Chapter 4

Multi-asset options

In this chapter we introduce the idea of higher dimensionality by describing the Black-Scholes theory for options on more than one underlying asset. This theory is perfectly straightforward; the only new idea is that correlated random walks and the corresponding multifactor version of Ito Lemma.

4.1 Pricing model

4.1.1 Two-asset options

Consider a European option whose payoff, denoted by \( f(S_1, S_2) \), depends on two assets \( S_1 \) and \( S_2 \). The basic building block for option pricing with one underlying is the lognormal random walk

\[
\frac{dS}{S} = \mu dt + \sigma dW.
\]

This is readily extended to a world containing two assets via models for each underlying

\[
\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dW_1
\]

\[
\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dW_2
\]

As before, we can think of \( dW_i, i = 1, 2 \) as a random number drawn from a Normal distribution with mean zero and standard deviation \( dt^{1/2} \) so that

\[
E(dW_i) = 0 \quad \text{and} \quad E[dW_i^2] = dt
\]
but the random numbers \( dW_1 \) and \( dW_2 \) are correlated:

\[ E[dW_1dW_2] = \rho dt \]

Here \( \rho \) is the correlation coefficient between the two random walks.

Let \( V(S_1, S_2, t) \) be the option value. Since there are two sources of uncertainty, we construct a portfolio of one long option position, two short positions in some quantities of underlying assets:

\[ \Pi = V - \Delta_1 S_1 - \Delta_2 S_2. \]

Consider the increment

\[ d\Pi = dV - \Delta_1 dS_1 - \Delta_2 dS_2. \]

Here we need the Ito Lemma involving two variables.

\[
\begin{align*}
dV &= \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] dt \\
&\quad + \frac{\partial V}{\partial S_1} dS_1 + \frac{\partial V}{\partial S_2} dS_2
\end{align*}
\]

Actually the two dimensional Ito Lemma can be derived by using Taylor series and the rules of thumb: \( dW_i^2 = dt, i = 1, 2 \), and \( dW_1dW_2 = \rho dt \).

Taking \( \Delta_1 = \frac{\partial V}{\partial S_1} \) and \( \Delta_2 = \frac{\partial V}{\partial S_2} \) to eliminate risk, we then have

\[
\begin{align*}
d\Pi &= \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} \right] dt.
\end{align*}
\]

Then the portfolio is riskless and then earn riskless return, namely

\[ d\Pi = r\Pi = r(V - \frac{\partial V}{\partial S_1} S_1 - \frac{\partial V}{\partial S_2} S_2) dt. \]

So we arrive at an equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + r \frac{\partial V}{\partial S_1} S_1 + r \frac{\partial V}{\partial S_2} S_2 - rV = 0. \tag{4.1}
\]

The solution domain is \( \{ S_1 > 0, S_2 > 0, t \in [0, T) \} \), and the final condition is

\[ V(S_1, S_2, T) = f(S_1, S_2) \tag{4.2} \]

(4.1-4.2) form a complete model. Well-known payoffs are the following:
4.1. PRICING MODEL

\[ f(S_1, S_2) = \begin{cases} 
(max(S_1, S_2) - X)^+, & \text{maximum call} \\
(X - max(S_1, S_2))^+, & \text{maximum put} \\
(min(S_1, S_2) - X)^+, & \text{minimum call} \\
(X - min(S_1, S_2))^+, & \text{minimum put} \\
(S_1 - S_2 - X)^+, & \text{spread option} 
\end{cases} \] (4.3)

If the assets pay continuous dividends, then (4.1) is replaced by

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + (r - q_1) S_1 \frac{\partial V}{\partial S_1} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - r V = 0.
\] (4.4)

where \( q_1 \) and \( q_2 \) are dividend yields of two assets, respectively.

