Abstract: We are concerned with the optimal investment and consumption in a continuous-time setting for a constant absolute risk aversion (CARA) investor who faces proportional transaction costs and finite horizon. This is a singular stochastic control problem and the associated value function is governed by a variational inequality equation with gradient constraints. In terms of a novel approach developed by Dai and Yi [J. Differential Equations, 246 (2009), pp. 1445-1469], we provide a comprehensive theoretical analysis on the optimal investment and consumption strategy. It turns out that the optimal investment strategy is characterized by the resulting free boundaries which exhibit similar behaviors as in Dai et al. [SIAM J. Control Optim., 48 (2009), pp. 1134-1154] where the constant relative risk aversion utility is considered. Moreover, we show that in order to maintain optimal consumption, the CARA investor needs to invest more in risky asset. In addition, with the CARA utility, the optimal consumption may be negative. To ensure positive consumption for any positive liquidated wealth in the presence of transaction costs, the sufficient and necessary condition is that the discount rate is not less than the risk-free rate.

Key Words: Optimal investment and consumption, portfolio choice, transaction costs, constant absolute risk aversion

1 Introduction

Markowitz (1952) employed the (single-period) mean-variance analysis to understand and quantify the trade-off between risk and return in a portfolio of stocks. This work marked the start of modern finance. Merton (1969, 1971) initialized the study of portfolio choice in a continuous-time Brownian-motion-driven setting and replaced the quadratic utility, implicitly used in the mean-variance analysis, by more general increasing and concave utility functions. Magil and
Constantinides (1976) introduced transaction costs to Merton’s model and provided a fundamental insight that there is a no-trading region in the presence of transaction costs.

Since then, portfolio choice with transaction costs has been extensively studied. Constantinides (1986) considered an infinite horizon problem where the investor maximizes discounted utility of intermediate consumption. Davis and Norman (1990) addressed the same problem and presented a rigorous mathematical formulation of free boundary problem, where the free boundaries correspond to the optimal investment strategy. Shreve and Soner (1994) further conducted a comprehensive theoretical analysis on the optimal strategy by using the notion of viscosity solutions. Akian, Menaldi and Sulem (1996) and Kabanov and Kluppelberg (2004) considered an extension to multiple risky assets. Taksar, Klass and Assaf (1988) and Dumas and Luciano (1991) studied the maximization of the expected utility of terminal utility as time to maturity goes to infinity. Jang et al. (2007) considered the lifetime consumption and portfolio rule in a regime switching market. Portfolio choice with fixed and proportional transaction costs or more general transaction cost structure was studied in Constantinides (1979), Morton and Pliska (1995), Bielecki and Pliska (2000), Sulem and Oksendal (2002), Liu (2004), Pliska and Suzuki (2004), etc. Some numerical methods were proposed by Gennotte and Jung (1994), Atkinson, Pliska and Wilmott (1997), Bielecki et al. (2004), Muthuraman (2006), Muthuraman and Kumar (2006), Dai and Zhong (2010), etc. Asymptotical analysis was presented to examine the optimal strategy in Atkinson and Wilmott (1995), Janecek and Shreve (2004), etc. Using a martingale approach, Cvitanic and Karatzas (1996) and Cvitanic and Wang (2001) proved the existence of an optimal solution. Other existence results were obtained in Bouchard (2002), Guasoni (2002), Guasoni and Schachermayer (2004), etc. Recently, Kallsen and Muhle-Karbe (2010) and Gerhold, Muhle-Karbe and Schachermayer (2011) studied the optimal strategy by determining a shadow price which is the solution to the dual problem.

Most existing literature focuses on the infinite horizon problem with (proportional) transaction costs. When finite horizon is involved, the problem turns out to be very challenging because the resulting free boundaries vary with time. Theoretical analysis on the finite horizon problem has become possible only very recently. Liu and Loewenstein (2002) examined the finite horizon investment strategy by virtue of a sequence of analytical solutions to the infinite horizon investment problem. Dai and Yi (2009) considered the same problem and developed a novel partial differential equation approach: an equivalent double obstacle problem was utilized to study the optimal strategy. Dai et al. (2009) and Dai, Xu and Zhou (2010) employed the novel approach and fully characterized the optimal strategy for the utility maximization with investment-consumption\(^1\) and the continuous-time mean-variance analysis, respectively. It is worthwhile pointing out that the novel approach is robust and can also work in a more realistic

\(^1\)A technical condition assumed in Dai et al. (2009) was recently removed by Dai and Yang (2011).
market environment (e.g. Dai, Wang and Yang (2011) with a regime switching market, Dai, Jin and Liu (2011) with position limits). All these papers assume that the investor is of constant relative risk aversion (CRRA).

