Chapter 3

American options and early exercise

American options are contracts that may be exercised early, prior to expiry. These options are contrasted with European options for which exercise is only permitted at expiry. Most traded stock and futures options are American style, while most index options are European.

3.1 Pricing models

3.1.1 Continuous-time model for American options

We now consider the pricing model for American options. Here we take into account a put as an example. Let $V = V(S,t)$ be the option value. At expiry, we still have

$$V(S,T) = (X-S)^+. \tag{3.1}$$

The early exercise feature gives the constraint

$$V(S,t) \geq X - S. \tag{3.2}$$

As before, we construct a portfolio of one long American option position and a short position in some quantity Δ, of the underlying.

$$\Pi = V - \Delta S.$$

With the choice $\Delta = \frac{\partial V}{\partial S}$, the value of this portfolio changes by the amount

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt$$

In the Black-Scholes argument for European options, we set this expression equal to riskless return, in order to preclude arbitrage. However, when the option in the portfolio is of American style, all we can say is that we can earn no more than the risk-free rate on our portfolio, that is,

$$d\Pi \leq r\Pi dt = r(V - S\frac{\partial V}{\partial S})dt.$$

The reason is the holder of the option controls the early exercise feature. If he/she fails to optimally exercise the option, the change of the portfolio value would be less than riskless return. Thus we arrive at an inequality

$$(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \leq r(V - S\frac{\partial V}{\partial S})dt$$

or

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0. \tag{3.3}$$

Remark 4 *For American options, the long/short position is asymmetrical. The holder of an American option is given more rights, as well as more headaches: when should he exercise? Whereas the writer of the option can do no more than sit back and enjoy the view. The writer of the American option can make more than the risk-free rate if the holder does not exercise optimally. A question: what happens if the portfolio is composed of a long position in some quantity of the underlying and one short American option?*

It is clear that (3.1-3.3) are insufficient to form a model because solution is not unique. We need to exploit more information. Note that if $V(S,t) > X - S$, which implies that the option should not be exercised at the moment, then the equality holds in the inequality (3.3), namely

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \text{ if } V > X - S.$$

If $V(S,t) = X - S$, of course we still have the inequality, that is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \text{ if } V = X - S.$$

The above two formulas imply that at least one holds in equality between (3.2-3.3). So we arrive at a complete model:

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &\leq 0, \ V \geq X - S \\
\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV\right][V - (X - S)] &= 0, \ (S,t) \in D \\
V(S,T) &= (X - S)^+
\end{aligned}$$

It can be shown that there exists a unique solution to the model.

A succinct expression of the above model is

$$\begin{aligned}\min\left\{-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}-rS\frac{\partial V}{\partial S}+rV,V-(X-S)\right\} &= 0,\ (S,t)\in D\\ V(S,T) &= (X-S)^+\end{aligned}$$

For American call options, we similarly have

$$\begin{aligned}\min\left\{-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}-rS\frac{\partial V}{\partial S}+rV,V-(S-X)\right\} &= 0,\ (S,t)\in D\\ V(S,T) &= (S-X)^+\end{aligned}$$

We claim the price function of European call option $C(S,t)$ just satisfies the above model. Indeed, $C(S,t)>S-X$ for $t<T$ and $C(S,t)$ clearly satisfies the Black-Scholes equation. So $C(S,t)$ must be the (unique) solution to the American option pricing model. The result $C(S,t)>S-X$ implies that the option should never be exercised before expiry.

Remark 5 *From the view point of probabilistic approach, we have (for an American put)*

$$V(S,t)=\max_{t'}\widehat{E}\left[e^{-r(t'-t)}(X-S_{t'})^+|S_t=S\right], \tag{3.4}$$

where t' is a stopping time. Intuitively $t'(.)$ can be thought of as a strategy to exercise the option. The option's value corresponds to the optimal exercise strategy. Mathematically we can show the equivalence between (3.4) and the above PDE model.

