Convergence Analysis of Binomial Tree Method for American-Type Path-Dependent Options

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Abstract. The pricing models of American-type path-dependent options are of degenerate parabolic obstacle problems. The binomial tree method is the most popular approach to pricing options. For some special cases, this method is modified in order to make it feasible. The main purpose of this paper is, using numerical analysis and the notion of viscosity solutions, to show the uniform convergence of the binomial tree method and a modified binomial tree method (i.e., forward shooting grid method) for American-type path-dependent options. Numerical experiments are given to demonstrate our conclusion.

1. Introduction

1.1. What are path-dependent options?

A vanilla call/put option is a contract which gives the holder the right to buy/sell the underlying asset such as stock, exchange rate, commodity, etc., by a certain date (expiry) for a predetermined price (strike price). A European-type option is an option that can be exercised only at expiry, while an American-type option is the one that may be exercised at any time prior to expiry.

A path-dependent option is an option whose payoff at expiry depends on the “past history” of the underlying asset price as well as its spot price. Well-known examples of path-dependent options are Asian options and lookback options. The terminal payoff of an Asian option depends on some form of averaging of the underlying asset price over the whole period of the option’s lifetime. According to the way of averaging, Asian options are separated into two classes: Asian arithmetic options and Asian geometric options. A lookback option is another type of path-dependent options, whose terminal payoff depends on the historical maximum or minimum of the underlying asset price.

Let $S_t$ be a random walk of the underlying asset price. Let $A_t$ be a path-dependent variable, then the value function of a path-dependent option depends on $S_t$, $A_t$ and time $t$, namely

\[ V = V(S_t, A_t, t), \]

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where

\[ A_t = \begin{cases} \frac{1}{t} \int_0^t S_\tau d\tau, & \text{Asian arithmetic} \\ \exp\left(\frac{1}{t} \int_0^t \ln S_\tau d\tau\right), & \text{Asian geometric} \\ \max_{0 \leq \tau \leq t} S_\tau = A^\max_t \\ \min_{0 \leq \tau \leq t} S_\tau = A^\min_t \end{cases}, \text{lookback} \]

At expiry \( V = V(S_T, A_T, T) \) is known as the payoff of the option. For example, for the fixed strike option,

\[ V(S_T, A_T, T) = \begin{cases} (A_T - X)^+, & \text{Asian call} \\ (X - A_T)^+, & \text{Asian put} \\ A_T^\max - X, & \text{lookback call} \\ X - A_T^\min, & \text{lookback put} \end{cases} \]

And for the floating strike,

\[ V(S_T, A_T, T) = \begin{cases} (S_T - A_T)^+, & \text{Asian call} \\ (A_T - S_T)^+, & \text{Asian put} \\ S_T - A_T^\min, & \text{lookback call} \\ A_T^\max - S_T, & \text{lookback put} \end{cases} \]

**Problem:** Find the value of a path-dependent option during its lifetime

\[ V(S_t, A_t, t) =? \quad (0 \leq t < T) \]

**1.2. Models**

In the risk neutral world, the underlying asset price \( S \) is assumed to follow the log-normal diffusion process

\[ dS = rSdt + \sigma SdW_t, \quad (1.1) \]

where \( W_t \) is a standard Brownian motion

\[ E(dW_t) = 0, \quad Var(dW_t) = dt, \]

and \( r \) and \( \sigma \) represent the interest rate and volatility respectively.

Based on the arbitrage-free principle and Ito lemma we find that the valuation models for American-type path-dependent options are the following parabolic obstacle problems (see [15]):

\[
\begin{align*}
\min\{-\frac{\partial V}{\partial t} - \mathcal{L}V, V - \Lambda(S, A)\} &= 0, \quad t \in (0, T), \quad (S, A) \in D. \\
V(S, A, T) &= \Lambda(S, A) \\
\end{align*}
\]

(1.2)

where

\[
\mathcal{L}V = \begin{cases} \frac{1}{t}(S - A)\frac{\partial V}{\partial A} + \mathcal{L}_{BS}V, & \text{Asian arithmetic} \\ \frac{1}{t}(\ln S - \ln A)\frac{\partial V}{\partial A} + \mathcal{L}_{BS}V, & \text{Asian geometric} \\ \mathcal{L}_{BS}V, & \text{lookback} \end{cases}
\]
\[ \mathcal{L}_{BSV} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV, \]

\[ \Lambda(S, A) = \begin{cases} (S - A)^+, & \text{floating strike call} \\ (A - S)^+, & \text{floating strike put} \\ (A - X)^+, & \text{fixed strike call} \\ (X - A)^+, & \text{fixed strike put} \end{cases} \]

and

\[ D = \begin{cases} (0, \infty) \times (0, \infty), & \text{Asian arithmetic or Asian geometric} \\ \{(S, A) : 0 \leq S \leq A\}, & \text{floating (fixed) strike lookback put (call)} \\ \{(S, A) : 0 \leq A \leq S\}, & \text{floating (fixed) strike lookback call (put)}. \end{cases} \]