In this paper, we aim to study the optimal investment and consumption for a constant absolute risk aversion (CARA) investor who faces proportional transaction costs and finite horizon. The problem with infinite horizon has been considered by Liu (2004), but the finite horizon case remains open. The reason we are interested in the CARA utility lies in the separability of the utility function by which the multi-asset portfolio choice problem can be reduced to the single risky asset case provided that the assets are uncorrelated\(^2\) (cf. Liu (2004)). Yi and Yang (2008) made use of the approach developed in Dai and Yi (2009) to solve a sub-problem arising from the utility indifference pricing with transaction costs discussed in Davis, Panas and Zariphopoulou (1993). It should be pointed out that the sub-problem is essentially a finite horizon optimal investment problem with terminal CARA utility. The present paper will involve intermediate consumption and examine the optimal consumption and investment strategy. A key step is to make a transformation so as to utilize the approach developed in Dai and Yi (2009).

The rest of the paper is organized as follows. In Section 2 we present the problem formulation. In Section 3, we study the optimal investment and consumption strategy. Since the investment strategy is similar to that studied in Dai and Yi (2009) for the CRRA utility, our focus will be on the consumption strategy. In Section 4 we compare the investment strategy in the consumption case to that in the no-consumption case. Numerical results are presented as well for illustration. We conclude in the last section.

2 Problem formulation

Suppose there are two assets that an investor can trade\(^3\). The first asset ("the bond") is a money market account growing at a constant risk-free rate \(r < 1\).\(^4\) The second asset ("the stock") is a risky asset. Let \((\Omega, \mathcal{F}, p)\) be a probability space with a given filtration \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\). The stock price, denoted by \(S_t\), follows the geometric Brownian motion:

\[
dx_t = \alpha S_t dt + \sigma S_t dB_t,
\]

where \(\alpha > r\) and \(\sigma\) are respectively constant return rate and volatility of stock, and \(B_t\) is a one-dimension standard \(\mathcal{F}_t\)-Brownian motion.

Let \(x_t\) denote the amount invested in the bond and \(y_t\) denote the current value of the stock

\(^2\)Even if the assets are correlated, the strategy based on the single risky asset case can be used as a benchmark.

\(^3\)As mentioned earlier, the problem can be trivially extended to the multiple risky assets case provided that the assets are uncorrelated.

\(^4\)We will need \(r < 1\) to ensure (3.8) in Proposition 3.1. Since \(r\) is usually small, this assumption is almost without loss of generality.
holding. In the presence of transaction costs, \( x_t \) and \( y_t \) evolve according to
\[
\begin{align*}
\frac{dx_t}{dt} &= (r x_t - \kappa C_t)dt - (1 + \lambda)dL_t + (1 - \mu)dM_t, \\
\frac{dy_t}{dt} &= \alpha y_t dt + \sigma y_t dB_t + dL_t - dM_t,
\end{align*}
\]
respectively, where \( C_t \) is the consumption rate, \( \kappa = 0 \) or \( 1 \) indicates the no-consumption or consumption case, \( L_t \) and \( M_t \) are right continuous, non-negative and non-decreasing \( \mathcal{F}_t \)-adapted process with \( L_0 = M_0 = 0 \), representing cumulative dollar values for the purpose of buying and selling stock respectively. The constants \( \lambda \in [0, +\infty) \) and \( \mu \in [0, 1) \) account for proportional transaction costs incurred on purchase and sale of stock, and \( \lambda + \mu > 0 \).

Due to \( \alpha > r \), we can show that short selling in stock is never optimal. So, we always assume \( y_t \geq 0 \). At time \( t \), the investor’s wealth after liquidation is
\[
w_t = x_t + (1 - \mu) y_t \text{ for } y_t \geq 0.
\]

An investment and consumption strategy \((L, M, C)\) is admissible for \((x, y)\) starting from \( s \in [0, T) \) if \((x_t, y_t)\) given by (2.1)-(2.2) with \( x_s = x \) and \( y_s = y \) satisfies \( y_t \geq 0 \). We denote by \( \mathcal{A}_s(x, y) \) the set of all admissible strategies. The investor’s problem is to choose an admissible strategy so as to maximize the expected utility of intermediate consumptions (if any) and the terminal liquidated wealth,
\[
\begin{align*}
\sup_{(L, M, C) \in \mathcal{A}_0(x, y)} E_{x, y}^0 \left[ \int_0^T \kappa e^{-\delta s} U(C_s)ds + e^{-\delta T} U(w_T) \right],
\end{align*}
\]
subject to (2.1)-(2.2). Here \( \delta > 0 \) is the discount rate, \( E_{t}^{x, y} \) denotes the conditional expectation at time \( t \) given that initial endowment \( x_t = x, y_t = y \), and the utility function is
\[
U(w) = -e^{-\gamma w}
\]
with risk aversion \( \gamma > 0 \) being a constant.

