Calibration of Stochastic Volatility Models: An Optimal Control Approach

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Abstract

We aim to calibrate stochastic volatility models from option prices. We develop an optimal control approach to recover the risk neutral drift term of stochastic volatility. An efficient numerical algorithm is given. Numerical results and empirical studies are presented to demonstrate our algorithm. In contrast to existing literature, we do not assume that the stochastic volatility model has special structure, so our algorithm applies to calibration of general stochastic volatility models. In addition, our empirical results reveal that the risk neutral process of volatility recovered from market prices of options on S&P 500 index is indeed linearly mean-reverting.

Subject classifications: Finance: asset pricing; Dynamic programming-optimal control: applications.

Area of review: Financial engineering.

1 Introduction

Black and Scholes (1973) develop the celebrated option pricing model under the assumption that the underlying stock price follows a log-normal distribution with constant volatility. The model possesses some appealing properties, including the uniqueness of prices and the perfect replication of options. Unfortunately, the constant volatility assumption apparently conflicts with the volatility smile phenomenon observed in options markets.

As a remedy, Dupire (1994) considers the local volatility model, where volatility is a deterministic function of the underlying stock price and time. He derives an insightful equation, known as Dupire’s equation, revealing that there is a unique diffusion process consistent with the risk neutral densities derived from the market prices of options. In spite that a limited number of market

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prices may be insufficient to recover the entire volatility curve, Dupire’s equation provides very useful insights into the inverse problem of calibrating the local volatility model which has been extensively studied (e.g., Bouchouev and Isakov (1997,1999), Jiang et al. (2003), Egger and Engl (2005), Jiang and Bian (2012)).

The local volatility model inherits some nice properties from the Black-Scholes model, such as the law of unique price and perfect replication. However, Dumas et al. (1998) find that the out-of-sample performance of deterministic volatility models is poor. Empirical evidences by Brockman and Chowdhury (1997) and Buraschi and Jackwerth (2001) show that volatility is nondeterministic. This motives us to instead consider stochastic volatility models.

There is an extensive literature devoted to stochastic volatility models\(^1\), among which there are some named models, including Hull and White model (1987), Stein and Stein model (1991), Heston model (1993), etc. All of these models assume special structure of the volatility (or variance) process so as to obtain analytical pricing formulas for European vanilla options, which simplifies model calibration.

In this paper we consider calibration of general stochastic volatility models and aim to identify, through market prices of options, the risk neutral drift term of the volatility (or variance) process. The reason we only calibrate the drift term is that the volatility of the volatility (or variance) process does not change from the real world to the risk neutral world, whereas the drift term does. Thus the volatility of the volatility (or variance) process can be estimated by historical underlying stock prices whereas the risk-neutral drift term must be recovered from options market.

In contrast to standard literature, we assume that the risk neutral drift term of the volatility (or variance) process is a deterministic function of instantaneous volatility (or variance) and time and does not possess any special structure. As a consequence, analytical price formulas European options are unavailable. We shall formulate the calibration problem as a standard inverse problem of partial differential equations (PDEs) that can be attacked by an optimal control approach.

The contributions of the paper are summarized as follows. First, using the Dupire’s equation associated with stochastic volatility models, we formulate the calibration problem as a standard inverse problem of PDEs. Second, we solve the inverse problem in terms of an optimal control approach with Tikhonov regularization. We derive a necessary condition that the optimal solution satisfies. We further reduce the necessary condition, which plays a critical role in algorithm design. Third, by the reduced necessary condition, we propose a gradient descent algorithm to numerically

\(^1\)Note that for the stochastic volatility model, the law of unique price loses effect and perfect replication is no longer possible.
find the optimal solution. We highlight that our algorithm applies to general stochastic volatility models. Last but not least, we conduct an extensive numerical analysis to demonstrate the efficiency of our numerical algorithm. Moreover, by virtue of real market data, we reveal that the risk neutral drift term of the variance process of S&P 500 index is indeed linearly mean-reverting.

The rest of the paper is organized as follows. In Section 2, we formulate the calibration problem as a standard inverse problem of PDEs. In Section 3, we solve the inverse problem by an optimal control approach with the Tikhnonov regularization technique. We derive a necessary condition of the control problem by which we propose a numerical algorithm. Numerical and empirical results are presented in Section 4. We conclude in Section 5.

2 Problem Formulation

2.1 Calibration Problem

Let $X_t$ be the price process of the underlying stock. Under the risk-neutral world,

$$\frac{dX_t}{X_t} = (r-q)dt + \sigma_t dW_t^0,$$

$$\sigma_t = f(Y_t),$$

$$dY_t = b(Y_t, t)dt + \beta(Y_t)dW_t^1,$$

where $r, q$ represent the riskfree rate and the dividend yield of the underlying, respectively, $W_t^0$ and $W_t^1$ are standard 1-dimensional Brownian motions with constant correlation coefficient $\rho$, i.e., $dW_t^0 \cdot dW_t^1 = \rho dt$. We assume that $b(\cdot, \cdot)$ and $\beta(\cdot)$ are deterministic functions, and $Y_t$ is either the volatility (i.e. $f(Y_t) = Y_t$) or the variance (i.e. $f(Y_t) = \sqrt{Y_t}$) of the underlying.

Consider a European call option with strike price $K$ and maturity $T$. The option price, denoted by $V(x, y, t; K, T)$, satisfies the following equation:

$$\partial_t V + \mathcal{L}V = 0 \quad \text{in } x > 0, y > 0, t < T \quad (2.1)$$

with the terminal condition

$$V|_{t=T} = (x - K)^+, \quad (2.2)$$

where

$$\mathcal{L}V \equiv \frac{1}{2} x^2 f^2(y) \partial_{xx} V + \rho x f(y) \beta(y) \partial_{xy} V + \frac{1}{2} \beta^2(y) \partial_{yy} V + (r-q)x \partial_x V + b(y, t) \partial_y V - rV.$$

Apparently $V(x, y, t; K, T)$ depends on $b(y, t)$. 

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Denote the current time, current stock price, and current volatility by \( t^*, x^*, \) and \( \sigma^* \), respectively, and \( y^* \equiv f^{-1}(\sigma^*) \). Suppose we can observe the market prices of options with all maturities and strike prices, denoted by \( V^*(K, T) \). We consider the following calibration problem.

**Problem A:** Find a pair of functions \( b(y, t) \) and \( V(x, y, t; K, T) \) that satisfy (2.1)-(2.2) from market prices of options

\[
V(x^*, y^*, t^*; K, T) = V^*(K, T) \quad \text{for all } K, T.
\]

Note that the observed information \( V^*(\cdot, \cdot) \) is two dimensional, so we require that \( b(\cdot, \cdot) \), to be recovered, be also of two dimension. Regarding \( \beta \) and \( \rho \) that are given, there is no restriction on their functional forms. We use the present ones because it is easier to identify them by statistical approach.

It should be pointed out that Problem A is not a standard inverse problem of PDEs because the state variables of equation (2.1) are \( x, y, t \), but the data \( V^*(K, T) \) observed is with respect to \( K \) and \( T \).\(^2\) To be consistent with the data, we need to derive a PDE with state variables \( K, T \), as Dupire (1994) does.