In addition, for lookback options, one has an additional boundary condition at \( S = A \)

\[ \frac{\partial V}{\partial A}(S, S, t) = 0. \] (1.3)

### 1.3. Algorithms of binomial tree method

The binomial tree method (BTM), first proposed by Cox, Ross and Rubinstein (1979) [5], has become one of the most popular approaches to pricing options due to its simplicity and flexibility. In this section, let us extend this method to the valuation of path-dependent options (see [12] or [3]).

If \( N \) is the number of discrete time points, we have time points \( t_n = n \Delta t, \quad n = 0, 1, \ldots, N \) with \( \Delta t = T/N \). Let \( V^n(S, A) \) be the option price at time \( t_n \) with underlying asset value \( S \) and path-dependent variable \( A \). Here we might as well assume

\[ A = \begin{cases} \frac{1}{n} \sum_{i=1}^{n} S_{t_i}, & \text{Asian arithmetic} \\ \left( \prod_{i=1}^{n} S_{t_i} \right)^{1/n}, & \text{Asian geometric} \\ \max_{0 \leq i \leq n} S_{t_i}, & \text{floating (fixed) strike lookback put (call)} \\ \min_{0 \leq i \leq n} S_{t_i}, & \text{floating (fixed) strike lookback call (put)} \end{cases} \]

\( S_{t_i} \) stands for the underlying asset value of such path at time \( t_i, \quad i = 0, 1, \ldots, n \) (Note \( S_{t_n} = S \)). It is assumed that \( S \) will either jump up to \( Su \) with probability \( p \) or down to \( Sd \) with probability \( 1 - p \) at time \( t_{n+1} \). Consequently, \( A \) will become either \( A^u \) or \( A^d \), where

\[ A^u = \begin{cases} \frac{nA + Su}{n+1}, & \text{Asian arithmetic} \\ (A^n Su)^{1/(n+1)}, & \text{Asian geometric} \\ \max(A, Su), & \text{floating (fixed) strike lookback put (call) } (S \leq A) \\ A, & \text{floating (fixed) strike lookback call (put) } (S \geq A) \end{cases} \]

and \( A^d \) is given similarly. By no-arbitrage argument, one has for American-type path-dependent options

\[ V^n(S, A) = \max\{e^{-r \Delta t}[pV^{n+1}(Su, A^u) + (1 - p)V^{n+1}(Sd, A^d)], \Lambda(S, A)\}. \] (1.4)
where \( p = \frac{e^{r\Delta t} - d}{u - d} \). Setting \( ud = 1 \) and combining with stochastic differential equation (1.1), we get

\[
u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}
\]

and thus

\[
p = \frac{e^{r\Delta t} - e^{\sigma \sqrt{\Delta t}}}{e^{\sigma \sqrt{\Delta t}} - e^{-\sigma \sqrt{\Delta t}}}
\]

At expiry (i.e. \( T = N\Delta t \)) we have

\[
V^N(S, A) = \Lambda(S, A).
\]

Using the backward induction (1.4)-(1.5), option prices can be calculated. This is the so-called binomial tree method.

### 1.4. A modified binomial tree method (forward shooting grid method)

BTM for Asian options involves tracks of \( 2^n \) paths at time \( t = n\Delta t \). For Asian geometric options, fortunately, the number of possible geometric average values does not increase exponentially, because the geometric average of the underlying asset price is also lognormal distribution. But, for Asian arithmetic options, BTM is not feasible since the number of possible arithmetic average values increases exponentially with the number of timesteps. With interpolation technique, a remedy proposed by Hull and White (1993) is to restrict the possible average values to a set of predetermined values \([12]\). Barraquand and Pudet (1996) present a similar algorithm, known as the forward shooting grid method \([3]\). In the following we refer to their algorithms as modified binomial tree method (MBTM).