We would like to emphasize that consumption \( C_t \) is allowed to be negative and negative consumption means infusion of funds which also generates utility. This assumption, appearing implicitly in Merton (1969) and Liu (2004), permits an explicit optimal strategy in the absence of transaction costs (i.e., \( \lambda = \mu = 0 \)), which is presented as follows.

**Theorem 2.1** In the absence of transaction costs (i.e., \( \lambda = \mu = 0 \)), the optimal consumption \( C_t^* \) and the optimal dollar value \( y_t^* \) invested in stock can be written in explicit form as follows:
\[
\begin{align*}
C_t^* &= \left( \delta - r + \frac{(\alpha - r)^2}{2\sigma^2} \right) \frac{e^{r(T-t)} - 1 + r(r-1)(T-t)}{\gamma r (e^{r(T-t)} - 1 + r)} + \xi_1(t) w_t, \text{ for } \kappa = 1 \\
y_t^* &= \frac{\alpha - r}{\gamma \sigma^2} \frac{1}{\xi_\kappa(t)},
\end{align*}
\]
where
\[\xi_\kappa(t) = \begin{cases} 
\frac{r}{1-(1-r)e^{-r(t-T)}} & \text{for the consumption case } (\kappa = 1), \\
e^{r(T-t)} & \text{for the no-consumption case } (\kappa = 0).
\end{cases}\] (2.4)

Proof: The proof is similar to Merton (1969) which addresses the infinite horizon case.

For later use, we call \(\xi_\kappa(t)y_t = \frac{\alpha - r}{\gamma\sigma^2}\) the Merton line. From the above theorem, we can see that if \(\delta - r + \frac{(\alpha - r)^2}{2\sigma^2} \geq 0\) and \(w_t > 0\), then \(C^*_t > 0\), which means (positive) consumption. If \(\delta - r + \frac{(\alpha - r)^2}{2\sigma^2} < 0\), then \(C^*_t < 0\) for sufficiently small \(w_t > 0\), which means infusion of funds may be optimal even with positive wealth. Moreover, given the optimal strategy, we cannot ensure non-negative wealth process and/or non-negative consumption process in any case.

3 Optimal strategy with transaction costs

Now we consider the case with transaction costs (i.e., \(\lambda + \mu > 0\)). The value function associated with the control problem (2.3) turns out to be the viscosity solution of the following HJB equation (cf. Dai et al. (2009), Fleming and Soner (2006)):

\[
\begin{cases}
\min \{-\partial_t \varphi - \mathcal{L} \varphi, -(1 - \mu) \partial_x \varphi + \partial_y \varphi, (1 + \lambda) \partial_x \varphi - \partial_y \varphi\} = 0 \\
\varphi(T, x, y) = -e^{-\gamma(x+(1-\mu)y)}
\end{cases}
\] (3.1)

where
\[\mathcal{L} \varphi = \frac{1}{2}\sigma^2 y^2 \partial_{yy} \varphi + \alpha y \partial_y \varphi + rx \partial_x \varphi - \frac{\kappa}{\gamma} \left[1 - \log \left(\frac{\partial_x \varphi}{\gamma}\right)\right] \partial_x \varphi - \delta \varphi.
\]

In the consumption case \((\kappa = 1)\), the optimal consumption is
\[C^*_t(t, x, y) = -\frac{1}{\gamma} \log \left(\frac{\partial_x \varphi}{\gamma}\right).
\] (3.2)

As a remark, it can be shown problem (3.1) has a unique viscosity solution (cf. Davis, Panas and Zariphopoulou (1993)).

The explicit form of optimal strategy inspires us to make the following transformation so as to reduce the problem dimension:

\[z = \xi_\kappa(t)y, \quad \varphi(t, x, y) = -e^{-\gamma(\xi_\kappa(t)x+\phi(t,z))},\] (3.3)

where \(\xi_\kappa(t)\) is as given in (2.4). It follows

\[
\begin{cases}
\min \{-\partial_t \phi - \mathcal{L}_1 \phi, -(1 - \mu) \partial_x \phi, (1 + \lambda) \partial_x \phi - \partial_y \phi\} = 0 \\
\phi(T, z) = (1 - \mu)z
\end{cases}
\] (3.4)

\footnote{We point out that the transformation in the no-consumption case can be obtained by the separability of the associated value function. However, it is not obvious for the consumption case.}
\( \Omega \equiv [0, T) \times [0, +\infty) \), where
\[
\mathcal{L}_1 \phi = \frac{1}{2} \sigma^2 z^2 (\partial_{zz} \phi - \gamma (\partial_z \phi)^2) + (\alpha - r + \kappa \xi_\kappa(t)) z \partial_z \phi - \kappa \xi_\kappa(t) \phi + \frac{1}{\gamma} \left[ \delta - \kappa \xi_\kappa(t) (1 - \log \xi_\kappa(t)) \right].
\]

We point out that if \( \kappa = 0 \), then the problem (3.4) is exactly the same as that studied in Yi and Yang (2008) though they claimed their problem was from the option pricing problem with transaction costs studied in Davis, Panas and Zariphopoulou (1993).

