that $\tilde{C} > C$ and the whole portfolio is liquidated at time τ. By the monotonicity in consumption (e.g., Xiao 2012, Corollary 3.7), we obtain the desired result.

Part (ii) extends Propositions 4.1 and 4.2 in Ben Tahar, Soner, and Touzi (2010) using the same argument. First, we remark that, given any admissible strategy (C, L, M) of (2.2), $(C, (1-\alpha)L, M)$ is an admissible strategy for the corresponding tax-free model with initial wealth $z = x + (1-\alpha)y + \alpha k$. Hence, the upper bound in (2.5) holds. Second, we note that the lower bound in (2.5) is the value function of a tax-free model with initial wealth z and tax-deflated risky asset price S_t^α defined as

$$dS_t^\alpha = S_t^\alpha[\mu^\alpha(\iota_t)dt + \sigma^\alpha(\iota_t)dW_t],$$

where $\mu^\alpha(\iota_t) = (1-\alpha)\mu(\iota_t)$ and $\sigma^\alpha(\iota_t) = (1-\alpha)\sigma(\iota_t)$. Denote by z_t^*, $C_t^* = c_t^* z_t^*$, and ξ_t^* the corresponding optimal wealth, consumption stream, and proportion of wealth in the risky asset, respectively. It follows that

$$dz_t^* = [r(\iota_t)z_t^* - c_t^* z_t^*]dt + \xi_t^* z_t^* \big[\frac{dS_t^\alpha}{S_t^\alpha} - r(\iota_t)dt\big].$$

Using the argument in Section B.1 of Ben Tahar, Soner, and Touzi (2010), we can construct a sequence of admissible strategies (C^n, L^n, M^n) such that C_t^n is arbitrarily close to C_t^*. Indeed, consider the liquidation wealth process $z_t^n := x_t^n + (1-\alpha)y_t^n + \alpha_t k_t^n$. The dynamics of z_t^n are given by

$$dz_t^n = [r(\iota_t)z_t^n - C_t^n]dt + y_t^n\big[\frac{dS_t^\alpha}{S_t^\alpha} - r(\iota_t)dt\big] + r(\iota_t)\alpha y_t\big(1 - \frac{y_t^n}{k_t^n}\big)dt,$$

The approximation is then achieved by keeping y_t^n/k_t^n close to 1, y_t^n/z_t^n close to ξ_t^*, and $C_t^n = c_t^* z_t^n$. The desired result then follows by assuming a suitable stability property of (2.1) on the consumption process C_t^n.

Appendix B. HJB Equation

The value function $\varphi^{(i)}$ turns out to satisfy the following HJB equation:[22]

$$\max\left\{\mathcal{A}_i\varphi^{(i)} + \sum_{j\in\mathcal{I}} q_{ij}\varphi^{(j)},\ \mathcal{T}_i\varphi^{(i)},\ \mathcal{P}_i\varphi^{(i)}\right\} = 0 \ \text{ in } \ \Omega, \quad \forall i \in \mathcal{I}, \tag{B.1}$$

where

$$\begin{aligned}
\mathcal{A}_i\varphi &:= \frac{1}{2}\sigma_i^2 y^2\varphi_{yy} + \mu_i y\varphi_y + r_i x\varphi_x + F^*(\varphi_x, \varphi),\\
\mathcal{T}_i\varphi &:= [(1-\alpha)y + \alpha k]\varphi_x - y\varphi_y - k\varphi_k,\\
\mathcal{P}_i\varphi &:= -\varphi_x + \varphi_y + \varphi_k,
\end{aligned}$$

and $F^*(q,\varphi) := \sup_{C>0}(F(C,\varphi) - Cq)$ for any $q > 0$.

We make the following dimension reduction:[23]

$$\varphi^{(i)}(x,y,k) = (x + (1-\alpha)\,y + \alpha k)^{1-\gamma}\,\phi^{(i)}(b,\xi), \tag{B.2}$$

where ξ and b are as given in (2.12). The combination of (2.10) and (B.2) motivates us to make a further transformation[24]:

$$w^{(i)} = \frac{1}{1-\gamma}\log[(1-\gamma)\phi^{(i)}] - \log G_i. \tag{B.3}$$

We will identify $w^{(i)}$ with the value of deferral $\Delta^{(i)}$ as defined in (2.10). It is not hard to verify that $w^{(i)}$ satisfies

$$\begin{aligned}
\max\Bigg\{\frac{\sigma_i^2}{2}(\mathscr{L}_i w^{(i)} + f_i) + \frac{1}{1-\gamma}\sum_{j\in\mathcal{I}} q_{ij}\frac{G_j^{1-\gamma}}{G_i^{1-\gamma}}\left(e^{(1-\gamma)(w^{(j)}-w^{(i)})} - 1\right),&\\
-\,w_\xi^{(i)},\ \xi w_\xi^{(i)} + (1-b)w_b^{(i)}\Bigg\} = 0&
\end{aligned} \tag{B.4}$$

[22]Ben Tahar, Soner, and Touzi (2010) and Bian, Chen, and Dai (2015) show rigorously that the value function is a viscosity solution of the HJB equation in the single regime case with CRRA utility.