Adopting the notation of Barraquand and Pudet (1996) \([3]\), we present the algorithm of MBTM for Asian arithmetic options as follows. For \( \Delta t \) given, let

\[
\Delta Y = \rho \sigma \sqrt{\Delta t}.
\]

Here \( \rho \) is a quantization parameter for spacing in the average direction and \( 1/\rho \) is assumed to be an integer. Later we will see that, in order to guarantee convergence, \( \rho \) also depends on \( \Delta t \). Let discrete values of the asset price \( S \) and the arithmetic average price \( A \) be given by

\[
S^n_j = u^j \quad \text{and} \quad A^n_k = e^{k\Delta Y}
\]

for \( n = 0, \ldots, N \) and \( j, k \in \mathbb{Z} \). It is assumed that \((S^n_j, A^n_k)\) will either jump up to \((S^{n+1}_j, A^{n+1}_k)\) with probability \( p \) or down to \((S^{n+1}_{j-1}, A^{n+1}_k)\) with probability \( 1 - p \), where

\[
A^{n+1}_k = \frac{(n + 1)A^n_k + S^{n+1}_j}{n + 2}, \quad A^{n+1}_k = \frac{(n + 1)A^n_k + S^{n+1}_{j-1}}{n + 2}.
\]

Noting that \( A^{n+1}_k \) in general does not coincide with \( A^{n+1}_{k'} = e^{k'\Delta Y} \), for some integer \( k' \), some form of interpolation should be taken. For future reference, define

\[
k_{floor}^\pm = \text{floor}\left(\frac{\ln(A^{n+1}_k)}{\Delta Y}\right).
\]

Here \( \text{floor}(x) \) denotes the largest integer less than or equal to \( x \).
Let \( U^n(S^n_j, A^n_k) \) stand for option values at time \( t = n\Delta t, S = S^n_j, A = A^n_k \). The backward procedure of the MBTM for American-type Asian arithmetic option is described as follows:

\[
\begin{cases}
U^n(S^n_j, A^n_k) = \max \{ e^{-r\Delta t} \{ \Pi_A U^{n+1}(S^{n+1}_j, A^{n+1}_k) + (1 - p)\Pi_A U^{n+1}(S^{n+1}_j, A^{n+1}_k) \} \}, (A^n_k - X)^+ \\
U^N(S^N_j, A^N_k) = (A^N_k - X)^+
\end{cases}
\]

for \( n = N - 1, \ldots, 0; j, k \in \mathbb{Z} \), where \( \Pi_A \) is the interpolation operator. For example, for nearest lattice point, linear or quadratic interpolations, we can write

\[
\Pi_A U^{n+1}(S^{n+1}_j, A^{n+1}_k) = \alpha^- U^{n+1}(S^{n+1}_{j-1}, A^{n+1}_{k_{\text{floor}} - 1}) + \alpha^0 U^{n+1}(S^{n+1}_j, A^{n+1}_{k_{\text{floor}}}) + \alpha^+ U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k_{\text{floor}} + 1}),
\]

where \( \alpha \)'s are determined by the type of interpolation used and

\[
\alpha^- + \alpha^0 + \alpha^+ = 1, \quad 0 \leq \alpha^-, \alpha^0, \alpha^+ \leq 1.
\]

2. Convergence analysis of BTM and MBTM

Many authors show that the prices of European-type vanilla options computed from the BTM converge to their corresponding continuous-time PDE model values (See [10] and references therein). Amin & Khanna (1994) and Jiang & Dai (1998) produce the convergence proofs of the BTM for American-type vanilla options by using probabilistic approach and PDE approach, respectively [1][13]. By ignoring the effect of interpolation error, Barraquand & Pudet (1996) prove the convergence of the MBTM for European-type Asian arithmetic options [3]. Forsyth, Vetzal & Zvan (1998) investigate the propagation of interpolation error of the MBTM and point out that the grid quantization parameter (for the spacing of the arithmetic averages) should depend on the timestep size to guarantee the convergence [9]. Their analysis is for European-type options. In this section we will give a general framework to prove the convergence of BTM and MBTM for American-type path-dependent options by using the notion of viscosity solutions.

2.1. Main results

To begin with, let us recall the notion of viscosity solutions.

**Definition.** A function \( u \in \text{USC}(\mathcal{D} \times (0, T]) \) (resp. \( \text{LSC}(\mathcal{D} \times (0, T]) \)) is a viscosity subsolution (resp. supersolution) of the problem (1.2) (and (1.3) for lookback options) if \( u(S, A, T) \leq \Lambda(x) \) (resp. \( u(S, A, T) \geq \Lambda(x) \)), and whenever \( \phi \in C^{2,1}(\mathcal{D} \times (0, T]) \) and \( u - \phi \) attains its local maximum (resp. local minimum) at \((S, A, t) \in \mathcal{D} \times (0, T)\) we have

\[
\min \{-\frac{\partial \phi}{\partial t} - \mathcal{L}\phi, \phi - \Lambda\}_{(S, A, t)} \leq 0
\]

(resp.

\[
\min \{-\frac{\partial \phi}{\partial t} - \mathcal{L}\phi, \phi - \Lambda\}_{(S, A, t)} \geq 0.
\]