Denote
\[
V(t, z) = \partial_z \phi(t, z).
\]

It is easy to verify
\[
\frac{\partial}{\partial z} \mathcal{L}_1 \phi = \frac{1}{2} \sigma^2 z^2 \partial_{zz} V + (\alpha - r + \sigma^2 + \kappa \xi_\kappa(t)) z \partial_z V + (\alpha - r) V - \gamma \sigma^2 z V (z \partial_z V + V)
\]
\[
\equiv \mathcal{L}_2 V.
\]

Using the approach developed in Dai and Yi (2009), we can obtain the following proposition which plays a critical role in our theoretical analysis.

**Proposition 3.1** Let \( \phi(t, z) \) be the solution to problem (3.4). Denote \( V(t, z) = \partial_z \phi(t, z) \). Then \( V(t, z) \) is governed by the following double obstacle problem:
\[
\begin{aligned}
\partial_t V + \mathcal{L}_2 V &= 0, & \text{if } 1 - \mu < V < 1 + \lambda \\
\partial_t V + \mathcal{L}_2 V &\geq 0, & \text{if } V = 1 + \lambda \\
\partial_t V + \mathcal{L}_2 V &\leq 0, & \text{if } V = 1 - \mu \\
V(T, z) &= 1 - \mu
\end{aligned}
\]
\[(3.6)\]

in \( \Omega \), where the differential operator \( \mathcal{L}_2 \) is as given in (3.5). Moreover, \( V(t, z) \in W^{1,2}_p ([0, T] \times [\varepsilon, N]) \), for any \( 0 < \varepsilon < N, \ p > 1 \), and
\[
\begin{aligned}
\partial_t V &\leq 0, \quad & (3.7) \\
\partial_t V &\leq 0, \quad & (3.8) \\
z \partial_z V + V &\geq 0, \quad & (3.9)
\end{aligned}
\]

for all \( (t, z) \in \Omega \).

Proof: We will only prove (3.8) for \( \kappa = 1 \) because the proof of other parts is similar to that in Dai and Yi (2009). Denote \( u = \partial_t V \). Note that
\[
\frac{\partial}{\partial t} \mathcal{L}_2 V = \frac{1}{2} \sigma^2 z^2 \partial_{zz} u + (\alpha - r + \sigma^2 + \xi_1(t)) z \partial_z u + (\alpha - r) u - \gamma \sigma^2 z (z \partial_z V + V) u - \gamma \sigma^2 z V (z \partial_z u + u) + \xi_1'(t) z \partial_z V.
\]

Owing to the assumption \( r < 1 \), we have
\[
\xi_1'(t) = \frac{r^2 (1 - r) e^{-r(T-t)}}{(1 - (1 - r) e^{-r(T-t)})^2} > 0,
\]

which, combining with (3.7), leads to $\xi_1'(t) z\partial_z V \leq 0$. It is apparent that $\partial_t V \leq 0$ at $t = T$. Applying the maximum principle (cf. Friedman (1982)) yields the desired result. The proof is complete.

Compared with problem (3.4), problem (3.6) is easier to study. In what follows, we will make use of problem (3.6) to study the optimal investment strategy.

### 3.1 Optimal investment strategy

To study the optimal investment strategy, we only need to characterize the selling, buying, and no trading regions, which are defined as follows:

$$SR = \{(t, z) \in \Omega : V(t, z) = 1 - \mu\},$$
$$BR = \{(t, z) \in \Omega : V(t, z) = 1 + \lambda\},$$
$$NT = \{(t, z) \in \Omega : 1 - \mu < V(t, z) < 1 + \lambda\}.$$

The following theorem entirely characterizes the optimal investment strategy.