[23]Following the scaling invariance property, one should use $\hat{\xi} = y/(x + (1-\alpha)y + \alpha k)$ and $\hat{b} = k/(x + (1-\alpha)y + \alpha k)$ as state variables. To derive (B.2), we can further make a transformation: $\xi = (1-\alpha)\hat{\xi}$ and $b = \hat{b}/\hat{\xi}$.

[24]A similar transformation is used in Dai and Yi (2009) for the Merton problem with transaction costs.

in $Q := \{(b,\xi) \mid b \geqslant 0,\, \xi \geqslant 0\,\}$, where

$$
\begin{aligned}
\mathscr{L}_i w &= b^2 w_{bb} - 2b\xi(1-\xi)w_{b\xi} + \xi^2(1-\xi)^2 w_{\xi\xi} \\
&+ \Big[2\big(1-\frac{\mu_i}{\sigma_i^2}\big) - (1-\gamma)\big(2\xi + \xi(1-\xi)w_\xi - bw_b\big)\Big] bw_b \\
&- \Big[\big(2(\xi-\xi_i^*) - (1-\gamma)(2\xi + \xi(1-\xi)w_\xi - bw_b)\big)(1-\xi) - \frac{2r_i\alpha[1+(b-1)\xi]}{(1-\alpha)\sigma_i^2}\Big]\xi w_\xi \\
&+ \frac{2c_i^*}{\sigma_i^2}\big(U^*(e^w(1-\xi w_\xi),\kappa) - U^*(1,\kappa)\big), \\
U^*(q,\kappa) &= -\frac{1}{1-\kappa}q^{1-\kappa}, \qquad f_i(b,\xi) = \gamma\Big[\frac{2r_i\alpha\xi}{(1-\alpha)\gamma\sigma_i^2}(1-b) - (\xi-\xi_i^*)^2\Big].
\end{aligned}
\tag{B.5}
$$

Note that the term $f_i(b,\xi)$ in (B.5) has two components that are restrained by each other: the former corresponds to the benefit of tax deferral, while the latter represents the loss due to deviation from the Merton line. The optimal portfolio allocation results from a delicate trade-off between maximizing the former and reducing the latter. If $r_i = 0,\ \forall i \in \mathcal{I}$, then the former disappears and no constraint is imposed on reducing the latter. Hence, we again find that adopting the strategy $\xi \equiv \xi_i^*$ is optimal when the interest rate vanishes.

In terms of the HJB equation, we define the sell region $\mathbf{SR}_i$, the buy region $\mathbf{BR}_i$, and the no-trade region $\mathbf{NT}_i$ by:

$$
\begin{aligned}
\mathbf{SR}_i &:= \{(b,\xi) \mid 0 \leqslant b < 1,\ w_\xi^{(i)} = 0\}, \\
\mathbf{BR}_i &:= \{(b,\xi) \mid 0 \leqslant b < 1,\ \xi w_\xi^{(i)} + (1-b)w_b^{(i)} = 0\}, \\
\mathbf{NT}_i &:= \{(b,\xi) \mid 0 \leqslant b < 1,\ \xi w_\xi^{(i)} + (1-b)w_b^{(i)} < 0 < w_\xi^{(i)}\}.
\end{aligned}
$$

Numerical results and financial intuitions show that these regions can be described by trading boundaries $\xi^+(b)$ and $\xi^-(b)$ as given in subsection 2.3.

Appendix C. $\xi^\pm(1) =?$

Without loss of generality, we suppress the regime index i from now to Appendix G because we are concerned with the single regime case. It has been pointed out earlier that $\xi^{\pm}(\cdot) \equiv \xi^{\pm}(1) = \xi^*$ for $\alpha r = 0$. In general, we have

$$\xi^* \leqslant \xi^{\pm}(1) \leqslant \xi^* + O(\varepsilon^{8/3}) \ \text{ if } \alpha r \neq 0, \tag{C.1}$$

where ε is as given in (3.2).

Proof of (C.1): Let us first rewrite $\mathscr{L} w$ as follows:

$$\begin{aligned}
\mathscr{L} w &= b^2 w_{bb} - 2b\xi(1-\xi)w_{b\xi} + \xi^2(1-\xi)^2 w_{\xi\xi} + cbw_b + d\xi w_\xi - R_1 w, \\
c &= 2\Big(1 - \frac{\mu}{\sigma^2}\Big) - (1-\gamma)\Big(2\xi + \xi(1-\xi)w_\xi - bw_b\Big), \\
d &= -\Big[2(\xi - \xi^*) - (1-\gamma)\Big(2\xi + \xi(1-\xi)w_\xi - bw_b\Big)\Big](1-\xi) + \frac{2r\alpha[1+(b-1)\xi]}{(1-\alpha)\sigma^2} + R_2, \\
R_1 &= -\hat{K}\, \frac{U^*(q,\kappa) - U^*(1,\kappa)}{q-1}\Big|_{q=e^w(1-\xi w_\xi)} \frac{e^w - 1}{w} = \hat{K} + O(|w| + |\xi w_\xi|), \\
R_2 &= -\hat{K}\, \frac{U^*(q,\kappa) - U^*(1,\kappa)}{q-1}\Big|_{q=e^w(1-\xi w_\xi)} e^w \quad = \hat{K} + O(|w| + |\xi w_\xi|),
\end{aligned}$$