3.1.2 Continuous-dividend payment case

Let q be the continuous dividend yield. Denote by

$$\varphi(S)=\left\{\begin{array}{l}S-X\\ X-S.\end{array}\right.$$

Then the American option price function V satisfies

$$\begin{aligned}\min\left\{-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2S^2\frac{\partial^2V}{\partial S^2}-(r-q)S\frac{\partial V}{\partial S}+rV,V-\varphi\right\} &= 0,\ (S,t)\in D\\ V(S,T) &= \varphi^+\end{aligned}$$

3.1.3 Binomial model

Let T be the expiration date, $[0, N]$ be the lifetime of the option. If N is the number of discrete time points, we have time points $n\Delta t$, $n = 0, 1, ..., N$, with $\Delta t = T/N$. Let V_j^n be the option price at time point $n\Delta t$ with underlying asset price S_j. Suppose the underlying asset price S_j will move either up to $S_{j+1} = S_j u$ or down to $S_{j-1} = S_j d$ after the next timestep. Similar to the arguments in the continuous time case, we are able to derive the binomial tree method (BTM)

$$\begin{cases} V_j^n = \max\left\{\frac{1}{\rho}[pV_{j+1}^{n+1} + (1-p)V_{j-1}^{n+1}], \varphi_j\right\}, \\ \text{ for } j = -n, -n+2, ..., n \text{ and } n = 0, 1, ...N-1 \\ V_j^N = \varphi_j^+, \text{ for } j = -N, -N+2, ..., N \end{cases}$$

where $\varphi_j = \begin{cases} S_0 u^j - X \\ X - S_0 u^j \end{cases}$

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d},$$

$$\rho = e^{r\Delta t}, \; u = e^{\sigma\sqrt{\Delta t}}, \text{ and } d = e^{-\sigma\sqrt{\Delta t}}.$$

A question: what about the relation between continuous and discrete models for American options?

3.2 Free boundary problems

We still take a put for example. First we give definitions of Stopping Region E (or, Exercise Region) and Holding Region H (or, Continuous Region) :

$$\begin{aligned} E &= \{(S,t) \in D : V(S,t) = X - S\} \\ H &= D \backslash E = \{(S,t) \in D : V(S,t) > X - S\} \end{aligned}$$

3.2.1 *Optimal exercise boundaries

Lemma 1 *If $(S_1, t) \in E$, then $(S_2, t) \in E$ for all $S_2 \leq S_1$.*

Proof: It suffices to show that

$$V(S_2, t) + S_2 \leq V(S_1, t) + S_1, \text{ if } S_2 \leq S_1 \tag{3.5}$$

Indeed, (3.5) is equivalent to

$$V(S_2,t) - (X - S_2) \le V(S_1,t) - (X - S_1)$$

Since $V(S_1,t) - (X - S_1) = 0$ and $V(S_2,t) - (X - S_2) \ge 0$, we derive $V(S_2,t) = X - S_2$, which implies $(S_2,t) \in E$. (3.5) can be proved in terms of the binomial model. We omit the details..

Remark 6 *(3.5) can be rewritten as*

$$\frac{V(S_1,t) - V(S_2,t)}{S_1 - S_2} \ge -1.$$

As S_1 tends S_2, we have

$$\frac{\partial V}{\partial S} \ge -1.$$

Proposition 2 *(i)There exists a boundary $S^*(t)$, called the optimal exercise boundary hereafter, such that*

$$E = \{(S,t) \in D : S \le S^*(t)\}, \text{ and } H = \{(S,t) \in D : S > S^*(t)\}$$

(ii) $S^*(t)$ is monotonically increasing.
(iii) $S^*(T-) = \min(X, \frac{r}{q}X)$

Proof: Part i) can be derived from Lemma 1. To show part ii), it is not hard to prove $V(S,t)$ is monotonically decreasing w.r.t. t by using the binomial model (financial intuition: the larger the time to expiry, the larger the option value). Thus if $V(S,t_1) > X - S$, then $V(S,t_2) \ge V(S,t_1) > X - S$.