We call \( u \in C(\mathcal{D} \times (0, T]) \) is a viscosity solution of (1.2) (and (1.3) for lookback options) if it is both a viscosity subsolution and supersolution.
Lemma 2.1 Suppose $u$ and $v$ are viscosity subsolution and supersolution of problem (1.2) (and (1.3) for lookback options) respectively, then $u \leq v$. (see [7])

Theorem 2.2 The problem (1.2) (and (1.3) for lookback options) has a unique viscosity solution. (see [7])

Let $V^n(S, A)$ be the function defined by algorithms (1.4)-(1.5) in $\mathcal{D}$ for American-type path-dependent option. We now define the extension function $V^{\Delta t}(S, A, t)$ as follows: for $t \in [n\Delta t, (n+1)\Delta t]$, $n = 0, 1, \cdots, N - 1$,

$$V^{\Delta t}(S, A, t) = \frac{(n+1)\Delta t - t}{\Delta t} V^n(S, A) + \frac{t - n\Delta t}{\Delta t} V^{n+1}(S, A).$$

Let $U^n(S^n_j, A^n_k)$ be the values computed from the MBTM (1.10). Define the extension function $U^{\Delta t}(S, A, t)$, $S \geq 0, A \geq 0, t \in (0, T]$ as follows: for $S \in [u_j - 1/2, u_{j+1}/2)$, $A \in [e^{(k-1/2)\Delta Y}, e^{(k+1/2)\Delta Y}]$, $t \in [(n - 1/2)\Delta t, (n + 1/2)\Delta t)$,

$$U^{\Delta t}(S, A, t) = U^n(S^n_j, A^n_k).$$

The main results of this paper are the following:

Theorem 2.3 Suppose that $V(S, A, t)$ is the viscosity solution to the problem (1.2) (and (1.3) for lookback options). Then, as $\Delta t \to 0$, we have $V^{\Delta t}(S, A, t)$ converges uniformly to $V(S, A, t)$ in any bounded closed subdomain of $\mathcal{D} \times (0, T)$.

Theorem 2.4 Consider the MBTM with the nearest lattice point interpolation or the linear interpolation, where

$$\rho = \begin{cases} o(\Delta t^{1/2}) & \text{for nearest lattice point interpolation,} \\ o(1) & \text{for linear interpolation.} \end{cases}$$

Let $V(S, A, t)$ be the viscosity solution to the problem (1.2) for American-type Asian arithmetic option. Then, as $\Delta t \to 0$, we have $U^{\Delta t}(S, A, t)$ converges uniformly to $V(S, A, t)$ in any bounded closed subdomain of $\mathcal{D} \times (0, T)$.

Remark 1 The MBTM with quadratic interpolation is not a monotonic scheme. We cannot prove its convergence, though it is consistent with the corresponding PDE model (see Lemma 2.5).

We will only prove Theorem 2.4 since the proof of Theorem 2.3 is similar. The proof will be divided into three steps. Section 2.2 is devoted to the consistency analysis of the MBTM and PDE model. In section 2.3 we present an $l^\infty$-estimate for the approximate sequence defined by MBTM. In section 2.4 we show the convergence in the viscosity solution framework.

To illustrate method, we only consider Asian arithmetic fixed strike call options. Throughout the remainder of this paper, Asian arithmetic options always refer to fixed strike call options.
2.2. Consistency condition

First we make the following consistency definition.

**Definition** A scheme $\mathcal{H}u = H$ is pointwise consistent with the PDE $\mathcal{F}v = F$ at point $(x, t)$ if for any smooth function $\phi = \phi(x, t)$, $(\mathcal{F}\phi - F)_{(x,t)} - (\mathcal{H}(\phi - H))_{(y,s)} \to 0$ as $\Delta t \to 0$ and $(y, s) \to (x, t)$.

**Lemma 2.5** Under the assumption of

$$\rho = \begin{cases} o(\Delta t^{1/2}) & \text{for nearest lattice point interpolation,} \\ o(1) & \text{for linear interpolation,} \\ O(1) & \text{for quadratic interpolation,} \end{cases}$$

(2.1)

the MBTM (1.10) is consistent with the corresponding PDE (1.2).

Proof. Let us first ignore the effect of interpolation. Suppose that $V(S, A, t - \Delta t) = \max\{e^{-\Delta t}[pV(Su, A^u, t) + (1 - p)V(Sd, A^d, t)], (A - X)^+\}$.