### 2.2 Formulation as an Inverse Problem

Define \( \psi(x, y, t; K, \bar{y}, T) \) as the fundamental solution to (2.1), that is,

\[
\begin{align*}
\partial_t \psi + L \psi &= 0 \quad \text{in } x > 0, y > 0, t < T \\
\psi|_{t=T} &= \delta(x - K)\delta(y - \bar{y}).
\end{align*}
\]

From the well known property of fundamental solution, \( \psi(x^*, y^*, t^*; K, \bar{y}, T) \), as a function of \( K, \bar{y}, \) and \( T \), satisfies the adjoint equation of (2.1):

\[
\begin{align*}
\partial_T \psi - L^* \psi &= 0 \quad \text{in } K > 0, \bar{y} > 0, T > t^* \\
\psi|_{T=t^*} &= \delta(x^* - K)\delta(y^* - \bar{y}),
\end{align*}
\]

where

\[
L^* \psi \equiv \frac{1}{2} \partial_{KK}(K^2 f^2(\bar{y}) \psi) + \partial_{K\bar{y}}(\rho K f(\bar{y}) \beta(\bar{y}) \psi)
\]

\[
+ \frac{1}{2} \partial_{\bar{y}\bar{y}}(\beta^2(\bar{y}) \psi) - (r - q) \partial_K(K \psi) - \partial_{\bar{y}}(b(\bar{y}, T) \psi) - r \psi.
\]

\(^2\)Lagnado and Osher (1997) directly solve a non-standard inverse problem arising from calibration of the local volatility model. However their algorithm is rather complicated, as compared with that of Bouchouev and Isakov (1997) who use the Dupire’s equation.
In what follows, we suppress the dependence on $x^*, y^*, t^*$ for notational simplicity, namely,

$$
\psi(K, \bar{y}, T) \equiv \psi(x^*, y^*, t^*; K, \bar{y}, T),
$$
$$
V(K, T) \equiv V(x^*, y^*, t^*; K, T).
$$

The following proposition plays a critical role in formulating Problem A as a standard inverse problem.

**Proposition 1.** Let $\psi$ be the fundamental solution as defined above. Then $V(K, T)$ satisfies

$$
\partial_T V + (r - q)K \partial_K V + qV = \frac{1}{2}K^2 \int_0^\infty f^2(\bar{y})\psi(K, \bar{y}, T)d\bar{y} \quad \text{in } K > 0, \ T > t^* \quad (2.5)
$$

with the initial condition

$$
V|_{T=t^*} = (x^* - K)^+ . \quad (2.6)
$$

Furthermore, (2.5)-(2.6) permits an analytic solution:

$$
V(K, T) = e^{-q(T-t^*)} \left(x^* - e^{-(r-q)(T-t^*)}K\right)^+ + \frac{K^2}{2} \int_{t^*}^T \int_0^\infty e^{(-2r+q)(T-\tau)}f^2(\bar{y})\psi(Ke^{-(r-q)(T-\tau)}, \bar{y}, \tau)d\bar{y}d\tau. \quad (2.7)
$$

A proof of the proposition is in Appendix A.

We call Eq. (2.5) the modified Dupire’s equation associated with stochastic volatility models. We would like to relate the equation to the following result obtained by Derman and Kani (1998):

$$
\sigma^2_{\text{loc}}(K, T) = \mathbb{E} \left[ \sigma_T^2 \mid X_T = K \right], \quad (2.8)
$$

where

$$
\sigma^2_{\text{loc}}(K, T) = \partial_T V + (r - q)K \partial_K V + qV. \quad (2.9)
$$

Indeed, combining (2.8) with (2.9), we have

$$
\partial_T V + (r - q)K \partial_K V + qV = \frac{1}{2}K^2 \partial_{KK} V \mathbb{E} \left[ \sigma_T^2 \mid X_T = K \right]. \quad (2.10)
$$

By definition,

$$
\mathbb{E} \left[ \sigma_T^2 \mid X_T = K \right] = \int_0^\infty f^2(\bar{y}) \frac{\psi(K, \bar{y}, T)}{\int_0^\infty \psi(K, \bar{y}, T)d\bar{y}}d\bar{y}.
$$

It is worthwhile pointing out [see (A.2) in Appendix]

$$
\partial_{KK} V = \int_0^\infty \psi(K, \bar{y}, T)d\bar{y}. \quad (2.11)
$$
Combination of (2.10)-(2.11) yields the modified Dupire’s equation (2.5).

Clearly $\psi(\cdot, \cdot, \cdot)$ depends on $b(\cdot, \cdot)$, so does $V(\cdot, \cdot)$ via $\psi(\cdot, \cdot, \cdot)$. To emphasize the dependence on $b(\cdot, \cdot)$, we use the notations $\psi = \psi(\cdot, \cdot, \cdot; b)$ and $V = V(\cdot, \cdot; b)$. By Proposition 1, we can reformulate Problem A as follows.

**Problem B**: Find a triple of functions $b(\cdot, \cdot)$, $\psi(K, \overline{y}, T; b)$, and $V(K, T; b)$ that satisfy (2.3)-(2.4) and (2.7) from market prices of options

$$V(K, T; b) = V^*(K, T) \quad \text{for all } K, T.$$  

### 3 An Optimal Control Approach

In this section, we use an optimal control approach with the Tikhonov regularization technique [cf. Tikhonov et al. (1995)] to solve Problem B.

Let $t^* = T_0 < T_1 < T_2 < \cdots < T_N$ with $T_n = T_0 + nh$ and $h = (T_N - T_0)/N$. Given $b_0(\cdot) \equiv b(\cdot, T_0) \in B = H^1(\mathbb{R}^+) \equiv \{ v \in L^2(\mathbb{R}^+) : \partial_y v \in L^2(\mathbb{R}^+) \}$, we inductively construct two sequences of cost functional $J_n$ and optimal solution $b_n(\cdot)$, $n = 1, 2, \cdots, N$. That is, for each $n$, given $b_0(\cdot), b_1(\cdot), \cdots, b_{n-1}(\cdot) \in B$, we introduce the cost functional

$$J_n(\bar{b}) = \frac{\epsilon_1}{2} \| \bar{b} - b_{n-1} \|^2_{L^2(\mathbb{R}^+)} + \frac{\epsilon_2}{2} \| \partial_y \bar{b}(\overline{y}) \|^2_{L^2(\mathbb{R}^+)} + \frac{1}{2h^2} \| V_h(\cdot; T_n; \bar{b}^h) - V^*(\cdot, T_n) \|^2_{L^2(\mathbb{R}^+)}$$  

(3.1)

for $\bar{b} \in B$, where $\epsilon_1, \epsilon_2 > 0$ are two regularization parameters, and $V_h(\cdot, \cdot; \bar{b}^h)$ is as given by (2.7) with the coefficient

$$\bar{b}^h(\overline{y}, T) \equiv \begin{cases} \frac{T-T_n}{T-T_0} \bar{b}(\overline{y}) + \frac{T_n-T}{T_0-T} b_{n-1}(\overline{y}), & T_{n-1} \leq T \leq T_n \\ \frac{T-T_n}{T-T_0} b_k(\overline{y}) + \frac{T_n-T}{T_0-T} b_{k-1}(\overline{y}), & T_{k-1} \leq T \leq T_k \end{cases} \quad 1 \leq k \leq n-1.$$  

(3.2)

We aim to solve the following control problem: given $b_0(\cdot), b_1(\cdot), \cdots, b_{n-1}(\cdot) \in B$, find $b_n(\cdot) \in B$ such that

$$J_n(b_n) = \inf_{\bar{b} \in B} J_n(\bar{b}).$$  

(3.3)

The first two terms on the right hand side of (3.1) are the so-called Tikhonov regularization which are widely utilized to handle ill posed problems. It is worthwhile pointing out that the term $\frac{\epsilon_2}{2h^2} \| \partial_y \bar{b}(\overline{y}) \|^2_{L^2(\mathbb{R}^+)}$, first introduced by Jiang and Bian (2012) to recover the time-dependent local volatility, enables us to achieve certain regularity of $b(\cdot, \cdot)$ in $T$ which is needed for convergence analysis. However, for numerical experiments presented in Section 4, this term is actually removed and we find that it does not cause any oscillation.