As for part iii), it is not hard to show

$$S^*(T-) \ge \min\left(X, \frac{r}{q}X\right).$$

What remains is to show $S^*(T-) \le \min\left(X, \frac{r}{q}X\right)$ whose proof requires some advanced knowledge of PDEs and is omitted.

Remark 7 *$S^*(t)$ is called optimal exercise boundary because it is optimal to exercise the option exactly on the boundary. If $S < S^*(t)$, then $V(S,t) = X - S$ and*

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV < 0$$

or

$$d\Pi < r\Pi dt.$$

3.2.2 Formulation as a free boundary problem

In the continuation region $H = \{S > S^*(t)\}$, the price function of an American put satisfies the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \text{ for } S > S^*(t),\ t \in [0,T) \quad (3.6)$$

On $S = S^*(t)$ we have

$$V(S^*(t), t) = X - S^*(t). \quad (3.7)$$

The finial condition is

$$V(S,T) = (X-S)^+. \quad (3.8)$$

However, (3.6-3.8) cannot form a complete model because $S^*(t)$ is not known a prior as a function of time. As a matter of fact, $S^*(t)$ and $V(S,t)$ must be solved simultaneously. Therefore, we need an additional boundary condition

$$\frac{\partial V}{\partial S}(S^*(t), t) = -1. \quad (3.9)$$

The condition is often called the *high-contact* condition which means that the hedging ratio Δ is continuous across the optimal exercise boundary. (3.6-3.9) form a complete model that is called the free boundary problem.

3.2.3 Perpetual American options

Pricing perpetual American options can give us some insights in the understanding of free boundary problems. A perpetual American put can be exercised for a put payoff at any time. There is no expiry; that is why it is called a perpetual option. Note that the price function of such a option is independent of time, denoted by $P_\infty(S)$. It only depends on the level of the underlying. Actually $P_\infty(S)$ can be regarded as the limit of an American put price as the time to expiry tends to infinity, i.e.

$$P_\infty(S) = \lim_{(T-t)\to\infty} V(S,t;T) = \lim_{\tau\to\infty} \widetilde{V}(S,\tau).$$

where $\widetilde{V}(S,\tau) = V(S,t;T)$, $\tau = T-t$. Thanks to (3.6-3.9), $\widetilde{V}(S,\tau)$ satisfies

$$\begin{aligned}
\frac{\partial \widetilde{V}}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \widetilde{V}}{\partial S^2} - (r-q)S\frac{\partial \widetilde{V}}{\partial S} + r\widetilde{V} &= 0, \text{ for } S > S_*(\tau),\ \tau \in [0,T) \\
\widetilde{V}(S_*(\tau),\tau) &= X - S_*(\tau),\ \frac{\partial \widetilde{V}}{\partial S}(S_*(\tau),\tau) = -1 \\
\widetilde{V}(S,0) &= (X-S)^+
\end{aligned}$$

where $S_*(\tau) = S^*(T-\tau)$. Due to part ii) of Proposition 2, $S_*(\tau)$ is monotone, and we can denote

$$S_\infty = \lim_{\tau\to\infty} S_*(\infty).$$

Then $P_\infty(S)$ satisfies

$$-\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_\infty}{\partial S^2} - (r-q)S\frac{\partial P_\infty}{\partial S} + rP_\infty = 0, \text{ for } S > S_\infty \tag{3.10}$$

$$P_\infty(S_\infty) = X - S_\infty, \qquad \frac{\partial P_\infty}{\partial S}(S_\infty) = -1. \tag{3.11}$$

This is actually a free boundary problem with an ordinary difference equation. We now seek a solution of the form S^α to Eq. (3.11), where α satisfies

$$-\frac{1}{2}\sigma^2\alpha(\alpha-1) - (r-q)\alpha + r = 0$$

or

$$\frac{1}{2}\sigma^2\alpha^2 + (r - q - \frac{\sigma^2}{2})\alpha - r = 0.$$

The two solutions of the above equation are

$$\alpha_\pm = \frac{-(r-q-\frac{\sigma^2}{2}) \pm \sqrt{(r-q-\frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2}. \tag{3.12}$$

So the general solution of Eq. (3.11) is

$$AS^{\alpha_+} + BS^{\alpha_-}$$

where A and B are arbitrary constants.