By Taylor expansions and the following identities

$$e^{-\Delta t}[p(u - 1) + (1 - p)(d - 1)] = r\Delta t + O(\Delta t^2),$$

$$e^{-\Delta t}[p(u - 1)^2 + (1 - p)(d - 1)] = \sigma^2 \Delta t + O(\Delta t^2),$$

$$e^{-\Delta t}[p(u - 1)^3 + (1 - p)(d - 1)^3] = O(\Delta t^2),$$

$$e^{-\Delta t}[p(A^u - A) + (1 - p)(A^d - A)] = \frac{S - A}{t} \Delta t + O(\Delta t^2),$$

$$e^{-\Delta t}[p(u - 1)(A^u - A) + (1 - p)(d - 1)(A^d - A)] = O(\Delta t^2),$$

it is not hard to check that

$$\max\left\{-\frac{\partial V}{\partial t} - \frac{1}{t}(S - A)\frac{\partial V}{\partial A} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV, V - (A - X)^+\right\}_{(S, A, t)} = O(\Delta t).$$

(2.2)

We now consider the effect of interpolation error. Suppose the order of interpolation error is $\Delta Y^\beta$, namely

$$\Pi_A \varphi - \varphi = O(\Delta Y^\beta),$$

(2.3)

where $\beta = 1, 2, 3$ correspond to the nearest lattice point, linear and quadratic interpolation, respectively. Let $U(S, A, t)$ be a smooth function and satisfy

$$U(S, A, t - \Delta t) = \max\{e^{-\Delta t}[p\Pi_A U(Su, A^u, t) + (1 - p)\Pi_A U(Sd, A^d, t)], (A - X)^+\}.$$

By (2.2) and (2.3), it is not hard to check that

$$\max\left\{-\frac{\partial U}{\partial t} - \frac{1}{t}(S - A)\frac{\partial U}{\partial A} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - rS \frac{\partial U}{\partial S} + rU, U - (A - X)^+\right\}_{(S, A, t)} = O(\Delta t + \frac{\Delta Y}{\Delta t}) = O(\Delta t + \rho^\beta \Delta t^{\beta/2 - 1}).$$

This implies the conclusion.
2.3. $l^\infty$-estimate

In the following we will always suppose

\[ 0 < p < 1, \]

which is a fact for sufficiently small $\Delta t$.

Let us first define an auxiliary function sequence $W^n(S, A)$, $0 \leq S \leq A$, $0 \leq n \leq N$, which is computed from the following backward procedure

\[
\begin{align*}
W^n(S, A) &= \max \{ e^{-r\Delta t} [pW^{n+1}(Su, \max(Su, A)) + (1 - p)W^{n+1}(Sd, A)], e^{\alpha(N-n)\Delta t}A \} \\
W^N(S, A) &= A
\end{align*}
\]

(2.4)

for $S \leq A$, $\alpha > 0$.

**Lemma 2.6** Let $W^n(S, A)$, $0 \leq S \leq A$, $0 \leq n \leq N$ be the solution to (2.4) with $\alpha > 0$. Then we have

\[ W^n(S, A) \leq e^{\alpha T}(\max(A, \frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)})^{1/(\lambda_- - \lambda_+)}S + 1) \]

(2.5)

for sufficiently small $\Delta t$, where

\[ \lambda_\pm = \frac{r}{\sigma^2} + \frac{1}{2} \pm \sqrt{\left(\frac{r}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2\alpha}{\sigma^2}}. \]

(2.6)

**Remark 2** In Lemma 2.6, $\alpha > 0$ guarantees $\left(\frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)}\right)^{1/(\lambda_- - \lambda_+)} < \infty$.

Before proving Lemma 2.6 we inquire into some properties of the solution to the problem (2.4). By transformations

\[ x = \ln \frac{A}{S} \text{ and } W^n(x) = e^{-\alpha(N-n)\Delta t}W^n(S, A) \frac{S}{A}, \]

(2.7)

the numerical scheme (2.4) is reduced to

\[
\begin{align*}
W^n(x) &= \max \{ e^{-(r+\alpha)\Delta t}[pW^{n+1}(x - \sigma \sqrt{\Delta t})^+] + (1 - p)dW^{n+1}(x + \sigma \sqrt{\Delta t}), e^x \} \\
W^N(x) &= e^x
\end{align*}
\]

(2.8)

for $x \geq 0$.

**Lemma 2.7** Let $\bar{W}^n(x)$, $x \geq 0$, $0 \leq n \leq N$ be the solution to (2.8). Then for all $n \leq N$, we have

(a) $\bar{W}^{n+1}(x) \leq \bar{W}^n(x)$
(b) $\bar{W}^n(x_1) \leq \bar{W}^n(x_2)$ if $x_1 \leq x_2$
(c) $\bar{W}^n(x) = e^x$ if $x \geq (N - n)\sigma \sqrt{\Delta t}$. 

(2.9)
The proof of Lemma 2.7 is obvious.