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For ease of presentation, we assume a regular partition in $[t^*, T_N]$. 

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6
3.1 A Necessary Condition

Let us derive the necessary condition for optimality of problem (3.3). Assume that \( b_n(\cdot) \) is the optimal solution of (3.3). For any \( \omega \in B \), we denote \( b_\lambda = b_n + \lambda \omega \in B \), \( \lambda \in \mathbb{R} \), and define

\[
j(\lambda) = J_n(b_\lambda), \quad \lambda \in \mathbb{R}.
\]

Note that \( j(\lambda) \) reaches its minimum at \( \lambda = 0 \). It follows

\[
j'(0) = \frac{d}{d\lambda} J_n(b_\lambda)\big|_{\lambda=0} = 0,
\]

from which we can derive the following necessary condition.

**Proposition 2.** Let \( b_n \) be the minimizer of problem (3.3). Let \( b^h(\cdot, \cdot) \) be as given in (3.2) with \( b_n \) in place of \( \bar{b} \), and let \( \psi_h(K, \bar{y}, T) \) and \( V_h(K, T) \) be the solution to (2.3)-(2.4) and as given by (2.7), respectively, where the coefficient \( b(\cdot, \cdot) \) is replaced by \( b^h(\cdot, \cdot) \). Then \( b_n \) satisfies the following variational problem

\[
\int_0^\infty \left( \epsilon_1 \frac{b_n - b_{n-1}}{h} \cdot \omega + \epsilon_2 \partial_\tau b_n \cdot \partial_\tau \omega 
- \frac{1}{h^2} \int_{T_{n-1}}^{T_n} \int_0^\infty \frac{\tau - T_{n-1}}{h} g_h(K, \bar{y}, \tau) \partial_\tau \psi (\bar{y}(K, \bar{y}, \tau) \cdot \omega) \, d\tau \, dK \right) \, d\bar{y} = 0,
\]

for any \( \omega \in B \), where \( g_h(K, \bar{y}, \tau) \) satisfies

\[
\partial_\tau g_h + \mathcal{L}_h g_h = -\frac{K^2}{2} \left[ V_h(K e^{(r-q)(T_n-\tau)}; T_n) - V^*(K e^{(r-q)(T_n-\tau)}; T_n) \right] e^{(r-2q)(T_n-\tau)} f^2(\bar{y}) \quad (3.6)
\]

\[
K > 0, \bar{y} > 0, \tau \in [T_{n-1}, T_n),
\]

where \( \mathcal{L}_h \) is the operator \( \mathcal{L} \) with coefficient \( b^h(\cdot, \cdot) \) in place of \( b(\cdot, \cdot) \).

The proof is in Appendix B.

It is somewhat time consuming to find numerical solutions of (3.5)-(3.6) because (3.6) is a two dimensional time-dependent problem. To simplify computations, we send \( h \to 0 \) to obtain the following limiting equation of (3.5):

\[
\int_0^T d\tau \int_0^\infty \left\{ \epsilon_1 \partial_\tau b \cdot \omega + \epsilon_2 \partial_\tau b \cdot \partial_\tau \omega 
+ \frac{1}{2} f(\bar{y}) f'(\bar{y}) \int_0^\infty K^2 [V(K, \tau; b) - V^*(K, \tau)] \psi (K, \bar{y}, \tau; b) \, dK \cdot \omega \right\} \, d\bar{y} = 0 \quad (3.7)
\]
for any $\omega(\bar{y}, \tau) \in C_0^\infty((0, +\infty) \times (t^*, +\infty))$, where $b = b(\bar{y}, t), \psi(K, \bar{y}, T; b)$ and $V(K, T; b)$ are the solution to (2.3)-(2.4) and as given by (2.7), respectively. The derivation of (3.7) is in Appendix C.

(3.7) can be regarded as the necessary condition of the optimal control problem in continuous time. It is apparent that (3.7) is the weak formulation of the following equation:

$$
\epsilon_1 \partial_T b - \epsilon_2 \partial_y b + \frac{1}{2} f(\bar{y}) f'(\bar{y}) \int_0^\infty K^2[V(K, T) - V^*(K, T)] \psi(K, \bar{y}, T; b) \, dK = 0
$$

in $\bar{y} > 0, T > t^*$. We will design an algorithm based on a discretization of (3.8) to find the optimal solution.\footnote{We also tried solving (3.5)-(3.6) and obtained almost the same numerical results, though it was relatively time-consuming.}

3.2 Numerical Algorithm

We have seen that the optimal solution is determined by a system of equations (2.3)-(2.4), (2.7), and (3.8) with certain initial/boundary conditions. This requires the numerical solution of the system which can be accomplished using a finite difference method (see e.g., Quarteroni and Valli (1994)). Due to stability concern, we use the implicit finite difference method which requires that the solution domain be truncated into a bounded domain:

$$
K \in [K_{\text{min}}, K_{\text{max}}], \bar{y} \in [0, y_{\text{max}}], t \in [t^*, T_N].
$$

Appropriate boundary conditions will be prescribed.

To describe our algorithm, we use semi-discretization schemes. Let us choose $h > 0$ as the step size of time and let $t^* = T_0 < T_1 < T_2 < \cdots < T_N$ be the partition of $[t^*, T_N]$ with $T_n = T_0 + nh$. Denote

$$
b_n(\cdot) \equiv b(\cdot, T_n), \quad \psi_n(\cdot, \cdot) = \psi(\cdot, \cdot, T_n), \quad V_n(\cdot) = V(\cdot, T_n).
$$

Suppose $b_0(\cdot)$ is given with $\psi_0(K, \bar{y}) = \delta(x^* - K)\delta(y^* - \bar{y}), V_0(K) = (x^* - K)^+$. We inductively solve for $b_n(\cdot)$ as well as $\psi_n(\cdot, \cdot)$ and $V_n(\cdot)$, $n = 1, 2, \cdots, N$ through an implicit discretization of the system. Because (3.8) is nonlinear, an iterative procedure is required at each time step. Let $b_{n,k}$, $\psi_{n,k}$, and $V_{n,k}$ be the values at the $k$-th iteration of the $n$-th time step. The iteration procedure for the $n$-th time step is given as follows.

**Iteration procedure for the $n$-th time step:**

1. Choose a tolerance $\varepsilon > 0$ and an initial guess

$$
b_{n,0} : b_{n-1}(\cdot), \quad \psi_{n,0}(\cdot, \cdot) = \psi_{n-1}(\cdot, \cdot), \quad V_{n,0}(\cdot) = V_{n-1}(\cdot),
$$
and set $k = 1$.

2. Solve for $b_{n,k}$ in terms of an implicit discretization of (3.8):

$$
\epsilon_1 \frac{b_{n,k} - b_{n-1}}{h} - \epsilon_2 \partial_y b_{n,k} + F(\bar{y}, \psi_{n,k-1}, V_{n,k-1}) = 0 \text{ for } \bar{y} \in (0, y_{\text{max}}),
$$

(3.9)

where

$$
F(\bar{y}, \psi_{n,k-1}, V_{n,k-1}) = \frac{1}{2} f(\bar{y}) f'(\bar{y}) \int_{K_{\text{min}}}^{K_{\text{max}}} K^2 (V_{n,k-1}(K) - V^*(K, T_n)) \psi_{n,k-1}(K, \bar{y}) dK,
$$

and either Dirichlet or Neumann boundary conditions are given at $\bar{y} = 0, Y_{\text{max}}$ and are elaborated in the next section.