**Theorem 3.1** There are two monotonically decreasing functions $z_s(t) : [0, T) \rightarrow [0, +\infty]$ and $z_b(\tau) : [0, T) \rightarrow [0, \infty)$ such that

$$SR = \{(t, z) \in \Omega : z \geq z_s(t)\},$$
$$BR = \{(t, z) \in \Omega : z \leq z_b(t)\}.$$

Moreover, $z_s(t), z_b(t) \in C^\infty[0, T)$,

$$z_b(t) \leq \frac{\alpha - r}{\gamma \sigma^2(1 + \lambda)}, \quad (3.10)$$
$$z_s(t) \geq \frac{\alpha - r}{\gamma \sigma^2(1 - \mu)}, \quad (3.11)$$
$$\lim_{t \to T} z_s(t) = \frac{\alpha - r}{\gamma \sigma^2(1 - \mu)}, \quad (3.12)$$
$$z_b(t) = 0 \text{ if and only if } t \in [T - \frac{1}{\alpha - r} \log(\frac{1 + \lambda}{1 - \mu}), T). \quad (3.13)$$

Proof: These results are similar to those in Dai and Yi (2009) for the CRRA utility and their proofs are also similar. The existence of $z_s(t)$ and $z_b(t)$ follows from (3.7). In terms of (3.8), we obtain the monotonicity of $z_s(t)$ and $z_b(t)$. We can further use the maximum principle to obtain the strict monotonicity in $\{z > 0\}$. The smoothness of $z_s(t), z_b(t)$ can be proved by the approach in Soner and Shreve (1991) (see also Dai, Xu and Zhou (2010) and Yi and Yang (2008)). To show (3.10), we notice that for any $(t, z) \in BR$, the double obstacle problem gives

$$0 \leq (\partial_t + \mathcal{L}_2)(1 + \lambda) = \mathcal{L}_2(1 + \lambda) = (1 + \lambda)[(\alpha - r) - \gamma \sigma^2 z(1 + \lambda)],$$
which implies $BR \subset \{ z \leq \frac{\alpha - r}{\sigma^2(1 + \lambda)} \}$. This yields (3.10), which also implies that $z_b(t)$ is finite for all $t$. The proof of (3.11) is similar. The proof of (3.12) is analogous to that in Dai and Yi (2009). To show (3.13), we notice that at $z = 0$, (3.6) is reduced to

$$
\begin{align*}
\partial_t V + (\alpha - r)V|_{z=0} &= 0, \text{ if } 1 - \mu < V < 1 + \lambda \\
\partial_t V + (\alpha - r)V|_{z=0} &\geq 0, \text{ if } V = 1 + \lambda \\
\partial_t V + (\alpha - r)V|_{z=0} &\leq 0, \text{ if } V = 1 - \mu \\
V(T,0) &= 1 - \mu,
\end{align*}
$$

whose solution is

$$
V(t,0) = \begin{cases} 
(1 - \mu) e^{(\alpha - r)(T-t)}, & \text{for } t \geq T - \frac{1}{\alpha - r} \log \left( \frac{1 + \lambda}{1 - \mu} \right) \\
1 + \lambda, & \text{for } t < T - \frac{1}{\alpha - r} \log \left( \frac{1 + \lambda}{1 - \mu} \right).
\end{cases}
$$

This implies (3.13). The proof is complete.

**Remark 3.1** We can prove that $z_s(t)$ is finite for all $t$. Indeed, the case $\kappa = 0$ has been proved in Yi and Yang (2008) by studying the stationary solution to the double obstacle problem (3.6). Later we will show in Theorem 4.1 that $z_s(t)$ in the consumption case ($\kappa = 1$) is bounded by that in the no-consumption case ($\kappa = 0$). This implies the conclusion.

**Remark 3.2** In finance, $z_s(t)$ and $z_b(t)$ represent the optimal selling and buying boundaries, respectively. From (3.10) and (3.11), we infer that the Merton line is always in the no-trading region which is wider than $\frac{(\alpha - r)(1 + \mu)}{\gamma \sigma^2(1 - \mu)(1 + \lambda)}$. (3.13) means that buying stock is suboptimal as the investment horizon is short enough. This is because the investor would have less investment chances to offset transaction costs incurred. All these results are similar to those in Dai et al. (2009) where the CRRA utility is considered.

### 3.2 Optimal consumption strategy

Now let us focus on the consumption case $\kappa = 1$ and examine the optimal consumption strategy. We have seen from Theorem 2.1 that consumption is likely negative in the absence of transaction costs. We will investigate when consumption is positive in the presence of transaction costs.

Due to (3.2) and (3.3), it is easy to see

$$
C^*(t,x,y) = -\frac{1}{\gamma} \log \left( \xi_1(t)e^{-\gamma(\xi_1(t)x + \phi(t,z))} \right)
= \phi(t,z) - \frac{1}{\gamma} \log \xi_1(t) + \xi_1(t)x
$$

(3.14)

This motivates us to study $\phi(t,z) - \frac{1}{\gamma} \log \xi_1(t)$. Let us first introduce a lemma.