4.1.2 American feature:

Suppose that the option can be exercised early receiving the payoff. Then the pricing model is

\[
\min \{ -LV, V - f(S_1, S_2) \} = 0
\]

\[
V(S, T) = f(S_1, S_2)
\]

where \( L = \frac{\partial}{\partial t} + \frac{1}{2} \sigma_1^2 S_1^2 \frac{\partial^2}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2^2 \frac{\partial^2}{\partial S_2^2} + (r - q_1) S_1 \frac{\partial}{\partial S_1} + (r - q_2) S_2 \frac{\partial}{\partial S_2} - r
\)

4.1.3 Exchange option: similarity reduction

An exchange option gives the holder the right to exchange one asset for another. The payoff for this contract at expiry is \((S_1 - S_2)^+\). So the final condition is

\[
V(S_1, S_2, T) = (S_1 - S_2)^+.
\]

The governing equation is still (4.1).

This contract is special in that there is a similarity reduction. Let us postulate that the solution takes the form

\[
V(S_1, S_2, t) = S_2 H(\xi, t),
\]

where the new variable is

\[
\xi = \frac{S_1}{S_2}.
\]
If this is the case, then instead of finding a function $V$ of three variables, we only need find a function $H$ of two variables, a much easier task.

It follows

\[
\frac{\partial V}{\partial S_1} = S_2 \frac{\partial H}{\partial \xi} \frac{1}{S_2}, \quad \frac{\partial V}{\partial S_2} = H + S_2 \frac{\partial H}{\partial \xi} \left( -\frac{S_1}{S_2^2} \right) = H - \xi \frac{\partial H}{\partial \xi},
\]

\[
\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{S_1}{S_2^3} \frac{\partial^2 H}{\partial \xi^2}, \quad \frac{\partial V}{\partial t} = S_2 \frac{\partial H}{\partial t}.
\]

The partial differential equation now becomes

\[
\frac{\partial H}{\partial t} + \left[ \frac{1}{2} \sigma_1^2 \xi^2 - \rho \sigma_1 \sigma_2 \xi^2 + \frac{1}{2} \sigma_2^2 \xi^2 \right] \frac{\partial^2 H}{\partial \xi^2} + (r - q_1) \xi \frac{\partial H}{\partial \xi} + (r - q_2) \left( H - \xi \frac{\partial H}{\partial \xi} \right) = 0
\]

or

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \sigma' \xi^2 \frac{\partial^2 H}{\partial \xi^2} + (q_2 - q_1) \xi \frac{\partial H}{\partial \xi} - q_2 H = 0,
\]

where $\sigma' = \sqrt{\sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2}$. This equation is just the Black-Scholes equation for a single stock with $q_2$ in place of $r$, $q_1$ in place of the dividend yield on the single stock and with a volatility of $\sigma'$. Note that the final condition is

\[
H(\xi, T) = (\xi - 1)^+
\]

From this it follows that

\[
V(S_1, S_2, t) = S_1 e^{-\eta_1(T-t)} N(d_1') - S_2 e^{-q_2(T-t)} N(d_2'),
\]

where

\[
d_1' = \frac{\log(S_1/S_2) + (q_2 - q_1 + \frac{1}{2} \sigma'^2)(T-t)}{\sigma' \sqrt{T-t}}, \quad \text{and} \quad d_2' = d_1' - \sigma' \sqrt{T-t}
\]

**Remark 9** An exchange option is a kind of spread option with $X = 0$. If $X \neq 0$, the similarity reduction doesn’t work because the payoff cannot be reduced to a function of $\xi$ and $t$. 
4.1. PRICING MODEL

4.1.4 Options on many underlyings

Options with many underlyings are called basket options, options on baskets or rainbow options. We now extend the two-asset option pricing model to a general case. Suppose

\[ dS_i = \mu_i S_i dt + \sigma_i S_i dW_i. \]