3. Use the updated $b_{n,k}$ to solve for $\psi_{n,k}(\cdot, \cdot)$ in terms of an implicit discretization of (2.3):

$$
\frac{\psi_{n,k} - \psi_{n-1}}{h} - \mathcal{L}^* \psi_{n,k} = 0 \text{ for } K \in (K_{\text{min}}, K_{\text{max}}), \bar{y} \in (0, y_{\text{max}})
$$

with boundary conditions

$$
\psi_{n,k}(K_{\text{min}}, \cdot) = \psi_{n,k}(K_{\text{max}}, \cdot) = 0,
$$

$$
\psi_{n,k}(\cdot, 0) = \psi_{n,k}(\cdot, y_{\text{max}}) = 0,
$$

where the risk-neutral drift term of volatility process appearing in $\mathcal{L}^*$ takes the form of linear interpolation

$$
b(\cdot, T) \equiv \frac{T - T_{n-1}}{h} b_{n,k}(\cdot) + \frac{T_n - T}{h} b_{n-1}(\cdot), \ T_{n-1} \leq T \leq T_n.
$$

4. Find $V_{n,k}(\cdot)$ by the updated $\psi_{n,k}(\cdot)$ and a variation of (2.7)

$$
V_{n,k}(K) = e^{-qh} V_{n-1}(e^{-(r-q)h} K)
$$

$$
+ \frac{K^2}{2} \int_{T_{n-1}}^{T_n} e^{-2(r+q)(T_n - \tau)} \int_0^\infty f^2(\bar{y}) \psi(K e^{-(r-q)(T_n - \tau)}, \bar{y}, \tau) d\bar{y} d\tau.
$$

Using a simple numerical integration, we have

$$
V_{n,k}(K) = e^{-qh} V_{n-1}(e^{-(r-q)h} K)
$$

$$
+ \frac{K^2}{2} \int_0^\text{Y_{max}} \frac{h}{2} f^2(\bar{y}) \left[ e^{-(r+q)h} \psi_{n-1}(K e^{-(r-q)h}, \bar{y}) + \psi_{n,k}(K, \bar{y}) \right] d\bar{y}.
$$

5. Let $b_n = b_{n,k}$, $\psi_n = \psi_{n,k}$, $V_n = V_{n,k}$ and stop if $\| b_{n,k}(\cdot) - b_{n,k-1}(\cdot) \| < \varepsilon$. Otherwise set $k = k + 1$ and go to Step 2.

5Here we use the fully implicit scheme, so only $b_{n,k}$ is actually used. The form of interpolation will be needed when the Crank-Nicolson scheme is adopted.
To make the above iteration more smoothly, we introduce a ‘relaxed’ time scale $\kappa$ at Step 2 of the above procedure. That is, we replace (3.9) by

$$\frac{b_{n,k} - b_{n,k-1}}{\kappa} + \epsilon_1 \frac{b_{n,k} - b_{n-1}}{h} - \epsilon_2 \partial_{y}b_{n,k} + F(\bar{y}, \psi_{n,k-1}, V_{n,k-1}) = 0.$$  

(3.10)

Note that (3.10) can be rewritten as

$$b_{n,k} = b_{n,k-1} - \kappa \left[ \epsilon_1 \frac{b_{n,k} - b_{n-1}}{h} - \epsilon_2 \partial_{y}b_{n,k} + F(\bar{y}, \psi_{n,k-1}, V_{n,k-1}) \right].$$ 

(3.11)

Recalling the derivation procedure of the necessary condition, we can see that the last term $[\cdot]$ on the right hand side of (3.11) is essentially the Frechet derivative of the cost functional $J_n(\cdot)$. Hence, this is equivalent to employing the steepest descent method to find the minimizer of the optimization problem (3.3). To ensure stability, we still use an implicit scheme to solve (3.10).

A natural question is how to choose the values of regularity parameters $\epsilon_1, \epsilon_2$. Theoretically the smaller $\epsilon_1, \epsilon_2$, the more accurate the solutions. However decreasing the values of $\epsilon_1, \epsilon_2$ may cause numerical oscillation. Luckily our numerical experiments reveal that we can always choose $\epsilon_1 = 0$ which nonetheless does not incur any oscillation. The value of $\epsilon_2$ varies for different examples but is also very small.

The remaining problem is how to choose the initial guess $b_0(\cdot)$. By numerical experiments, we find that owing to $\epsilon_1 = 0$, the recovered $b(\cdot, \cdot)$ is insensitive to $b_0(\cdot)$. Hence, for all experiments presented in the next section, we always choose $b_0(\cdot) \equiv 0$.

4 Numerical Experiments and Market Tests

To test our algorithm, we consider an extended Hull-White model:

$$dX_t = (r - q)dt + \sigma_t dW^0_t,$$

$$\sigma_t = \sqrt{Y_t},$$

$$dY_t = b(Y_t, t)dt + \gamma Y_t dW^1_t$$

with $dW^0_t dW^1_t = \rho dt$. That is,

$$f(y) = \sqrt{y}, \quad \beta(y) = \gamma y.$$

Here $\rho$ and $\gamma$ are constants that can be estimated from historical data of the underlying. We implement the algorithm presented in Section 3.2 to recover $b(\cdot, \cdot)$.

---

6The reason we are interested in the Hull-White model is that it is consistent with the GARCH(1,1) model. In the extended Hull-White model considered here, the risk-neutral drift term of the variance process is a general function of instantaneous variance and time. We highlight that our approach applies to other stochastic volatility models.
We use a uniform mesh with $N_K$ subintervals in $[K_{min}, K_{max}]$ and $N_y$ subintervals in $[0, y_{max}]$, that is,

$$K_i = K_{min} + i \frac{K_{max} - K_{min}}{N_K}, \quad i = 0, 1, ..., N_K,$$

$$y_j = j \frac{y_{max}}{N_y}, \quad j = 0, 1, ..., N_y.$$

### 4.1 Numerical Experiments

To demonstrate the efficiency of our algorithm, we first carry out tests with exact solutions. Suppose the exact solution is $b^*(\cdot, \cdot)$ with which we solve (2.1)-(2.2) to get $V(x^*, y^*, t^*; K_i, T_n)$ for $i = 0, 1, ..., N_K; n = 0, 1, ..., N$. We then treat $V^*(K_i, T_j) \equiv V(x^*, y^*, t^*; K_i, T_j)$ as the market prices of options which are used to recover $b(\cdot, \cdot)$ through the algorithm presented in Section 3.2.

At Step 2 of the iteration procedure, the boundary conditions for $b_{n,k}$ are needed. We then use the exact boundary conditions. Both Dirichlet conditions and Neumann conditions are tested respectively, that is,

- **Dirichlet conditions:** $b_{n,k}(0) = b^*(0, T_n)$, $b_{n,k}(y_{max}) = b^*(y_{max}, T_n)$;
- **Neumann conditions:** $\partial_y b_{n,k}(0) = \partial_y b^*(0, T_n)$, $\partial_y b_{n,k}(y_{max}) = \partial_y b^*(y_{max}, T_n)$.

The default parameter values are given as follows:

$$x^* = 1, \quad y^* = 0.02, \quad t^* = 0, \quad K_{min} = 0.5, \quad K_{max} = 1.5, \quad y_{max} = 0.1, \quad N_K = 50, \quad N_y = 50,$$

$$r = 0.02, \quad q = 0, \quad \gamma = 0.1, \quad \rho = 0.5, \quad \varepsilon = 2 \times 10^{-9}, \quad \epsilon_1 = 0, \quad \epsilon_2 = 4 \times 10^{-5}, \quad \kappa = 2 \times 10^{-6}.$$

**Test 1.** $b^* = b^*(\bar{y})$.

We first consider a simple case in which $b^*$ is independent of time. Then we only need the prices of options with one maturity $T_1$ to do calibration. So, $N = 1$, and we take $T_1 = 1/12$.

We test three kinds of profile for $b$, corresponding constant, linear, and cubic dependence on $\bar{y}$, respectively.