where $\hat{K} := 2c^*/\sigma^2$, $O(|w| + |\xi w_\xi|)$ represents the same order of $|w| + |\xi w_\xi|$, and we use $U_q^*(1,\kappa) = -1$ when evaluating R_1 and R_2. We emphasize that the operator $\mathscr{L}$ can be treated as if it were linear. Let us first derive free boundary conditions at $\xi^{\pm}(b)$.

C.1. Free Boundary Conditions on $\xi^{\pm}(b)$.

In **BR**, $\xi w_\xi + (1-b)w_b = 0$. Then, there exists a function h such that $w(b,\xi) = h([b-1]\xi)$ for every $(b,\xi) \in$ **BR**,[25] and $0 \geqslant \mathscr{L} w + f = \mathscr{L} h + f$ in **BR** where

$$\mathscr{L} h = (1+\eta)^2 \xi^2 h''(\eta) + [2ab + cb + d(b-1)]\xi h'(\eta) - R_1 h(\eta)\Big|_{\eta=\xi[b-1]}.$$

Next, set $u(b,\xi) = w(b,\xi) - h([b-1]\xi)$. Then $u \leqslant 0$ in the whole space since $\xi u_\xi + (1-b)u_b \leqslant 0$. Hence, u attains a local maximum at $\xi = \xi^-(b)$. This implies that $\mathscr{L} u \leqslant 0$ at $\xi = \xi^-$ (in a certain weak sense). Consequently, at $(b, \xi^-(b)+0)$,

$$0 = \mathscr{L} w + f = \mathscr{L} u + \mathscr{L} h + f \leqslant \mathscr{L} h + f.$$

[25] We assume that $(b-1)\xi^-(b)$ is monotonic in $b \in [0,1]$.

Thus, $\mathscr{L}h + f = 0$ and $\mathscr{L}u = 0$ at $\xi = \xi^-(b)$. Therefore,

$$w(b,\xi) = h([b-1]\xi), \quad \mathscr{L}h + f \leqslant 0 \qquad \text{for} \quad \xi \in [0, \xi^-(b)],$$
$$\xi w_\xi + (1-b)w_b = 0, \quad \mathscr{L}w = \mathscr{L}h, \quad \mathscr{L}h + f = 0 \qquad \text{on} \ \ \xi = \xi^-(b).$$

In **SR**, $w_\xi \equiv 0$, so there exists a function g such that $w(b,\xi) = g(b)$ and

$$0 \geqslant \mathscr{L}w + f = \mathscr{L}g + f \qquad \forall\, \xi > \xi^+(b).$$

One can check that $\mathscr{L}g$ is a linear function of ξ:

$$\mathscr{L}g \;=\; b^2 g''(b) + \Big[2\Big(1 - \frac{\mu}{\sigma^2}\Big) + (1-\gamma)\Big(bg'(b) - 2\xi\Big)\Big] bg'(b) - R_1 g(b).$$

In a similar way, we can show

$$w(b,\xi) = g(b), \quad \mathscr{L}g + f \leqslant 0 \qquad \text{for} \ \ \xi \geqslant \xi^+(b),$$
$$w_\xi = 0, \quad \mathscr{L}w = \mathscr{L}g, \quad \mathscr{L}g + f = 0 \qquad \text{on} \ \ \xi = \xi^+(b).$$

C.2. **Estimate of** $\xi^\pm(1)$**.**

Due to (2.4), $w(1,\xi) \equiv$ *constant*. Combining with C^1 continuity of w, we have

$$w_b(1,\xi) = 0, \quad w_\xi(1,\xi) = 0, \quad g'(1) = 0, \qquad h'(0) = 0.$$

Now, sending $b \nearrow 1$ in the equation $(\mathscr{L}g + f)(b, \xi^+(b)) = 0$ and using $g'(1) = 0$ we obtain

$$g''(1-) - R_1 g(1) - \gamma(\xi^+(1) - \xi_*)^2 = 0.$$

Also, in $\mathscr{L}g + f \leqslant 0$ with fixed $\xi > \xi^+(b)$, sending $b \nearrow 1$ we obtain

$$g''(1-) - R_1 g(1) - \gamma(\xi - \xi_*)^2 \leqslant 0.$$

Comparing the equation and the inequality, we derive that $(\xi^+(1) - \xi_*)^2 \leqslant (\xi - \xi_*)^2$ for every $\xi > \xi^+(1)$, which implies the left-hand side inequality.