To simplify notation, (2.8) will also be written as

$$\hat{W}^n(x) = F(\Delta t)\hat{W}^{n+1}(x).$$  \hspace{1cm} (2.10)

**Lemma 2.8** For $\Delta t$ given, there exists unique element $\hat{W}_{\Delta t}(x)$ satisfying $\hat{W}_{\Delta t}(x) - e^x \in L^\infty(R^+)$ such that

$$\hat{W}_{\Delta t}(x) = F(\Delta t)\hat{W}_{\Delta t}(x).$$ \hspace{1cm} (2.11)

In addition, $\hat{W}_{\Delta t}(x)$ is a monotone function of $x$ and

$$\hat{W}^n(x) \leq \hat{W}_{\Delta t}(x).$$ \hspace{1cm} (2.12)

Proof: Let $\hat{W}^n(x) = \hat{W}^n(x) - e^x$. Then $\hat{W}^n(x)$ satisfies

$$\hat{W}^n(x) = F(\Delta t)(\hat{W}^{n+1}(x) + e^x) - e^x \hat{G}(\Delta t)\hat{W}^{n+1}(x).$$

By (2.9), $\hat{W}^n(x) \in L^\infty(R^+)$. Hence $\hat{G}(\Delta t)$ can be regarded as a mapping from $L^\infty(R^+)$ to $L^\infty(R^+)$. It is not hard to verify $\hat{G}(\Delta t)$ is a contraction mapping. Therefore, there exists a unique element $\hat{W}_{\Delta t}(x) \in L^\infty(R^+)$ such that

$$\hat{W}_{\Delta t}(x) = \hat{G}(\Delta t)\hat{W}_{\Delta t}(x).$$

This completes the proof by denoting $\hat{W}_{\Delta t}(x) = \hat{W}_{\Delta t}(x) + e^x$.

We now prove Lemma 2.6.

**Proof of Lemma 2.6.** Let $\Delta x = \sigma \sqrt{\Delta t}$ and $x_j = j\Delta x$, $j = 0, 1, \cdots$. It is not hard to see that $\hat{W}_{\Delta t}(x_j)$ satisfies the following difference equation of a free boundary problem:

$$\frac{\hat{W}_{\Delta t}(x_j)}{\hat{W}_{\Delta t}(x_{j-1})} = e^{-(r+\alpha)\Delta t}[pu\hat{W}_{\Delta t}(x_{j-1}) + (1 - p)d\hat{W}_{\Delta t}(x_{j+1})] + (1 - p)d\hat{W}_{\Delta t}(x_{j+1})], \text{ for } 1 \leq j < j_\infty \label{eq:2.13}$$

where $j_\infty$ is the point of free boundary to be determined. It is easy to check that the solution of (2.13) is

$$\hat{W}_{\Delta t}(x_j) = C_1\xi^j_1 + C_2\xi^j_2 \text{ for } 0 \leq j \leq j_\infty + 1,$$

where

$$\xi_{1,2} = \frac{e^{(r+\alpha)\Delta t} \pm \sqrt{e^{2(r+\alpha)\Delta t} - 4p(1 - p)}}{2(1 - p)d},$$

$$C_1 = \frac{\xi_2 u^{j_\infty} - u^{j_\infty+1}}{\xi_1^{j_\infty} (\xi_2 - \xi_1)}, \text{ } C_2 = \frac{\xi_1 u^{j_\infty} - u^{j_\infty+1}}{\xi_2^{j_\infty} (\xi_1 - \xi_2)},$$

and

$$j_\infty = \frac{1}{\ln \xi_2 - \ln \xi_1} \ln \left(\frac{e^{(r+\alpha)\Delta t} - pu - (1 - p)d\xi_2 \xi_1 - u}{(1 - p)d\xi_1 - (e^{(r+\alpha)\Delta t} - pu) \xi_2 - u}\right).$$
By symbol operation, one gets

$$\lim_{\Delta t \to 0} j_{\infty} = \frac{1}{\lambda_- - \lambda_+} \ln \frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)} < \infty,$$

where $\lambda_\pm$ are given by (2.6). Then for sufficiently small $\Delta t$,

$$W_{\Delta t}(x) \leq \max(e^x, \left(\frac{\lambda_-(\lambda_+ - 1)}{\lambda_+(\lambda_- - 1)}\right)^{1/(\lambda_- - \lambda_+)} + 1),$$

Together with (2.7) and (2.12), this implies (2.5). The proof is complete.

The following lemma plays a crucial role in the convergence proof of MBTM.

**Lemma 2.9** Let $U^n(S^n_j, A^n_k)$ be the American-type Asian option value computed from MBTM (1.10). Then for all $n, j$ and $k$

$$U^n(S^n_j, A^n_k) \leq W^n(S^n_j, \max(S^n_j, A^n_k))$$

(2.14)

where $W^n(S, A)$ be the solution to (2.4) with $\alpha > 0$.