1. $b^*(\bar{y}) = 0.2$;
2. $b^*(\bar{y}) = 2(0.08 - \bar{y})$;
3. $b^*(\bar{y}) = -(\bar{y} - 0.05)^3$.

For all three cases we use $b_0(\cdot) \equiv 0$ as an initial guess. The results are presented in Figure 1, Figure 2, and Figure 3, respectively. It can be observed that the recovered values coincide with the exact values perfectly.
Figure 1: Recovery of constant drift term. The left figure is with Dirichlet conditions while the right is with Neumann conditions. The dotted line is the initial guess \( b_0(\cdot) \equiv 0 \), the solid line is the exact drift term \( b^*(\cdot) = 0.2 \), and the circles are the drift term \( b(\cdot) \) recovered by our algorithm from option prices that are generated with the exact drift term.

**Test 2.** \( b^* = b^*(\bar{y}, T) \)

Now we examine the time-dependent case in which only Neumann conditions are given for (3.10). We need to use three months data, so we take \( N = 3 \), \( T_1 = \frac{1}{12} \), \( T_2 = \frac{2}{12} \), \( T_3 = \frac{3}{12} \).

The exact value \( b^*(\bar{y}, T) = -\frac{(\bar{y} - 0.05)^2}{1 + 2T} \) with which we generate the option prices as the market prices. We again use the initial value \( b_0(\cdot) \equiv 0 \) and Neumann boundary conditions, and the results are shown in Figure 4. We can see that our algorithm still performs very well.

**Test 3. Stability with data**

To investigate the stability of our algorithm, we add a white noise to the ‘observed’ data. The whole procedure is elaborated below. We consider the case in Test 1 with the exact risk-neutral drift term \( b^*(\bar{y}) = 2(0.08 - \bar{y}) \). As before we solve (2.1)-(2.2) with \( b^*(\cdot) \) to get \( V(x^*, y^*, t^*; K_i, T_n) \).

For any \( i = 0, 1, \ldots, N_K \), \( n = 0, 1, \ldots, N \), we let

\[
V^*(K_i, T_n) = \left(1 + \frac{\xi_{i,n}}{50}\right) V(x^*, y^*, t^*; K_i, T_n),
\]

where \( \xi_{i,n} \) is a random variable drawn from the standardized normal distribution. Then we use \( V^*(K_i, T_n) \) to do calibration. The results, presented in Figures 5, verify the stability of our algorithm.
Figure 2: Recovery of mean-reverting drift term. The left figure is with Dirichlet conditions while the right is with Neumann conditions. The dotted line is the initial guess $b_0(\cdot) \equiv 0$, the solid line is the exact drift term $b^*(\bar{y}) = 2(0.08 - \bar{y})$, and the circles are the drift term $b(\cdot)$ recovered by our algorithm from option prices that are generated with the exact drift term.

### 4.2 Market Tests

We now test the performance of our algorithm with real market data. Here the underlying asset is SPY, an exchange traded fund (ETF) on SP 500 index. To calibrate the extended Hull-White model, we use the quoted prices of call options on SPY with maturities $T_1 = 28/09/2012$, $T_2 = 20/10/2012$, $T_3 = 17/11/2012$, which are documented in Appendix D.\(^7\) All quoted prices are of $t^* = 30/08/2012$, and the SPY price on the day is $x^* = 140.49$. The default parameter values are given as follows.

\[
\begin{align*}
y^* &= 0.024, \quad K_{\min} = 0.4x^*, \quad K_{\max} = 1.4x^*, \quad y_{\max} = 0.1, \\
N_K &= 100, \quad N_y = 50, \quad r = 0.005, \quad q = 0.03, \quad \gamma = 0.1, \quad \rho = -0.5, \\
\epsilon_1 &= 0, \quad \epsilon_2 = 5 \times 10^{-4}x^2, \quad \varepsilon = 10^{-4}, \quad \kappa = 10^{-2}.
\end{align*}
\]

We first use the prices of the options with maturity $T_1$ to recover the time-independent $b(\cdot)$, where one time step is needed, i.e., $N = 1$. As before we need to prescribe the boundary conditions for $b_{n,k}$ in Step 2 of the iteration procedure. To examine the effect of boundary conditions on calibrations, we consider the following three (Neumann) boundary conditions:

1. $\partial_y b_{1,k}(0) = -2$, $\partial_y b_{1,k}(y_{\max}) = -2$;
2. $\partial_y b_{1,k}(0) = -3$, $\partial_y b_{1,k}(y_{\max}) = -3$;
3. $\partial_y b_{1,k}(0) = -4$, $\partial_y b_{1,k}(y_{\max}) = -4$.

\(^7\)Bid and ask prices are recorded, and we use their average as market prices for calibration.
Figure 3: Recovery of cubic drift term. The left figure is with Dirichlet conditions while the right is with Neumann conditions. The dotted line is the initial guess \( b_0(\cdot) \equiv 0 \), the solid line is the exact drift term \( b^*(\bar{y}) = -(\bar{y} - 0.05)^3 \), and the circles are the drift term \( b(\cdot) \) recovered by our algorithm from option prices that are generated with the exact drift term.

3. \( \partial_{\bar{y}} b_{1,k}(0) = -3, \partial_{\bar{y}} b_{1,k}(y_{max}) = -2 \).

For all cases we use the initial guess \( b_0(\cdot) \equiv 0 \). The recovered results are reported in Figures 6-8. In each figure, the left shows the recovered \( b(\cdot) \) while the right presents the market prices of options used for calibration (dots) and the option prices generated from the recovered \( b(\cdot) \) (solid line). We observe that the boundary conditions significantly affect the recovered \( b(\cdot) \). However, for all cases, the option prices generated from the recovered \( b(\cdot) \) coincide with the market prices very well. This implies that the solution to the calibration problem is not unique. Moreover, using a least-square fitting, we can fit the recovered \( b(\cdot) \) by a straight line \( \alpha(\theta - \bar{y}) \), where \( (\alpha, \theta) = (1.997, 0.007), (2.998, 0.012), (2.308, 0.105) \) for the above three boundary conditions, respectively. This indicates that the calibrated risk-neutral variance process is linearly mean-reverting.

Now we further use the real market data on \( T_2 \) and \( T_3 \) to calibrate \( b(\cdot, \cdot) \), with boundary conditions \( \partial_{\bar{y}} b_{n,k}(0) = -3, \partial_{\bar{y}} b_{n,k}(y_{max}) = -3 \). The calibration results are shown in Figure 9, where the left presents calibrated \( b(\cdot, T_i), i = 1, 2, 3 \), and the right presents the market option prices (dots) and the option prices generated from the recovered \( b(\cdot, \cdot) \) (solid line). Again we can observe that our algorithm performs very well. The recovered \( b(\cdot, T_i) \) can be fitted by straight lines \( \alpha_i(\theta_i - \bar{y}) \).
Figure 4: Recovery of time-varying drift term. Neumann conditions are given, and the initial guess $b_0(\cdot) \equiv 0$. The dotted, solid, and dashed lines are the exact drift term $b^*(\bar{y}, T) = -(\bar{y} - 0.05)^3/(1 + 2T)$ with $T = 1/12, 2/12, 3/12$, respectively, and the circles, dots, and plus are the drift terms $b(\cdot, 1/12), b(\cdot, 2/12), b(\cdot, 3/12)$ recovered by our algorithm from option prices generated with the exact drift term.

with

\[
\alpha_1 = 2.998, \quad \theta_1 = 0.012, \\
\alpha_2 = 2.990, \quad \theta_2 = 0.093, \\
\alpha_3 = 2.978, \quad \theta_3 = 0.091,
\]
respectively.