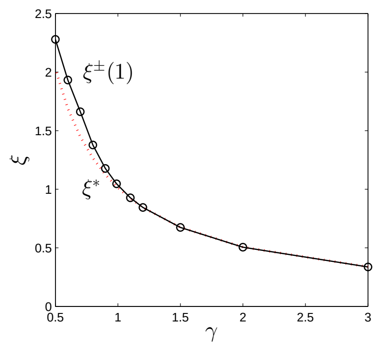
In a similar manner, using $\mathscr{L}h + f \leqslant 0$ for $\xi < \xi^-(b)$ and $\mathscr{L}h + f = 0$ for $\xi = \xi^-(b), b \in [0,1)$ we derive

$$h''(0-) = \frac{R_1 g(1) + \gamma(\xi^-(1) - \xi^*)^2}{\xi^-(1)^2} \leqslant \frac{R_1 g(1) + \gamma(\xi - \xi^*)^2}{\xi^2} \quad \forall\, \xi \in [0, \xi^-(1)).$$

As a function of ξ, the right-hand side of the inequality is decreasing in $(0, \xi^* + R_1 g(1)/(\gamma\xi^*)]$ and increasing in $[\xi^* + R_1 g(1)/(\gamma\xi^*), \infty)$, so the above inequality implies that $\xi^-(1) \leqslant \xi^* + R_1 g(1)/(\gamma\xi^*)$. Noticing $g(1) = w(1, \cdot) = O(\varepsilon^{8/3})$ (see (D.1)), we then derive the right-hand side inequality.

In Figure 9, we plot $\xi^{\pm}(1)$ against b for varying γ. It can be seen that $\xi^{\pm}(1)$ is apparently higher than the Merton line ξ^* for small γ.

Figure 9. $\xi^{\pm}(1)$ versus ξ^*



The dotted line represents the Merton line ξ^*, while the solid line with circle represents $\xi^{\pm}(1)$, the tax-adjusted optimal fraction of wealth in stock at $b = 1$, which is obtained numerically by the penalty method. Parameter values: $r = 0.03$, $\beta = 0.05$, $\alpha = 0.24$, $\mu = 0.08$, $\sigma = 0.2$, and $\kappa = 1.43$.

Appendix D. Asymptotic Analysis when $\xi^* \neq 1$

The differential operator $\mathcal{L}$ in (B.4) is degenerate at $\xi = 1$, which results in different expansions for $\xi^* \neq 1$ and $\xi^* = 1$. Let us first concentrate on the case $\xi^* \neq 1$, which is our primary interest.

Proposition 3. *Assume $\xi^* \neq 1$. Let ε be as given in (3.2), and let $\xi^+(\cdot)$ and $\xi^-(\cdot)$ be the optimal sell and buy boundaries, respectively. Denote $A := 4/[3\xi^*(\xi^* - 1)^2]$ and $m_* = 0.774765$.*

i) We have the following approximation for the value of deferral w:

$$w(b, \xi) = \frac{\gamma\sigma^2 A^{2/3} m_* \varepsilon^{8/3}}{2K} + o(\varepsilon^{8/3}). \tag{D.1}$$

ii) We have an approximation to the optimal trading boundaries $\xi^{\pm}(b)$, $b \in [0,1)$:

$$\xi^{\pm}(b) \quad \approx \quad \xi^* \pm \varepsilon\sqrt{1-b}\Big(\frac{1-b}{2(A\varepsilon)^{2/3}+1-b}\Big)^{1/6}. \tag{D.2}$$

iii) We have an approximation to the optimal consumption strategy:

$$\begin{aligned}\frac{C^*}{x+(1-\alpha)y+\alpha k} &= K[1+(1-\kappa)w]+o(\varepsilon^{8/3}) \\ &= K+\frac{1}{2}(1-\kappa)\gamma\sigma^2 m_* A^{2/3}\varepsilon^{8/3}+o(\varepsilon^{8/3}).\end{aligned} \tag{D.3}$$

Note that, for larger ε, we can modify m_* in (D.2) to obtain remarkable accuracy, as shown in Appendix E.

Proof of Proposition 3: We will utilize asymptotic analysis.

D.1. **Asymptotic Expansion.**

Near the tip $(1, \xi^{\pm}(1))$, we use the stretched variable

$$q = \frac{\xi-\xi^*}{\delta^2}, \qquad p = \frac{1-b}{\delta},$$

where

$$\delta = (A\varepsilon)^{2/3}. \tag{D.4}$$

Define

$$q_\delta^{\pm}(p) := \frac{\xi^{\pm}(b)-\xi^*}{\delta^2}\Big|_{b=1-\delta p}.$$