**Lemma 3.1** Define

$$
f(t) \equiv \phi(t,0) - \frac{1}{\gamma} \log \xi_1(t).
$$

i) If $\delta < r$, then $f(t) < 0$ for any $t \in [T - \frac{1}{\alpha - r} \log \left( \frac{1 + \lambda}{1 - \mu} \right), T]$.

ii) If $\delta \geq r$, then $f(t) \geq 0$ for all $t$.
Proof: For convenience, we denote \( B(t) = \phi(t, 0) \). By (3.13), we have for \( t \geq T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu} \),

\[
\phi_t + \mathcal{L}_1 \phi|_{z=0} = 0
\]
or equivalently

\[
B'(t) - \xi_1(t) B(t) + \frac{1}{\gamma} [\delta - \xi_1(t) (1 - \log \xi_1(t))] = 0.
\]

(3.15)

Then we can infer

\[
f'(t) = B'(t) - \frac{\xi_1'(t)}{\gamma \xi_1(t)}
\]

\[
= \xi_1(t) B(t) - \frac{1}{\gamma} [\delta - \xi_1(t) (1 - \log \xi_1(t))] - \frac{\xi_1'(t)}{\gamma \xi_1(t)}
\]

\[
= \xi_1(t) f(t) - \frac{\delta - \gamma}{\gamma}
\]

(3.16)

for \( t \geq T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu} \), where the second equality is due to (3.15). (3.16) can be rewritten as

\[
\left[ \left( e^{r(T-t)} - 1 + r \right) f(t) \right]' = \frac{r - \delta}{\gamma} \left( e^{r(T-t)} - 1 + r \right).
\]

If \( \delta < r \), then

\[
\left[ \left( e^{r(T-t)} - 1 + r \right) f(t) \right]' > 0 \text{ for } t \geq T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu}
\]

which combines with \( f(T) = 0 \) to yield part i).

Now let us prove part ii). Owing to (3.12), we have

\[
\partial_z \phi = 1 + \lambda \text{ for } z \leq z_b(t) \text{ and } t < T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu}.
\]

It follows

\[
\phi(z, t) = B(t) + (1 + \lambda) z \text{ for } z \leq z_b(t) \text{ and } t < T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu}.
\]

From Proposition 3.1 and the Sobolev embedding theorem, we infer \( V \) and \( \partial_z V \) are continuous, so are \( \partial_z \phi \) and \( \partial_{zz} \phi \). We can further obtain the continuity of \( \partial_t \phi \) by the smoothness of \( z_b(t) \) and \( z_s(t) \) (cf. Dai and Yi (2009)). Then, at \( z = z_b(t), t < T - \frac{1}{\alpha-r} \log \frac{1+\lambda}{1-\mu} \), (3.4) can be rewritten as

\[
B'(t) - \frac{1}{2} \gamma \sigma^2 (z_b(t))^2 (1 + \lambda)^2 + (\alpha - r) z_b(t) (1 + \lambda) - \xi_1(t) B(t) + \frac{1}{\gamma} [\delta - \xi_1(t) (1 - \log \xi_1(t))] = 0,
\]

that is

\[
B'(t) - \xi_1(t) B(t) + \frac{1}{\gamma} [\delta - \xi_1(t) (1 - \log \xi_1(t))]
\]

\[
= \frac{1}{2} \gamma \sigma^2 (1 + \lambda)^2 z_b(t) \left[ z_b(t) - \frac{2(\alpha - r)}{\gamma \sigma^2 (1 + \lambda)} \right],
\]

which leads to

\[
B'(t) - \xi_1(t) B(t) + \frac{1}{\gamma} [\delta - \xi_1(t) (1 - \log \xi_1(t))] \leq 0.
\]

(3.17)
Here we have used (3.11). Thanks to (3.15), we have (3.17) for any $t$. Using similar arguments as in the proof of part i), we infer
\[
\left[ (e^{r(T-t)} - 1 + r) f(t) \right]' \leq \frac{r - \delta}{\gamma} (e^{r(T-t)} - 1 + r) \leq 0
\]
for all $t$ and $\delta > r$. Due to $f(T) = 0$, this gives the desired result.

**Theorem 3.2** Let $C^*(t, x, y)$ be the optimal consumption as given in (3.2). Then

\[
C^*(t, x, y) \geq 0 \quad \text{for all } (t, \xi(t)y) \in NT \text{ and } x + (1 - \mu) y \geq 0
\]

if and only if $\delta \geq r$.