5 Conclusion

In this article, we propose an optimal control approach to recover, from the options market, the risk-neutral drift term of the volatility or variance process in the stochastic volatility model. In contrast to the existing literature, the drift term does not possess any special structure and analytical pricing formulas for European options are unavailable.

We first present a modified Dupire’s equation associated with stochastic volatility models, which allows us to formulate the calibration problem as a standard inverse problem of partial differential equations. We then use an optimal control approach with Tikhonov regularization to solve the inverse problem. A necessary condition that the optimal solution satisfies is derived.
Figure 5: Stability analysis of the algorithm. The exact drift term is $b^*(\bar{y}) = 2(0.08 - \bar{y})$. The left figure shows the calibration results, where the dotted line is the initial guess $b_0(\cdot) \equiv 0$, the solid line is the exact drift term, and the circles are the drift term $b(\cdot)$ recovered by our algorithm from option prices generated with the exact drift term and white ‘noise’. The right figure presents the option prices generated with the exact drift term (solid line), the prices with white ‘noise’ (dots), and the prices generated with the recovered drift term (circles).

We further simplify the necessary condition and then propose a gradient descent algorithm to numerically find the optimal solution. Our algorithm can be applied to calibrate general stochastic volatility models. An extensive numerical analysis is presented to demonstrate the efficiency of our numerical algorithm. It is worthwhile pointing out that the boundary conditions required in the algorithm significantly affect the calibration results, but in all cases the option prices generated from the stochastic volatility model with the recovered drift term coincide very well with the market prices of options. Moreover, we find that the variance process of S&P 500 index recovered from the options market is indeed linearly mean-reverting.

Appendix A: Proof of Proposition 1

Let us first prove (2.5). The original idea stems from Dai and Wu (2002). Denote

$$u = \partial_{KK} V. \tag{A.1}$$

Differentiating (2.1)-(2.2) w.r.t. $K$ twice, we obtain

$$\partial_t u + \mathcal{L} u = 0 \quad \text{in } x > 0, y > 0, t \in [0, T)$$

$$u|_{t=T} = \delta(x - K).$$

Here $\delta(x - K)$ is the Dirac’s delta function concentrated at $x = K$. Note that $\delta(x - K)$ is a line
source, instead of a point source, since the space is two dimensional. Therefore,

\[ u(x, y, t; K, T) = \int_{0}^{\infty} \psi(x, y, t; K, \bar{y}, T) d\bar{y} \quad (A.2) \]

by which, we integrate (2.3) with respect to \( \bar{y} \) to get

\[
\frac{\partial T}{\partial u} = \frac{1}{2} \frac{\partial K}{\partial K} \left( K^2 \int_{0}^{\infty} f^2(\bar{y}) \psi(x, y, t; K, \bar{y}, T) d\bar{y} \right) + \frac{\partial K}{\partial K} [\rho K f(y) \beta(\bar{y}) \psi(x, y, t; K, \bar{y}, T)]|_{0}^{\infty} \\
+ \frac{1}{2} \frac{\partial K}{\partial K} [\beta^2(\bar{y}) \psi(x, y, t; K, \bar{y}, T)]|_{0}^{\infty} - (r - q) \frac{\partial K}{\partial K} (Ku) \\
- (b(\bar{y}, T) \psi(x, y, t; K, \bar{y}, T))]|_{0}^{\infty} - ru \quad \text{in } K > 0, T > t.
\]

Since \( \psi \) decays fast enough at \( \bar{y} = 0, \infty \), we deduce

\[
\frac{\partial T}{\partial u} = \frac{1}{2} \frac{\partial K}{\partial K} \left( K^2 \int_{0}^{\infty} f^2(\bar{y}) \psi(x, y, t; K, \bar{y}, T) d\bar{y} \right) - (r - q) \frac{\partial K}{\partial K} (Ku) - ru \quad \text{in } K > 0, T > t.
\]

Integrating this equation twice w.r.t. \( K \) and restricting attention to \( x^*, y^*, t^* \), we get (2.5).

Given \( \psi \), (2.5) is a first-order linear equation. Now let us find the analytical solution of (2.5) subject to terminal condition (2.6). By transformation

\[ H(Z, T) = e^{qT} V(K, T), \quad Z = \log K, \]

we have

\[
\frac{\partial H}{\partial T} + (r - q) \frac{\partial H}{\partial Z} = \frac{1}{2} e^{qT} e^{2Z} \int_{0}^{\infty} f^2(\bar{y}) \psi(e^Z, \bar{y}, T) d\bar{y}
\]

Figure 6: Market test with boundary conditions \( \partial_y b_{1,k}(0) = -2 \), \( \partial_y b_{1,k}(y_{\text{max}}) = -2 \). The quoted prices of call options on SPY with maturity \( T_1 = 28/09/2012 \) are used. The current time \( t^* = 30/08/2012 \), and the SPY price on the day is 140.49. The left figure presents the calibrated \( b(\cdot) \) while the right figure presents the market prices of options used for calibration (circles) and the option prices generated from the recovered \( b(\cdot) \) (solid line). The recovered \( b(\cdot) \) can be fitted by a straight line \( \alpha(\theta - \bar{y}) \) with \( \alpha = 1.997 \) and \( \theta = 0.007 \).
in $Z \in (-\infty, \infty), T > t^\star$. Consider the characteristic line $Z(T) = Z(t^\star) + (r - q)T$ along which we have
\[
\frac{dH(Z(T), T)}{dT} = \frac{1}{2} e^{\gamma T} e^{2Z(T)} \int_0^\infty f^2(\gamma) \psi(e^{Z(T)}, \gamma, T) d\gamma.
\]
It follows
\[
H(Z(T), T) = H(Z(t^\star), t^\star) + \frac{1}{2} \int_{t^\star}^T e^{\gamma \tau} e^{2Z(\tau)} \int_0^\infty f^2(\gamma) \psi(e^{Z(\tau)}, \gamma, \tau) d\gamma d\tau.
\]
Since $Z(\tau) = Z(T) - (r - q)(T - \tau)$, we have
\[
H(Z(T), T) = H(Z(T) - (r - q)(T - t^\star), t^\star) + \frac{1}{2} \int_{t^\star}^T e^{\gamma \tau} e^{2[Z(T) - (r - q)(T - \tau)]} \int_0^\infty f^2(\gamma) \psi(e^{Z(T) - (r - q)(T - \tau)}, \gamma, \tau) d\gamma d\tau.
\]
Hence,
\[
H(Z, T) = H(Z - (r - q)(T - t^\star), t^\star) + \frac{1}{2} \int_{t^\star}^T e^{\gamma \tau} e^{2[Z - (r - q)(T - \tau)]} \int_0^\infty f^2(\gamma) \psi(e^{Z - (r - q)(T - \tau)}, \gamma, \tau) d\gamma d\tau.
\]
Changing to the original variables gives the desired result.

Appendix B: Proof of Proposition 2
Figure 8: Market test with boundary conditions $\partial_y b_{1,k}(0) = -3$, $\partial_y b_{1,k}(y_{\text{max}}) = -2$. The quoted prices of call options on SPY with maturity $T_1 = 28/09/2012$ are used. The current time $t^* = 30/08/2012$, and the SPY price on the day is 140.49. The left figure presents the calibrated $b(\cdot)$ while the right figure presents the market prices of options used for calibration (circles) and the option prices generated from the recovered $b(\cdot)$ (solid line). The recovered $b(\cdot)$ can be fitted by a straight line $\alpha(\theta - \bar{y})$ with $\alpha = 2.308$ and $\theta = 0.105$.