Here \( S_i \) is the price of the \( i \)th asset, \( i = 1, 2, \ldots, n \), and \( \mu_i \) and \( \sigma_i \) are the drift and volatility of that asset respectively and \( dW_i \) is the increment of a Brownian motion. We can still continue to think of \( dW_i \) as a random number drawn from a Normal distribution with mean zero and standard deviation \( dt^{1/2} \) so that

\[ E(dW_i) = 0 \quad \text{and} \quad E(dX_i^2) = dt \]

and the random numbers \( dW_i \) and \( dW_j \) are correlated:

\[ E[dW_i dW_j] = \rho_{ij} dt, \]

here \( \rho_{ij} \) is the correlation coefficient between the \( i \)th and \( j \)th random walks. The symmetric matrix with \( \rho_{ij} \) as the entry in the \( i \)th row and \( j \)th column is called the correlation matrix. For example, if we have four underlyings \( n = 4 \) and the correlation matrix will look like this:

\[
D = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & 1 & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & 1 & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & 1
\end{pmatrix}
\]

Note that \( \rho_{ii} = 1 \) and \( \rho_{ij} = \rho_{ji} \). The correlation matrix is positive definite, so that \( y^T D y \geq 0 \).

To be able to manipulate functions of many random variables we need a multidimensional version of Ito’s lemma. If we have a function of the variables \( S_1, S_2, \ldots, S_n \) and \( t \), \( V(S_1, S_2, \ldots, S_n, t) \), then

\[
dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right) dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i.
\]

We can get to this same result by using Taylor series and the rules of thumb:

\[
dW_i^2 = dt \quad \text{and} \quad dW_i dW_j = \rho_{ij} dt.
\]

The pricing model for basket options is straightforward. Still set up a portfolio consisting of one basket option, and short a number \( \Delta_i \) of each of
the asset $S_i$, employ the multidimensional Ito's Lemma, take $\Delta_i = \frac{\partial V}{\partial S_i}$ to eliminate the risk, and set the return of the portfolio equal to the risk-free rate. We are able to arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - q_i) S_i \frac{\partial V}{\partial S_i} - r V = 0.$$ 

Here $q_i$ is the dividend yield on the $i$th asset. The final condition is

$$V(S_1, S_2, ..., S_n, t) = f(S_1, S_2, ..., S_n)$$

The analytic solution to the above model is available, but involves multiple integral, as in the case of two-asset options. (See Page 154, Wilmott (1998))

4.2 Quantos

There is one special, and very important type of multi-asset option. This is the cross-currency contract called a quanto. The quanto has a payoff defined with respect to an asset or an index (or an interest rate) in one country, but then the payoff is converted to another currency payment. The general form of its payoff can be expressed as

$$f(S_S, S).$$

Here $S_S$ is the exchange rate between the domestic currency and the foreign currency (for example, dollar-yen rate, number of dollars per yen), and $S$ is the level of some foreign asset (for example, the Nikkei Dow index). Note that the quanto contract is measured in domestic currency, but $S$ is in foreign currency. So this contract is exposed to the exchange rate and the asset. We assume

$$dS_S = \mu_S S_S dt + \sigma_S S_S dW_S$$

and

$$dS = \mu S dt + \sigma S dW$$

with a correlation coefficient $\rho$ between them.

Let $V(S_S, S, t)$ be the quanto option value in US dollar. Construct a portfolio consisting of the quanto, hedged with the foreign currency and the asset:

$$\Pi = V(S_S, S, t) - \Delta_S S_S - \Delta SS_S.$$ 

Note that every term in this equation is measured in domestic currency (dollar). $\Delta_S$ is the number of the foreign currency (yen) we hold short, so $-\Delta_S S_S$ is the dollar value of that yen. Similarly, with the term $-\Delta SS_S$ we
have converted the yen-denominated index $S$ into dollars, $\Delta$ is the amount of the index held short.