**Proof:** Note that

\[
C^*(t, x, 0) = f(t) + \xi_1(t)x.
\]

By part i) of Lemma 3.1, we infer $C^*(t, x, 0) < 0$ for sufficiently small $x > 0$ and $t > T - \frac{1}{\alpha - r} \log \frac{1 + \lambda}{1 - \mu}$. This implies the only-if-part. Next we prove the if-part. It is easy to see

\[
\phi(t, z) = \phi(t, 0) + \int_0^z V(t, \eta)d\eta \geq \phi(t, 0) + (1 - \mu) z = \phi(t, 0) + (1 - \mu) \xi_1(t)y.
\]

So,

\[
C^*(t, x, y) \geq \phi(t, 0) + (1 - \mu) \xi_1(t)y - \frac{1}{\gamma} \log \xi_1(t) + \xi_1(t)x
\]

\[
= f(t) + \xi_1(t)(x + (1 - \mu) y) > 0,
\]

where the last inequality is due to $x + (1 - \mu) y > 0$ and part ii) of Lemma 3.1. The proof is complete.

**Remark 3.3** In contrast, Theorem 2.1 indicates that in the absence of transaction costs, the sufficient and necessary condition to ensure positive consumption for any positive wealth is $\delta > r - \frac{(\alpha - r)^2}{2\sigma^2}$, which is different from that in Theorem 3.2 with transaction costs. Examining the proof of Lemma 3.1 reveals that it is partially because in the presence of transaction costs, the investor is inclined to take no transaction with small stock holding as time goes to maturity.

### 4 Comparison between the consumption case and the no-consumption case

This section is devoted to the comparison of investment strategies between the consumption case and the no-consumption case. It is worth pointing out that the transformed variable $z$ in the
consumption case differs from that in the no-consumption case. For the purpose of comparison, we define the optimal buy and sell boundaries in the $y$-$t$ plane according to the transformation (3.3):

$$y_b(t) = \frac{z_b(t)}{\xi_\kappa(t)}, \quad y_s(t) = \frac{z_s(t)}{\xi_\kappa(t)},$$

where $\xi_\kappa(t)$ is as given in (2.4).

**Theorem 4.1** Let $y^0_b(t)$ and $y^1_b(t)$ ($y^0_s(t)$ and $y^1_s(t)$) be the optimal buy (sell) boundaries for the no consumption case and the consumption case, respectively. Then

$$y^0_b(t) \leq y^1_b(t) \leq e^{r(T-t)} - 1 + \frac{r}{y^0_b(t)}, \quad (4.1)$$

$$y^0_s(t) \leq y^1_s(t) \leq e^{r(T-t)} - 1 + \frac{r}{y^0_s(t)}. \quad (4.2)$$

**Proof:** First, let us prove the right hand side inequalities of (4.1)-(4.2), which are equivalent to

$$z^1_b(t) \leq z^0_b(t), \quad (4.3)$$

$$z^1_s(t) \leq z^0_s(t), \quad (4.4)$$

where $z^0(t)$ and $z^1(t)$ are the optimal boundaries in the $z$-$t$ plane for the no-consumption case and the consumption case. Let $V_\kappa$ be the solution to the double obstacle problem (3.6) with $\kappa = 0, 1$. Noticing

$$\xi_\kappa(t) z \partial_z V_1 \leq 0,$$

we apply the maximum principle to get

$$V_0(t, z) \geq V_1(t, z) \text{ for all } t \text{ and } z.$$ 

It follows

$$V_0(t, z) > 1 - \mu \text{ if } V_1(t, z) > 1 - \mu,$$

$$V_1(t, z) < 1 + \lambda \text{ if } V_0(t, z) < 1 + \lambda,$$

from which we infer (4.3) and (4.4). The right hand side inequalities of (4.1)-(4.2) then follow.

It remains to show the left hand side inequalities of (4.1)-(4.2). Let us make a transformation

$$\tilde{V}_\kappa(t, y) = V_\kappa(t, z), \quad y = \frac{z}{\xi_\kappa(t)},$$

where $\xi_\kappa(t)$ is given in (2.4). Then the double obstacle problem (3.6) is transformed to

$$\begin{align*}
\partial_t \tilde{V}_\kappa + \mathcal{G}_\kappa \tilde{V}_\kappa &= 0, \quad \text{if } 1 - \mu < \tilde{V}_\kappa < 1 + \lambda \\
\partial_t \tilde{V}_\kappa + \mathcal{G}_\kappa \tilde{V}_\kappa &\geq 0, \quad \text{if } \tilde{V}_\kappa = 1 + \lambda \\
\partial_t \tilde{V}_\kappa + \mathcal{G}_\kappa \tilde{V}_\kappa &\leq 0, \quad \text{if } \tilde{V}_\kappa = 1 - \mu \\
V(T, z) &= 1 - \mu \quad (4.5)
\end{align*}$$
in $\Omega$, where

$$\mathcal{J}_n V_n = \frac{1}{2} \sigma^2 y^2 \partial_{yy} V_n + (\alpha + \sigma^2) y \partial_y V_n + (\alpha - r) V_n - \gamma \sigma^2 \xi_n(t) y V_n(y \partial_y V_n + V_n).$$