(3.4) is equivalent to

$$\frac{d}{d\lambda} \left( \int_0^\infty \frac{\epsilon_1}{2h} [b_\lambda - b_{\lambda-1}]^2 + \frac{\epsilon_2}{2} |\partial_y b_\lambda(y)|^2 dy + \frac{1}{2h^2} \int_0^\infty |V_h(K, T_n; b_\lambda^h) - V^*(K, T_n)|^2 dK \right)\bigg|_{\lambda=0} = 0,$$  

(B.1)

where $V_h(K, T_n; b_\lambda^h)$ is as given by (2.7) with the coefficient $b_\lambda^h(\cdot, \cdot) \equiv b^h(\cdot, \cdot) + \lambda \omega(\cdot)$. Setting $\xi(K, T) = \frac{dV_h(K, T, b_\lambda^h)}{d\lambda}|_{\lambda=0}$, we infer from (B.1) that

$$\int_0^\infty \epsilon_1 \frac{b_n - b_{n-1}}{h} \cdot \omega + \epsilon_2 \partial_y b_n(y) \partial_y \omega(y) dy + \frac{1}{h^2} \int_0^\infty (V_h(K, T_n) - V^*(K, T_n)) \xi(K, T_n) dK = 0 \quad (B.2)$$

for any $\omega \in B$ and $T_{n-1} \leq T \leq T_n$. By (2.5), $\xi$ satisfies

$$\begin{cases}
\partial_T \xi + (r - q)K \partial_K \xi + q\xi = \frac{K^2}{2} \int_0^\infty \frac{f^2(y)\eta(K, y, T)}{d\lambda} dy, & T_{n-1} \leq T \leq T_n, \\
\xi(K, T_{n-1}) = 0,
\end{cases}$$

with $\eta(K, y, T) = \frac{d\psi_h(K, y, T; b_\lambda^h)}{d\lambda}|_{\lambda=0}$, where $\psi_h(K, y, T; b_\lambda^h)$ is the solution to (2.3)-(2.4) with the coefficient $b_\lambda^h$. Similar to (2.7), we have an analytical expression form for $\xi$,

$$\xi(K, T) = \int_{T_{n-1}}^T K^2 \frac{2}{e^{-(2r-q)(T-\tau)}} \int_0^\infty f^2(y)\eta(Ke^{-(r-q)(T-\tau)}, y, \tau) \partial_y dy d\tau. \quad (B.3)$$

Note that $\eta(K, y, T)$ is the solution to the following problem:

$$\begin{cases}
\partial_T \eta - \mathcal{L}_h^\kappa \eta = -\partial_y (\psi_h \omega) \frac{T-T_{n-1}}{h}, & T_{n-1} < T \leq T_n, \\
\eta(K, y, T_{n-1}) = 0.
\end{cases}$$
Figure 9: Market test with boundary conditions $\partial^2 b_{n,k}(0) = -3$, $\partial^2 b_{n,k}(y_{max}) = -3$, $n = 1, 2, 3$. The quoted prices of call options on SPY with maturities $T_1 = 28/09/2012$, $T_2 = 20/10/2012$, and $T_3 = 17/11/2012$ are used. The current time $t^* = 30/08/2012$, and the SPY price on the day is 140.49. The left figure presents the calibrated $b(\cdot, T_i)$, $i = 1, 2, 3$ while the right figure presents the market prices of options used for calibration and the option prices generated from the recovered $b(\cdot)$. The recovered $b(\cdot, T_i)$ can be fitted by a straight line $\alpha_i(\theta_i - \overline{y})$ with $(\alpha_1, \theta_1) = (2.998, 0.012)$, $(\alpha_2, \theta_2) = (2.990, 0.093)$, $(\alpha_3, \theta_3) = (2.978, 0.091)$, respectively.

Let $G_h$ be the Green function associated with operator $L^*_h$. By Green’s formula,

$$
\eta(K, \overline{y}, T) = \int_{T_{n-1}}^{T} d\overline{T} \int_{0}^{\infty} d\overline{K} \int_{0}^{\infty} G_h(K, \overline{y}, T; \overline{K}, \overline{y}, \overline{T}) \partial \overline{y}(\overline{y} - \psi) \frac{\overline{T} - T_{n-1}}{h} d\overline{y}. \tag{B.4}
$$

We now compute the third term on the left-hand side of (B.2):

$$
(*) = \int_{0}^{\infty} (V_h(K, T_n) - V^*(K, T_n)) \xi(K, T_n) dK. \tag{B.5}
$$

Plugging (B.3) into (B.5), we have

$$
(*) = \int_{0}^{\infty} (V_h(K, T_n; b^h) - V^*(K, T_n)) \left[ \int_{T_{n-1}}^{T_n} \frac{K^2}{2} e^{(-2r+q)(T_n-\tau)} \int_{0}^{\infty} f^2(\overline{y}) \right. $$

\hspace{1cm} \left. \times \eta(K e^{-(r-q)(T_n-\tau)}, \overline{y}, \tau) d\overline{y} d\tau \right] dK $$

\hspace{1cm} = \int_{0}^{\infty} dK \int_{T_{n-1}}^{T_n} d\tau \int_{0}^{\infty} d\overline{y} \left[ (V_h(K e^{(r-q)(T_n-\tau)}, T_n) - V^*(K e^{(r-q)(T_n-\tau)}, T_n)) \right.$$

\hspace{1cm} \left. \times \frac{K^2}{2} e^{(r-2q)(T_n-\tau)} f^2(\overline{y}) \eta(K, \overline{y}, \tau) \right],
$$

where we have used a change of variable on $K$ in the second equality. Substituting (B.4) into the
above equation, we obtain

\[ (*) = \int_0^\infty dK \int_{T_{n-1}}^{T_n} d\tau \int_0^\infty d\mathcal{G} \left[ V_h(K e^{(r-q)(T_n-\tau)}, T_n) - V^*(K e^{(r-q)(T_n-\tau)}, T_n) \right] \frac{K^2}{2} e^{(r-2q)(T_n-\tau)} \]

\[ f^2(\mathcal{G}) \int_0^T d\tilde{T} \int_0^\infty d\tilde{K} \int_0^\infty d\bar{y} G_h(K, \bar{y}, \tilde{T} ; \tilde{K}, \bar{y}, \tilde{\tau}) \partial_{\bar{y}}(\psi\omega, \tilde{T} - T_{n-1}) \]

\[ = \int_{T_{n-1}}^{T_n} d\tilde{T} \int_0^\infty d\tilde{K} \int_0^\infty d\bar{y}(\psi\omega, \tilde{T} - T_{n-1}) g_h(\tilde{K}, \bar{y}, \tilde{T}), \]

where the order of integrations is changed in the second equality and

\[ g_h(\tilde{K}, \bar{y}, \tilde{T}) = \int_{\tilde{T}}^{T_n} d\tau \int_0^\infty d\tilde{K} \int_0^\infty d\bar{y} G_h(K, \bar{y}, \tilde{T} ; \tilde{K}, \bar{y}, \tilde{\tau}) \]

\[ \left( V_h(K e^{(r-q)(T_n-\tau)}, T_n) - V^*(K e^{(r-q)(T_n-\tau)}, T_n) \right) \frac{K^2}{2} e^{(r-2q)(T_n-\tau)} f^2(\bar{y}). \]

It is easy to see \( g_h(\tilde{K}, \bar{y}, \tilde{T}) \) satisfies (3.6). Substituting the term \((*)\) into (B.2) yields the desired result. The proof is completed.