The change in the value of the portfolio is due to the change in the value of its components and the interest received on the yen:

$$dV = \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S \frac{\partial^2 V}{\partial S \partial \bar{S}} \frac{\partial V}{\partial \bar{S}} + \frac{1}{2} \sigma^2 \bar{S}^2 \frac{\partial^2 V}{\partial \bar{S}^2} \right] dt$$

$$+ \left( \frac{\partial V}{\partial S} \frac{dS}{\partial S} \bar{S} + \frac{\partial V}{\partial \bar{S}} \frac{d\bar{S}}{\partial \bar{S}} S - \frac{\partial V}{\partial \bar{S}} \frac{d\bar{S}}{\partial S} S - \frac{\partial V}{\partial S} \frac{dS}{\partial \bar{S}} \right) \frac{d\bar{S}}{\partial \bar{S}} S d\bar{S}$$

where the term $-\Delta \bar{S} S \sigma^2 \frac{\partial V}{\partial \bar{S}}$ is the interest received by the yen holding, and $-\rho \sigma S \sigma \Delta S \frac{\partial V}{\partial \bar{S}} d\bar{S}$ is due to the increment of the product $-\Delta S \bar{S}$. We now choose

$$\Delta = \frac{1}{\bar{S}} \frac{\partial V}{\partial S}$$

and $\Delta \bar{S} = \frac{\partial V}{\partial \bar{S}} S - \frac{\partial V}{\partial \bar{S}} \frac{dS}{\partial \bar{S}} S$ to eliminate the risk in the portfolio. Setting the return on this riskless portfolio equal to the US risk-free rate of interest $r$, since $V$ is measured entirely in dollars, yields

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma S \bar{S} \frac{\partial^2 V}{\partial S \partial \bar{S}} \frac{\partial V}{\partial \bar{S}} + \frac{1}{2} \sigma^2 \bar{S}^2 \frac{\partial^2 V}{\partial \bar{S}^2} dt + (r - r_f) S \frac{\partial V}{\partial S} \frac{dS}{\partial S} + (r_f - \rho \sigma S) S \frac{\partial V}{\partial \bar{S}} \frac{d\bar{S}}{\partial \bar{S}} - rV = 0. \tag{4.5}$$

This completes the formulation of the pricing equation. The equation is valid for any contract with underlying measured in one currency but paid in another. The final conditions on $t = T$

$$V(S, S, T) = f(S, S).$$

Notice that these parameters correspond to two-asset options with continuous dividend payments (i.e. Eqn (4.4)), where under the risk-neutral world, the underlying assets follow

$$\frac{dS_1}{S_1} = (r - q_1) dt + \sigma_1 dW_1$$
\[
\frac{dS_2}{S_2} = (r - q_2)dt + \sigma_2 dW_2
\]
with \(\rho dt = E(dW_1dW_2)\). For quanto options (i.e. Eqn (4.5)), the underlyings follow in the risk-neutral world
\[
\frac{dS_8}{S_8} = (r - r_f)dt + \sigma_8 dW
\]
\[
\frac{dS}{S} = (r_f - \rho \sigma_8 \sigma)dt + \sigma dW
\]
\[
= (r - (r - r_f + \rho \sigma_8 \sigma))dt + \sigma dW.
\]
with \(\rho dt = E(dW_8 W)\). Therefore, in this case, \(q_1 = r_f\) and \(q_2 = r_f - \rho \sigma_8 \sigma\).

### 4.3 Numerical Methods

#### 4.3.1 Binomial tree methods

Suppose \((S_1, S_2)\) will move to \((S_1u_1, S_2u_2)\) with probability \(p_1\), \((S_1d_1, S_2u_2)\) with probability \(p_2\), and \((S_1d_1, S_2d_2)\) with probability \(p_3\) after the next timestep. Then the binomial model for two-asset options is
\[
V(S_1, S_2, t) = e^{-r\Delta t}[p_1 V(S_1u_1, S_2u_2, t + \Delta t) + p_2 V(S_1u_1, S_2d_2, t + \Delta t) + p_3 V(S_1d_1, S_2d_2, t + \Delta t)]
\]
where \(p_i\) for \(i = 1, 2, 3, 4\), \(u_i, d_i\) for \(i = 1, 2\) are chosen to be consistent with the continuous-time model. One choice for these parameters is given as follows
\[
u_i = e^{\sigma_i \sqrt{\Delta t}}, \quad d_i = \frac{1}{u_i} \quad \text{for} \quad i = 1, 2.
\]