We write the solution in the form

$$w = \gamma\varepsilon^2\Big\{\frac{\delta m_*}{\hat{K}}+\delta^3 g_\delta(p)+\delta^5 v^\delta(p,q)\Big\}, \tag{D.5}$$

where m_* is a positive constant to be determined, $g_\delta(0)=0$, and $v^\delta(p,q)=0$ for $q \geq q_\delta^+(p)$. Then

$$\begin{aligned}\frac{\mathscr{L}w+f}{\gamma\varepsilon^2\delta} &= a v^\delta_{qq}+(1-\delta p)^2 g''_\delta - m_* + p - A^2q^2 + O(\delta), \\ \frac{\xi w_\xi+(1-b)w_b}{\gamma\varepsilon^2\delta^3} &= \xi_* v^\delta_q - p g'_\delta(p) + O\left(\delta\right),\end{aligned}$$

where $a = [\xi^*(\xi^*-1)]^2$. Sending $\delta \searrow 0$, we see that $(v^0, g_0) := \lim_{\delta\searrow 0}(V^\delta, g_\delta)$ is the solution of the limit problem for the tip:

$$\begin{cases} \max\{av^0_{qq} + g_0'' - m_* + p - A^2q^2,\ -v^0_q,\ \xi^* v^0_q - pg_0'\} = 0 \\ v^0(p,\infty) = 0,\ g_0(0) = 0,\ g_0'(0) = 0 \end{cases}$$

in $p \in \mathbb{R}^+, q \in \mathbb{R}$. Denote by $q_0^\pm(p)$ the resulting free boundary. One finds that $v^0_q = 0$ for $q \geqslant q_0^+(p)$ is the global minimum and $v^0_q = pg_0'/\xi^*$ for $q \leqslant q_0^-(p)$ is the global maximum. Thus,

$$v^0_{qq}(p, q_0^\pm(p)) = 0, \qquad q_0^\pm(p) = \pm\frac{\sqrt{\Phi_0(p)}}{A}, \qquad \Phi_0(p) := p + g_0'' - m_*. \tag{D.6}$$

The variational inequality can be written as

$$\begin{aligned} v^0 &\equiv 0 & \forall\, q \geqslant q_0^+(p), \\ av^0_{qq}(p,q) &= A^2q^2 - \Phi_0(p) & \forall\, q \in (q_0^-(p), q_0^+(p)), \\ \xi^* v^0_q(p,q) &= pg_0'(p) & \forall\, q \leqslant q_0^-(p). \end{aligned}$$

It then follows that when $q \in [q_0^-(p), q_0^+(p)]$,

$$\begin{aligned} v^0_q(p,q) &= \frac{A^2}{a}\Big\{\frac{q^3 - q_0^+(p)^3}{3} - q_0^+(p)^2(q - q_0^+(p))\Big\}, \\ v^0(p,q) &= \frac{A^2}{a}\Big\{\frac{q^4 - q_0^+(p)^4}{12} - \frac{q_0^+(p)^3}{3}(q - q_0^+(p)) - \frac{q_0^+(p)^2}{2}(q - q_0^+(p))^2\Big\}. \end{aligned}$$

The boundary condition $\xi^* v^0_q = pg_0'(p)$ at $q = q_0^-(p) = -q_0^+(p)$ gives

$$pg_0'(p) = \frac{4\xi^*}{3a}\, A^2\, q_0^+(p)^3 = \Phi_0^{3/2}(p) = \big(p + g_0''(p) - m_*\big)^{3/2},$$

since $A = 4\xi^*/(3a)$. The equation for g_0 becomes

$$\begin{cases} g_0''(p) = m_* - p + \big(pg_0'(p)\big)^{2/3} & \forall\, p > 0, \\ g_0(0) = 0, \quad g_0'(0) = 0, & \lim_{p\to\infty} g_0(p) \textit{ exists}. \end{cases} \tag{D.7}$$

Using a shooting method with m_* being the shooting parameter, one can numerically show that (D.7) admits a unique solution pair (m_*, g_0), where $m_* \approx 0.774765$. In addition, we find that a good approximation of $g_0'(p)$ is

$$G_0'(p) := \frac{p}{\sqrt{2+p}}.$$

Our numerical results reveal

$$\max_{p\in(0,8)}\left|G_0'(p)-g_0'(p)\right|\leqslant 0.01,\qquad \sup_{p\in(0,8)}\left|\frac{G_0'(p)}{g_0'(p)}-1\right|\leqslant 0.09.$$

(D.6) indicates that the free boundary is given by $q=\pm\sqrt{\Phi_0(p)}/A$ with

$$\Phi_0(p)=g_0''+p-m_*=\Big(pg_0'(p)\Big)^{2/3}=p\Big(\frac{g_0'(p)}{\sqrt{p}}\Big)^{2/3}.$$

In the original variable, this gives the approximation

$$\begin{aligned}\xi^{\pm} &\approx \xi^*\pm\frac{\delta^2\sqrt{\Phi_0(p)}}{A}=\xi^*\pm\frac{\delta^2\sqrt{p}}{A}\Big(\frac{g_0'(p)}{\sqrt{p}}\Big)^{1/3}\\ &= \xi^*\pm\varepsilon\sqrt{1-b}\Big(\frac{g_0'(p)}{\sqrt{p}}\Big)^{1/3}\Big|_{p=\frac{1-b}{(A\varepsilon)^{2/3}}}.\end{aligned}$$

Replacing g_0 by G_0 yields the expansion.