**Appendix C: Derivation of the necessary condition (3.7) in continuous time**

We restrict our attention to a bounded domain \( Q \equiv I \times \Omega \times (T_{\text{min}}, T_N) \equiv (0, K_{\text{max}}) \times (0, Y_{\text{max}}) \times (T_{\text{min}}, T_N) \), with any fixed \( T_{\text{min}} > t^* \). Without loss of generality, we always impose homogeneous boundary conditions on the boundary of \( Q \) for illustration. We still use the previous notations except \( T_0 = T_{\text{min}} \). Problem (3.5) is restated as follows:

Given \( b_0, \ldots, b_{n-1} \in H_0^1(\Omega) \), find \( b_n \in H_0^1(\Omega) \) such that for any \( \omega \in H_0^1(\Omega) \),

\[ \int_{\Omega} \left( \frac{b_n - b_{n-1}}{h} \cdot \omega + \epsilon_2 \partial_{\bar{y}} b_n \cdot \partial_{\bar{y}} \omega \right. \]

\[ - \frac{1}{h^2} \int_{T_{n-1}}^{T_n} \int_I \frac{\tau - T_{n-1}}{h} g_h(K, \bar{y}, \tau; b^h) \left( \psi_h(K, \bar{y}, \tau; b^h) \cdot \omega \right) \bar{y} dK d\tau \]

\[ \left. d\bar{y} = 0, \right. \]

where \( g_h \) and \( \psi_h \) are both given homogeneous boundary conditions on \( Q \), and \( V_h(0, t) = e^{-q(T-t)} x^* \), \( V_h(K_{\text{max}}, t) = 0 \).

As before, \( b^h(\cdot, \cdot) \) is constructed from \( b_n(\cdot) \), \( n = 0, ..., T_N \) by piecewise linear interpolation in time. We aim to show that as \( h \to 0 \), there is a subsequence of \( b^h(\cdot, \cdot) \) that weakly converges to the solution to the following problem:

---

*The reason we require \( T_{\text{min}} > t^* \) is that we want to avoid the non-smooth initial conditions at \( t^* \) for \( \psi \) and \( V \). Note that we are only interested in Eq. (3.8) for \( T > t^* \) and the test function \( \omega \) in (3.7) is \( C_0^\infty([0, +\infty) \times (t^*, +\infty)) \).

*Other boundary conditions can be treated in a similar way. We point out that a rigorous proof relies on formulating the original problem in a bounded region with appropriate boundary conditions.*
For any $T \geq T_{\min}$ and any $\omega(\bar{y}, T) \in S \equiv H^1(\Omega \times (T_{\min}, T_N)) \cap L^\infty((T_{\min}, T_N); H^1_0(\Omega))$, 
\[
\int_{T_{\min}}^T d\tau \int_\Omega \left( \epsilon_1 \frac{\partial b}{\partial \tau} \cdot \omega + \epsilon_2 b_{\bar{y}} \cdot \omega_{\bar{y}} + \frac{1}{2} f(\bar{y}) f'(\bar{y}) \int_I K^2 (V(K, \tau) - V^*(K, \tau)) \psi(K, \bar{y}; \cdot, b) \right) dK d\bar{y} = 0,
\]
where the boundary conditions for $\psi$ and $V$ are similar to those of $\psi_h$ and $V_h$, respectively.

We now give a sketch of the proof under the assumption that $b^h$ is uniformly bounded in $H^1(\Omega \times (T_{\min}, T_N))$. We can show that there exist some $\alpha \in (0, 1)$ and some positive constant $C$ independent of $h$ and $n$, such that
\[
\|b^h\|_{C^\alpha(\Omega \times (T_{\min}, T_N))} \leq C,
\]
where $C^\alpha(\Omega \times (T_{\min}, T_N))$ is a space of all Hölder continuous functions. Then there exists a subsequence of $b^h$, still denoted by $b^h$, weakly converges to a function denoted by $b(\bar{y}, T)$ in $H^1(\Omega \times (T_{\min}, T_N))$ and uniformly in $C^0(\Omega \times (T_{\min}, T_N))$ by Rellich’s Theorem and Arzela-Ascoli’s Theorem. Hence
\[
\sum_{k=1}^n h \int_\Omega \left( \epsilon_1 \frac{b_k - b_{k-1}}{h} \omega(\cdot, t_k) + \epsilon_2 b_{\bar{y}} b_k \partial_{\bar{y}} \omega(\cdot, t_k) \right) d\bar{y} \rightarrow \int_{T_{\min}}^T d\tau \int_\Omega \left( \epsilon_1 \frac{\partial b}{\partial \tau} \omega + \epsilon_2 b_{\bar{y}} \omega_{\bar{y}} \right) d\bar{y}
\]
for any $\omega(\bar{y}, T) \in S$.

It remains to show
\[
- \sum_{k=1}^n \frac{1}{h} \int_\Omega \int_{T_{k-1}}^{T_k} \int_I \frac{\tau - T_{k-1}}{h} g_h(K, \bar{y}; \tau; b^h) \left( \psi_h(K, \bar{y}; \cdot, b^h) \right) dK d\tau d\bar{y} \rightarrow \int_{T_{\min}}^T d\tau \int_\Omega \frac{1}{2} f(\bar{y}) f'(\bar{y}) \int_I K^2 (V(K, \tau) - V^*(K, \tau)) \psi(K, \bar{y}; \cdot, b) \omega dK d\bar{y}. \tag{C.1}
\]
The following result plays a critical role: Let $\phi_1(K, \bar{y}, \tau)$ and $\phi_2(K, \bar{y}, \tau)$ be the solutions of the problems
\[
\partial_\tau \phi_1 + L_h \phi_1 = -\frac{T(K, \bar{y}, \tau)}{h}, \quad \tau \in (T_{k-1}, T_k), \quad (K, \bar{y}) \in I \times \Omega
\]
\[
\phi_1(K, \bar{y}, T_k) = 0,
\]
and
\[
\partial_\tau \phi_2 + L_h \phi_2 = 0, \quad \tau \in (T_{k-1}, T_k), \quad (K, \bar{y}) \in I \times \Omega
\]
\[
\phi_2(K, \bar{y}, T_k) = T(K, \bar{y}, T_k),
\]

\[22\]
respectively, and both are subject to homogeneous boundary conditions. If $T$ is smooth, then there exists some $\alpha_1 \in (0, 1)$ such that

$$||\phi_1 - \phi_2||_{C^0(I \times \Omega \times [T_{k-1}, T_k])} \leq Ch^{\alpha_1},$$

where $C > 0$ is independent of $h$. Using this result and continuity, we can rewrite the left hand side of (C.1) as

$$-\sum_{k=1}^{n} \int_{\Omega} \int_{T_{k-1}}^{T_k} \int_{I} \frac{\tau - T_{k-1}}{h} K^2 \left[ V_h(K, T_k) - V^*(K, T_k) \right] f^2(\bar{y}) \left( \psi_h(K, \bar{y}, \tau; b^h) \cdot \omega \right) dK d\tau d\bar{y}$$

where we have neglected a term of order $h^{\alpha_1}$. We can show $V_h(K, T; b^h)$, $\psi_h(K, \bar{y}, T; b^h)$ and $\partial_y \psi_h(K, \bar{y}, T; b^h)$ converge to $V(K, T; b)$, $\psi(K, \bar{y}, T; b)$ and $\partial_y \psi(K, \bar{y}, T; b)$ uniformly in $C^0(I \times \Omega \times (T_{\min}, T_N))$, respectively. Taking the limit on the right hand side of (C.2) yields

$$-\frac{1}{2} \int_{T^*}^{T_n} d\tau \int_{\Omega} \int_{I} \frac{K^2}{2} [V(K, \tau) - V^*(K, \tau)] f^2(\bar{y}) (\psi(K, \bar{y}, \tau; b) \cdot \omega) dK d\bar{y}$$

where the integration by parts with respect to $\bar{y}$ is used in the last equality. This gives the desired result.

### Appendix D: Market Data

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Table 1: Quoted prices of call options on SPY recorded on 30/08/2012: Expire at close Friday, 28/09/2012.
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### Table 3: Quoted prices of call options on SPY recorded on 30/08/2012: Expire at close Friday, 17/11/2012.

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## References