Proof. Clearly for all $n, j$

$$U^n(S^n_j, A^n_{k_1}) \leq U^n(S^n_j, A^n_{k_2}) \text{ if } A^n_{k_1} \leq A^n_{k_2}. \tag{2.15}$$

Hence it suffices to show that

$$U^n(S^n_j, A^n_k) \leq W^n(S^n_j, A^n_k) \text{ for } S^n_j \leq A^n_k.$$ 

Suppose it is true for $n + 1$. We claim

$$\Pi A U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k^+}) \leq W^{n+1}(S^{n+1}_{j+1}, \max(S^{n+1}_{j+1}, A^{n+1}_{k})) \text{ for } S^{n+1}_j \leq A^{n+1}_k. \tag{2.16}$$

Indeed, since $1/\rho$ is assumed to be an integer, due to (1.6) and (1.7), there is an integer $k'$ such that

$$\max(S^{n+1}_{j+1}, A^n_k) = A^{n+1}_{k'} = S e^{k' \Delta Y}.$$ 

Owing to (1.8), one has

$$A^{n+1}_{k^+} \leq A^{n+1}_{k'}. \tag{2.17}$$

If $A^{n+1}_{k^+} = A^{n+1}_{k'_{floor}}$, (2.16) is clear. If $A^{n+1}_{k^+} \neq A^{n+1}_{k'_{floor}}$, one has by (1.9) and (2.17)

$$A^{n+1}_{k'_{floor} - 1} \leq A^{n+1}_{k'_{floor}} \leq A^{n+1}_{k'_{floor} + 1} \leq A^{n+1}_{k'}. \tag{2.18}$$

From (2.15) and (2.18), it follows

$$U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k'_{floor} - 1}) \leq U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k'_{floor}}) \leq U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k'_{floor} + 1}) \leq U^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k'}) \leq W^{n+1}(S^{n+1}_{j+1}, A^{n+1}_{k'}) = W^{n+1}(S^{n+1}_{j+1}, \max(S^{n+1}_{j+1}, A^{n+1}_k)),$$
where the last inequality is due to the assumption of induction. In virtue of (1.11) and (1.12), (2.16) is derived. Similarly one gets
\[ \Pi_A U^{n+1}(S^n_{j-1}, A^n_k) \leq W^{n+1}(S^n_{j-1}, A^n_k). \]
Then for \( S^n_j \leq A^n_k \)
\[
W^n(S^n_j, A^n_k) = \max \left\{ e^{-\Delta t}[pW^{n+1}(S^n_{j+1}, \max(S^n_{j+1}, A^n_k)) + (1 - p)\Pi_A U^{n+1}(S^n_{j-1}, A^n_k)], e^{\alpha(N-n)\Delta t} A^n_k \right\} 
\geq \max \left\{ e^{-\Delta t}[p\Pi_A U^{n+1}(S^n_{j+1}, A^n_k) + (1 - p)\Pi_A U^{n+1}(S^n_{j-1}, A^n_k)], (A^n_k - X)^+ \right\} = U^n(S^n_j, A^n_k).
\]
The proof is complete.

2.4. Convergence

We now employ the notion of viscosity solutions to show Theorem 2.4. The idea is based on [2] and [13].

Proof of Theorem 2.4. Define
\[
V^*(S, A, t) = \limsup_{\Delta t \rightarrow 0, (x,y,z) \rightarrow (S,A,t)} U_{\Delta t}(x,y,z), \\
V_*(S, A, t) = \liminf_{\Delta t \rightarrow 0, (x,y,z) \rightarrow (S,A,t)} U_{\Delta t}(x,y,z).
\]
Due to Lemma 2.6 and 2.9, we infer that \( V^* \) and \( V_* \) are well defined. It is obvious that \( V^* \in USC \) and \( V_* \in LSC \), and \( V_*(S, A, t) \leq V^*(S, A, t) \). If we show that \( V^* \) and \( V_* \) are viscosity subsolution and supersolution of (1.2) respectively, then in terms of comparison principle (Lemma 2.1) we deduce \( V^*(S, A, t) \leq V_*(S, A, t) \) and thus \( V^*(S, A, t) = V_*(S, A, t) = V(S, A, t) \), which comes to conclusion.