D.2. **An Approximation to the Optimal Consumption Strategy.**

By (2.11), (B.2), and (B.3), we can express the optimal consumption as

$$\begin{aligned}\frac{C^*}{x+(1-\alpha)y+\alpha k} &= c^*e^{(1-\kappa)w}\big(1-\xi w_\xi\big)^{-\kappa}\\ &= K[1+(1-\kappa)w]+o(\varepsilon^{8/3})\end{aligned}$$

by noting that $c^*=K$ and $\xi w_\xi=O(\varepsilon^4)$ (see (D.5)). The proof is complete.

Appendix E. A Modified m_* for Larger ε

By the derivation of (D.7), we can see that $m_*=\lim\limits_{\delta\to 0}m_*^\delta$, where m_*^δ, together with $g_\delta(\cdot)$, is the pair solution to the following ordinary differential equation problem:

$$\begin{cases}(1-\delta p)^2g_\delta''(p)+p-m_*^\delta-(pg_\delta'(p))^{2/3}=0,\quad p\in[0,\dfrac{1}{\delta}),\\ g_\delta(0)=0,\ g_\delta'(0)=0,\ \lim\limits_{p\to\frac{1}{\delta}}g_\delta(p)\text{ exists,}\end{cases}\tag{E.1}$$

with δ as given in (D.4). (E.1) has no closed form solution. Again using a shooting method with m_*^δ being the shooting parameter, we can numerically obtain m_*^δ, as shown in Figure 10, which is a decreasing function of δ. For a larger ε, we can replace m_* by m_*^δ to achieve a better approximation effect. Indeed, (D.1) implies that the

FIGURE 10. The Constant m_*^δ in (E.1) against δ

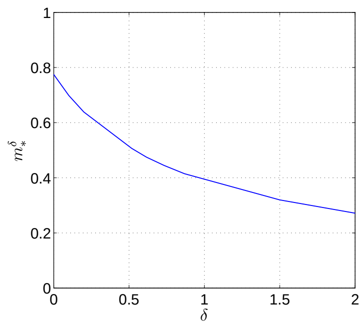
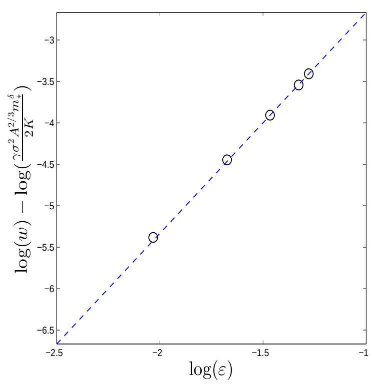


FIGURE 11. The Order of the Value of Deferral



The circles represent numerical values of $w = w(1, \cdot)$ obtained from the penalty method with $\alpha = 8\%, 16\%, 24\%, 32\%$, and 40%, respectively. The dashed line is the reference line $\log(\varepsilon) \mapsto \frac{8}{3}\log(\varepsilon)$. Default parameters values: $\beta = 0.05$, $r = 0.03$, $\mu = 0.08$, $\sigma = 0.20$, $\gamma = 3$, and $\kappa = 1.43$.

value of deferral w is of order $O(\varepsilon^{8/3})$. We then plot in Figure 11 the following function

$$\log(\varepsilon) \mapsto \log(w(1, \cdot)) - \log(\frac{\gamma\sigma^2 A^{2/3} m_*^\delta}{2K}) \tag{E.2}$$

and the reference line $\log(\varepsilon) \mapsto \frac{8}{3}\log(\varepsilon)$, where $w(1, \cdot)$ is numerically obtained from the penalty method. The figure demonstrates not only the order, but also the remarkable accuracy of expansion (D.1) with m_* replaced by m_*^δ.

Appendix F. Asymptotic Analysis at $\xi^* \approx 1$

Proposition 4. *Assume $\xi_* \approx 1$. Let ε be as given in (3.2), and let $\xi^+(\cdot)$ and $\xi^-(\cdot)$ be the optimal sell and buy boundaries, respectively. Denote $\hat{c} = 2\left(\gamma - \frac{\mu}{\sigma^2}\right)$, and*

$$\nu := \frac{1-\hat{c}}{2} + \sqrt{\left(\frac{1-\hat{c}}{2}\right)^2 + \hat{K}}, \tag{F.1}$$

We have an approximation to the optimal trading boundaries $\xi^\pm(b)$, $b \in [0,1]$:

$$\begin{aligned} \xi^+(b) &\approx \hat{\xi}(b), \\ \xi^-(b) &\approx \hat{\xi}(b) - \varepsilon\sqrt{\frac{(1-b)(1-b^{\nu-1})}{1-\hat{c}/\hat{K}}}, \end{aligned} \tag{F.2}$$

where

$$\hat{\xi}(b) := \xi^* + \frac{\varepsilon^2(1-b)}{2\xi^*} \tag{F.3}$$

Because ε^2 is small, Proposition 4 implies that the optimal selling boundary $\xi^+(b)$ is very close to the Merton line ξ^* when $\xi_* \approx 1$.