Clearly, for the perpetual American put the coefficient A must be zero; as $S \to \infty$ then value of the option must tend to zero. What about B? Here we need to take advantage of the condition (3.11)

$$\begin{aligned} BS_\infty^{\alpha_-} &= X - S_\infty \\ \alpha_- BS_\infty^{\alpha_- -1} &= -1 \end{aligned}$$

to get

$$\begin{aligned} S_\infty &= \frac{\alpha_-}{\alpha_- - 1}X \\ B &= \frac{X - S_\infty}{S_\infty^{\alpha_-}}. \end{aligned}$$

So, we obtain the perpetual American option price function

$$P_\infty(S) = (X - S_\infty)\left(\frac{S}{S_\infty}\right)^{\alpha_-}$$

A by-product of the above calculation is

Proposition 3

$$\lim_{\tau\to\infty} S_*(\tau) = \frac{\alpha_-}{\alpha_- - 1}X$$

3.2.4 *Put-call symmetry relations

For European options, we have the well-known put-call parity

$$C_E(S,t) - P_E(S,t) = Se^{-q(T-t)} - Xe^{-r(T-t)}.$$

However, such a parity doesn't hold for American options. Let us explain the reason from the view point of PDE. For European options, both $C_E(S,t)$ and $P_E(S,t)$ satisfy the Black-Scholes equations. Due to the linearity of the equation, $C_E(S,t) - P_E(S,t)$ also satisfies the equation, that is

$$\left[\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r-q)S\frac{\partial}{\partial S} - r\right](C_E - P_E) = 0,$$

together with the terminal condition

$$(C_E - P_E)(S,T) = S - X.$$

On the other hand, the Black-Scholes equation with the terminal condition has the unique solution $Se^{-q(T-t)} - Xe^{-r(T-t)}$. This leads to

$$C_E(S,t) - P_E(S,t) = Se^{-q(T-t)} - Xe^{-r(T-t)}.$$

However, for American options, their governing equation is nonlinear so that we cannot obtain such a parity relation.

But, the following put-call symmetry relation holds for both European and American options:

$$C(S,t;X,r,q) = P(X,t;S,q,r) \tag{3.13}$$

where the underlying price and the strike price in the put formula become the strike price and the underlying price in the call formula, respectively. Also, the roles of q and r are interchanged, like S and X.

The financial explanation is given as follows. We may consider a call option with the payoff $(S-X)^+$ as providing the right to exchange one asset X with dividend yield r for another asset S with dividend yield q. Similarly, a put call with payoff $(X_2-S_2)^+$ is regarded as a right to exchange one asset S_2 with dividend yield q_2 for another asset X_2 with dividend yield r_2. Suppose

$$X_2 = S,\ S_2 = X,\ r_2 = q \text{ and } q_2 = r.$$

then the two have the same payoff and thus the same price, namely,

$$\begin{aligned} C(S,t;X,r,q) &= P(S_2,t;X_2,r_2,q_2) \\ &= P(X,t;S,q,r). \end{aligned}$$