We only need to show that \( V^* \) is a subsolution of (1.2). It can be shown that \( V^*(S, A, T) = (A-X)^+ \). Suppose that for \( \phi \in C^{2,1}(D \times (0,T)) \), \( V^* - \phi \) attains a local maximum at \((S_0, A_0, t_0) \in D \times (0,T) \) and \( (V^* - \phi)(S_0, A_0, t_0) = 0 \). We might as well assume that \((S_0, A_0, t_0) \) is a strict local maximum on \( B_r = \{ t_0 \leq t \leq t_0 + r, |S - S_0| \leq r, |A - A_0| \leq r \} \), \( r > 0 \). By the definition of \( V^* \), there exists a sequence \( u_{\Delta t_k}(S_k, A_k, t_k) \) such that \( \Delta t_k \rightarrow 0, (S_k, A_k, t_k) \rightarrow (S_0, A_0, t_0), U_{\Delta t_k}(S_k, A_k, t_k) \rightarrow V^*(S_0, A_0, t_0) \) when \( k \rightarrow \infty \). Assuming that \((\hat{S}_k, \hat{A}_k, \hat{t}_k) \) is a global maximum point of \( U_{\Delta t_k} - \phi \) on \( B_r \), we can deduce that there is a subsequence \( U_{\Delta t_k}(\hat{S}_k, \hat{A}_k, \hat{t}_k) \), such that
\[
\Delta t_k \rightarrow 0, (\hat{S}_k, \hat{A}_k, \hat{t}_k) \rightarrow (S_0, A_0, t_0), \\
(U_{\Delta t_k} - \phi)(\hat{S}_k, \hat{A}_k, \hat{t}_k) \rightarrow (V^* - \phi)(S_0, A_0, t_0) \text{ as } k \rightarrow \infty. \tag{2.19}
\]
Therefore
\[
U_{\Delta t_k}(\cdot, \cdot, \hat{t}_k + \Delta t_k) \leq \phi(\cdot, \cdot, \hat{t}_k + \Delta t_k) + (U_{\Delta t_k} - \phi)(\hat{S}_k, \hat{A}_k, \hat{t}_k) \text{ in } B_r.
\]
and quadratic interpolation, respectively. The algorithms were coded in Fortran and prices by the MBTM with the nearest lattice point interpolation, linear interpolation considered American-type Asian arithmetic fixed strike call options and computed options In this section we present some numerical experiments to demonstrate our conclusion. We

yield the desired result because of

Here we have used the monotonicity of the MBTM with nearest point interpolation or 

convergence analysis is available. However, for safety, we recommend using the MBTM

is adopted, given any fixed \( \rho \), the MBTM seems to be convergent as \( \Delta t \to 0 \), though no convergence analysis is available. However, for safety, we recommend using the MBTM with linear interpolation due to its desired monotonicity. The MBTM with quadratic interpolation improves the order of consistency with the corresponding PDE model, but loses the monotonicity, which may spoil the convergence at some occasions.

3. Numerical experiments

In this section we present some numerical experiments to demonstrate our conclusion. We considered American-type Asian arithmetic fixed strike call options and computed options prices by the MBTM with the nearest lattice point interpolation, linear interpolation and quadratic interpolation, respectively. The algorithms were coded in Fortran and computations were performed on a personal computer (Intel 266).

We considered two cases. We computed option values using \( \rho = 1.0, 0.5, 0.1, N = T/\Delta t = 50, 100, 200, \) respectively. Numerical results are presented in Table 1 (Case 1) and Table 2 (Case 2). Let us see the case 1. The correct price for this option is about 5.82. Observe that when \( \rho \) is fixed, the MBTM with nearest lattice point interpolation clearly diverges as \( \Delta t \to 0 \) (\( N \to \infty \)). If linear interpolation is used, it seems that for \( \rho \) fixed, the MBTM tends to a value which is a little higher than the true price, as \( \Delta t \to 0 \). On the other hand, if \( \Delta t \) (i.e. \( N \)) is fixed, for small \( \rho = 0.1 \), option values computed from the MBTM with linear interpolation or the nearest lattice point interpolation are close to the true price. This implies that \( \rho \) should be chosen according to (2.1). The results of case 2 are also in agreement with our convergence analysis. If the quadratic interpolation is adopted, given any fixed \( \rho \), the MBTM seems to be convergent as \( \Delta t \to 0 \), though no convergence analysis is available. However, for safety, we recommend using the MBTM with linear interpolation due to its desired monotonicity. The MBTM with quadratic interpolation improves the order of consistency with the corresponding PDE model, but loses the monotonicity, which may spoil the convergence at some occasions.
### Table 1: Computation of MBTM

(Case 1: $\sigma = 0.4$, $r = 0.1$, $T = 0.25$, $S = 100$, $X = 100$, correct price $\approx 5.82$)

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<th>Quadratic Interpolation</th>
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### Table 2: Computation of MBTM

(Case 2: $\sigma = 0.5$, $r = 0.1$, $T = 5.0$, $S = 100$, $X = 100$, correct price $\approx 34.20$)

<table>
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### References