Proof of Proposition 4: We rewrite $f(b,\xi)$ as

$$f(b,\xi) = \gamma\Big[-(\xi - \hat{\xi}(b))^2 + \varepsilon^2(1-b) + O(\varepsilon^4)\Big].$$

Since the operator $\mathscr{L}$ is degenerate at $\xi = 1$, we use the change of variables

$$\zeta = \frac{\xi - \hat{\xi}(b)}{\varepsilon}, \qquad \zeta_\varepsilon^\pm(b) = \frac{\xi^\pm - \hat{\xi}(b)}{\varepsilon}, \qquad w(b,\xi) = \gamma\left[\varepsilon^2 g_\varepsilon(b) + \varepsilon^3 v^\varepsilon(b,\zeta)\right],$$

where $g_\varepsilon'(1) = 0$, and

$$v^\varepsilon(b,\zeta) = 0 \text{ when } \zeta \geq \zeta_\varepsilon^+(b).$$

Then problem (B.4) can be written as

$$\max\Big\{\mathbf{A}_\varepsilon[g_\varepsilon, v^\varepsilon], \quad -v^\varepsilon_\zeta, \quad v^\varepsilon_\zeta + (1-b)g_\varepsilon' + O(\varepsilon)\Big\} = 0, \tag{F.4}$$

where $\mathbf{A}_\varepsilon[g_\varepsilon, v^\varepsilon] = b^2 g_\varepsilon''(b) + \hat{c}bg_\varepsilon' - R_1 g_\varepsilon - \zeta^2 + 1 - b + R_2 v^\varepsilon_\zeta + O(\varepsilon)$.

Sending $\varepsilon \searrow 0$ in (F.4), we obtain, for the limit $(G,V) = \lim_{\varepsilon\searrow 0}(g_\varepsilon, v^\varepsilon)$,

$$\max\Big\{\hat{K}V_\zeta - \zeta^2 + \psi(b), \quad -V_\zeta, \quad V_\zeta + (1-b)G'\Big\} = 0 \tag{F.5}$$

for $b \in [0,1]$, $\zeta \in \mathbb{R}$, where $\psi(b) \equiv b^2G'' + \hat{c}bG' - \hat{K}G + 1 - b$, $G'(1) = 0$, and $V(b,+\infty) = 0$. Again, we denote by $\zeta^\pm(b)$ the resulting sell and buy boundaries.

First, we claim $\psi(b) = 0$. Assuming the contrary, we then have that $\psi(b) = \zeta^+(b)^2 > 0$ and when $\zeta > \zeta^+(b)$,

$$0 = V_\zeta \leq \frac{1}{\hat{K}} \left[\zeta^2 - \psi(b)\right].$$

We then deduce $\zeta^+(b) > 0$. On the other hand, it is easy to see that

$$V_\zeta = \frac{1}{\hat{K}} \left[\zeta^2 - \psi(b)\right] \text{ for } \zeta \in (\zeta^-(b), \zeta^+(b)]$$

and V_ζ achieves the global minimum at $(b, \zeta^+(b))$. Thus, we must have $\zeta^+(b) \leq 0$, which contradicts the earlier conclusion. As a byproduct, we have also proven

$$\zeta^+(b) = 0. \tag{F.6}$$

Therefore, the function G has to be the solution of

$$b^2 G'' + \hat{c} b G' - \hat{K} G + 1 - b = 0 \text{ in } (0,1)$$

with $G'(1) = 0$. Note that the above equation leads to $G(0) = 1/\hat{K}$. Thus, the unique solution is

$$G(b) = \frac{1}{\hat{K}} + \frac{b - b^\nu/\nu}{\hat{c} - \hat{K}}, \qquad G'(b) = \frac{1 - b^{\nu-1}}{\hat{c} - \hat{K}}, \tag{F.7}$$

where ν is as given in (F.1).

Substituting (F.7) into (F.5) and noticing (F.6) and C^1 continuity of V at $\zeta = \zeta^-(b)$, we then find

$$V(\zeta, b) = \begin{cases} 0 & \text{if } \zeta \geqslant 0, \\ \zeta^3/(3\hat{K}) & \text{if } \zeta^-(b) \leqslant \zeta < 0, \\ \left(\zeta - \frac{2}{3}\zeta^-(b)\right) \zeta^-(b)^2/\hat{K} & \text{if } \zeta < \zeta^-(b). \end{cases}$$

Using $V_\zeta + (1-b)\, G' = 0$ at $\zeta = \zeta^-(b)$, we have

$$\zeta^-(b)^2/\hat{K} = -(1-b)\, G' = -\frac{(1-b)\,(1 - b^{\nu-1})}{\hat{c} - \hat{K}},$$

namely, $\zeta^-(b) = -\sqrt{\dfrac{(1-b)(1-b^{\nu-1})}{1 - \hat{c}/\hat{K}}}$. This completes the proof.