Clearly

$$\{(t, y) \in \Omega : \mathcal{V}_n(t, y) = 1 - \mu\} = \{(t, y) \in \Omega : y \geq y^*_n(t)\}$$

$$\{(t, y) \in \Omega : \mathcal{V}_n(t, y) = 1 + \lambda\} = \{(t, y) \in \Omega : y \leq y^*_n(t)\}.$$

To complete the proof, we only need to show

$$\mathcal{V}_0(t, y) \leq \mathcal{V}_1(t, y) \text{ for all } t \text{ and } y.$$  \hspace{1cm} (4.6)

Note that

$$\mathcal{J}_1 \mathcal{V}_1 = \mathcal{J}_0 \mathcal{V}_1 + \gamma \sigma^2 (\xi_0(t) - \xi_1(t)) y \mathcal{V}_1(y \partial_y \mathcal{V}_1 + \mathcal{V}_1).$$

It is easy to verify

$$\xi_0(t) - \xi_1(t) = \frac{e^{r(T-t)} - 1}{1 - (1-r)e^{-r(T-t)}} > 0.$$

and

$$y \partial_y \mathcal{V}_1 + \mathcal{V}_1 = z \partial_z V + V \geq 0,$$

where the last inequality is due to (3.9). So

$$\gamma \sigma^2 (\xi_0(t) - \xi_1(t)) y \mathcal{V}_1(y \partial_y \mathcal{V}_1 + \mathcal{V}_1) \geq 0.$$

Applying the maximum principle yields the desired result (4.6). The proof is complete.

For illustration, we plot the optimal buy and sell boundaries against time $t$. The parameter default values are given as follows

$$r = 0.01, \alpha = 0.07, \sigma = 0.3, \mu = \lambda = 0.01, \gamma = 0.5, T = 3.$$

Figure 1 presents the boundaries in $z - t$ plane. We can see $z^1_0(t) \leq z^0_0(t)$ and $z^1_0(t) \leq z^0_0(t)$, which verify (4.3)-(4.4) or equivalently the right hand side inequalities of (4.1)-(4.2). Observe that these boundaries are monotonically decreasing in time. Note that the Merton line $\frac{\alpha - r}{\gamma \sigma^2} = 1.35$ and $T - \frac{1}{\alpha - r} \log \frac{1 + \lambda}{1 - \mu} = 2.7$. The behaviors of the boundaries are consistent with the results in Theorem 3.1. In Figure 2, we plot the buy and sell boundaries in $y - t$ plane. We can see that the buy (sell) boundary in the consumption case is higher than the buy (sell) boundary in the no-consumption case, which verifies the left hand side inequalities of (4.1)-(4.2). This indicates that a CARA investor needs to invest more in stock in order to maintain optimal consumption. Such a result is different from the one in Dai et al. (2009) that a CRRA investor would like to invest more in bank account to maintain optimal consumption. We believe it is because the present problem permits bankruptcy and investing more in stock can earn more to consume in the long run under the assumption $\alpha > r$. 

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Figure 1: Optimal buy and sell boundaries against time in $z - t$ plane

Parameter default values: $r = 0.01$, $\alpha = 0.07$, $\sigma = 0.3$, $\mu = \lambda = 0.01$, $\gamma = 0.5$, $T = 3$.

Figure 2: Optimal buy and sell boundaries against time in $y - t$ plane

Parameter default values: $r = 0.01$, $\alpha = 0.07$, $\sigma = 0.3$, $\mu = \lambda = 0.01$, $\gamma = 0.5$, $T = 3$. 
5 Conclusion

In this paper, we study the optimal investment and consumption strategy in a continuous-time setting for a constant absolute risk aversion (CARA) investor who faces proportional transaction costs and finite horizon. This is a singular stochastic control problem and the associated value function is governed by a variational inequality equation with gradient constraints. The key point is to make a transformation that enables us to use the novel approach developed by Dai and Yi (2009) to obtain an equivalent double obstacle problem. It turns out that the optimal investment strategy is characterized by the free boundaries, arising from the double obstacle problem, which exhibit similar behaviors as in Dai et al. (2009) where a constant relative risk aversion (CRRA) utility is considered. Moreover, we show that the CARA investor needs to invest more in risky asset to maintain optimal consumption, which is in contrast to the result in Dai et al. (2009) that the CRRA investor needs to invest more in risk-free asset to maintain optimal consumption. In addition, with the CARA utility, the optimal consumption is likely negative. To ensure a positive consumption for any positive liquidated wealth in the presence of transaction costs, the sufficient and necessary condition is that the discount rate is not less than the risk-free rate. This condition also differs from that in the absence of transaction costs.

References