Next, we would like to establish the put-call symmetry relation for the optimal exercise prices for American put and call options. Let $S_P^*(t;r,q)$ and $S_c^*(t;r,q)$ denote the optimal exercise boundaries for American put and call options on a continuous dividend payment stock, respectively. We assert

$$S_c^*(t;r,q)S_p^*(t;q,r) = X^2 \tag{3.14}$$

Indeed, due to the homogeneity, Eq. (3.13) can be rewritten as

$$C(S,t;X,r,q) = \frac{S}{X}P(\frac{X^2}{S},t;X,q,r).$$

According to the definition of $S_c^*(t;r,q)$, we have

$$\begin{aligned} \frac{S}{X}P(\frac{X^2}{S},t;X,q,r) &= C(S,t;X,r,q) = S-X,\ \text{for } S \geq S_c^*(t;r,q) \\ \frac{S}{X}P(\frac{X^2}{S},t;X,q,r) &= C(S,t;X,r,q) > S-X,\ \text{for } S < S_c^*(t;r,q). \end{aligned}$$

or, equivalently

$$\begin{aligned} P(\frac{X^2}{S},t;X,q,r) &= X-\frac{X^2}{S},\ \text{for } S \geq S_c^*(t;r,q) \\ P(\frac{X^2}{S},t;X,q,r) &> X-\frac{X^2}{S},\ \text{for } S < S_c^*(t;r,q). \end{aligned}$$

By denoting $S_1 = \frac{X^2}{S}$, we then get

$$\begin{aligned} P(S_1,t;X,q,r) &= X-S_1,\ \text{for } S_1 \leq \frac{X^2}{S_c^*(t;r,q)} \\ P(S_1,t;X,q,r) &> X-S_1,\ \text{for } S_1 > \frac{X^2}{S_c^*(t;r,q)} \end{aligned}$$

which implies $\frac{X^2}{S_c^*(t;r,q)}$ is the optimal exercise boundary for American put option with interest rate q and dividend yield r, namely

$$S_p^*(t;q,r) = \frac{X^2}{S_c^*(t;r,q)}.$$

Applying Eqs (3.13-3.14) and Proposition (2) leads to

Proposition 4 *Let $S_c^*(t;r,q)$ be the optimal exercise boundary for American call options. Then*
(i) $S_c^*(t)$ is monotonically decreasing.
(ii) $S_c^*(T-;r,q) = \max(X, \frac{r}{q}X)$

Remark 8 *When $q = 0$, then $S_c^*(T-;r,0) = \infty$. Because of the monotonicity of $S_c^*(t)$, we deduce $S_c^*(t;r,0) = \infty$ for all t, which means there is no optimal exercise boundary, that is, early exercise should never happen.*

In addition, it is not hard to get the closed form price function of the perpetual American option by using the put-call symmetry relation.

Proposition 5 *Suppose $q > 0$. Let $C_\infty(S)$ be the price function of the perpetual American option. Then*

$$C_\infty(S) = (S_{\infty,c} - X)\left(\frac{S}{S_{\infty,c}}\right)^{\alpha_+}$$

where α is given by (3.12), and

$$S_{\infty,c} = \frac{\alpha_+}{\alpha_+ - 1}X.$$

3.3 Bermudan options

In a standard American option, exercise can take place at any time during the life of the option and the exercise price is always the same. In practice, the American options that are traded in the over-the-counter market do not always have these standard features.

One type of nonstandard American option is known as a Bermudan option. In this early exercise is restricted to certain dates $t_1 < t_2 < ... < t_n$ during the life of the option ($t_i \in [0,T)$, $i = 1,2,...,n$). Note that at any interval (t_i, t_{i+1}), $i = 0, 1, ..., n$ (let $t_0 = 0$, $t_{n+1} = T$), there is no early exercise right. So

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0, \text{ for } t \in (t_i, t_{i+1}), i = 0, 1, ..., n$$

At $t = t_i$, due to the early exercise feature, one has

$$V(S, t_i) = \max(V(S, t_i^+), \varphi(S)), \text{ for } i = 1, 2, ...n$$

At expiry,

$$V(S, T) = \varphi^+.$$

These form a complete model. Also, it is easy to implement by the binomial tree method.

A question: if there is no dividend payment during the life of the option, whether is there a chance to exercise a Bermudan call option prior to the expiry?