Appendix G. The CEWL Incurred by Capital Gains Tax

The CEWL defined in (2.8) can be rewritten as

$$\overline{\Delta} \approx -\log(1-\overline{\Delta}) = \log \frac{\overline{G}}{G} - w. \tag{G.1}$$

It is easy to verify

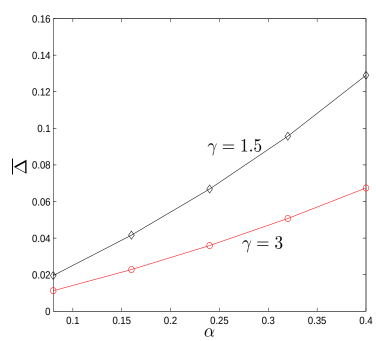
$$\log \frac{\overline{G}}{G} = \frac{1}{1-\kappa} \log \frac{\overline{K}}{K} = \frac{\gamma\sigma^2\varepsilon^2}{2K} + O(\varepsilon^4).$$

Combining with (D.1), we have

$$\overline{\Delta} \approx \frac{\gamma\sigma^2\varepsilon^2}{2K} - \frac{\gamma\sigma^2 A^{2/3} m_* \varepsilon^{8/3}}{2K},$$

from which we infer that the CEWL incurred by capital gains tax depends on not only the tax rate α, but also other parameters involved in ε. An extreme case is $r = 0$, where there is no wealth loss. In Figure 12, we plot the CEWL against α for $\gamma = 1.5$ and $\gamma = 3$, respectively. It can be seen that the CEWL can be as high as 13% when $\alpha = 0.4$ and $\gamma = 1.5$. The CEWL can be even higher for some other parameter values.[26]

Figure 12. The CEWL against α



The line with diamond and the line with circle represent the CEWL $\overline{\Delta}$ against α for $\gamma = 1.5$ and $\gamma = 3$, respectively, where $\overline{\Delta}$ is as given in (G.1) with $w = w(1, \cdot)$ computed from the penalty method. Default parameter values: $\beta = 0.05$, $r = 0.03$, $\mu = 0.08$, $\sigma = 0.2$, $\gamma = 3$, and $\kappa = 1.43$.

[26]For example, we may decrease the value of β or increase the value of r.

Appendix H. Proof of Proposition 2

Using a similar argument as in Appendix D, we can obtain

$$w_\lambda^{(i)}(b,\xi) \approx \frac{\gamma}{c_{i,\lambda}^*}\theta_i m_i, \tag{H.1}$$

where $\theta_i = \frac{\sigma_i^2}{2} A_i^{2/3}\varepsilon_i^{8/3}$ with $A_i = 4/[3\xi_i^*(\xi_i^*-1)^2]$ and

$$\varepsilon := \sqrt{\frac{2\alpha r_i \xi_i^*}{(1-\alpha)\gamma\sigma_i^2}} = \frac{\sqrt{2\alpha r_i(\mu_i - r_i/(1-\alpha))/(1-\alpha)}}{\gamma\sigma_i^2},$$

and $\{m_i, i\in\mathcal{I}\}$ is the solution of the following linear system:

$$\Big(1+\frac{1-\gamma}{\kappa-1}(\frac{K_i}{c_{i,\lambda}^*}-1)\Big)\theta_i m_i - \sum_{j\in\mathcal{I}}\frac{G_j^{1-\gamma}}{G_i^{1-\gamma}}\frac{q_{ij}}{c_{j,\lambda}^*}\theta_j m_j = \theta_i m_*^{\delta_i}, \quad i\in\mathcal{I}, \tag{H.2}$$

By (H.1), we can rewrite (H.2) as

$$\begin{pmatrix} M_{11}^\lambda & 1-M_{11}^\lambda \\ 1-M_{22}^\lambda & M_{22}^\lambda \end{pmatrix}\begin{pmatrix} c_{1,\lambda}^* w_\lambda^{(1)} \\ c_{2,\lambda}^* w_\lambda^{(2)} \end{pmatrix} \approx \gamma \begin{pmatrix} \theta_1 m_*^{\delta_1} \\ \theta_2 m_*^{\delta_2} \end{pmatrix}, \tag{H.3}$$

where the right-hand side is independent of λ. Note that $M_{11}^\lambda = M_{22}^\lambda = 1$ when $\lambda = \mathbf{0} := (0,0)$. Then, (H.3) reduces to

$$\begin{pmatrix} c_{1,\mathbf{0}}^* w_{\mathbf{0}}^{(1)} \\ c_{2,\mathbf{0}}^* w_{\mathbf{0}}^{(2)} \end{pmatrix} \approx \gamma \begin{pmatrix} \theta_1 m_*^{\delta_1} \\ \theta_2 m_*^{\delta_2} \end{pmatrix}. \tag{H.4}$$

Since we identify $w_\lambda^{(i)}$ with $\Delta_\lambda^{(i)}$, the combination of (H.3) and (H.4) yields the desired result.

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