\[
p_1 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \sigma_2^2}{\sigma_1} + \frac{r - q_2 - \sigma_2^2}{\sigma_2} \right) \sqrt{\Delta t} + \rho \right]
\]
\[
p_2 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \sigma_2^2}{\sigma_1} - \frac{r - q_2 - \sigma_2^2}{\sigma_2} \right) \sqrt{\Delta t} - \rho \right]
\]
\[
p_3 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \sigma_2^2}{\sigma_1} - \frac{r - q_2 - \sigma_2^2}{\sigma_2} \right) \sqrt{\Delta t} + \rho \right]
\]
\[
p_4 = \frac{1}{4} \left[ 1 + \left( \frac{r - q_1 - \sigma_2^2}{\sigma_1} + \frac{r - q_2 - \sigma_2^2}{\sigma_2} \right) \sqrt{\Delta t} - \rho \right]
\]
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\[
p_4 = \frac{1}{4} \left[ 1 + \left( - \frac{r - q_1 - \sigma_1^2}{\sigma_1} + \frac{r - q_2 - \sigma_2^2}{\sigma_2} \right) \sqrt{\Delta t} - \rho \right].
\]

We refer interested students to Kwok (1998) [pp 207-208] for derivation of the above parameters. It should be pointed out that we can also use the finite difference method to determine these parameters.

4.3.2 Monte-Carlo simulation

The amount of computation of BTM grows exponentially with the number of underlyings. We will have to give up BTM if the number of underlyings is greater than 3, and instead employ Monte-Carlo simulation which is relatively more efficient as the number of underlyings increases.

Monte-Carlo simulation is based on the risk-neutral valuation result. The expected payoff in a risk-neutral world is calculated using a sampling procedure. It is then discounted at the risk-free interest rate.

Suppose in a risk-neutral world

\[
dS_i = \tilde{\mu}_i S_i dt + \sigma_i S_i dW_i, \quad (1 \leq i \leq n)
\]

As in the single-variable case, the life of the derivative must be divided into \( N \) subintervals of length \( \Delta t \). The discrete version of the process for \( S_i \) is then

\[
S_i(t + \Delta t) - S_i(t) = \tilde{\mu}_i S_i \Delta t + \sigma_i S_i \epsilon_i \sqrt{\Delta t}, \quad (4.6)
\]

where \( \epsilon_i \) is a random sample from a standard normal distribution. The coefficient of correlation between \( \epsilon_i \) and \( \epsilon_j \) is \( \rho_{ij} \) for \( 1 \leq i, j \leq n \). One simulation trial involves obtaining \( N \) samples of the \( \epsilon_i \) (\( 1 \leq i \leq n \)) from a multivariate standardized normal distribution. These are substituted into equation (4.6) to produce simulated paths for each \( S_i \) and enable a sample value for the derivative to be calculated.

Note that correlated samples \( \epsilon_i \) (\( 1 \leq i \leq n \)) from standard normal distributions are required. We only give a procedure for \( n = 2 \). For \( n \geq 3 \), we refer interested readers to Appendix. Independent samples \( x_1 \) and \( x_2 \) from a univariate standardized normal distribution are easily obtained. The required samples \( \epsilon_1 \) and \( \epsilon_2 \) are then calculated as follows:

\[
\begin{align*}
\epsilon_1 &= x_1 \\
\epsilon_2 &= \rho x_1 + \sqrt{1 - \rho^2} x_2
\end{align*}
\]

where \( \rho \) is the coefficient of correlation.
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Remark 10  (1) At each time step, we need to find \( \epsilon_i \) (1 \( \leq \) i \( \leq \) 2).

(2) The number of simulation trials \( M \) carried out depends on the accuracy required. In general, we take \( M = 5000 \) or \( 10000 \).

Remark 11  The amount of computation of Monto-Carlo simulation grows only linearly with the number of underlyings. The main drawback of Monte Carlo simulation is that it cannot easily handle situations where there are early exercise opportunities.

4.3.3 *Generation of correlated samples

Consider the situation where we require \( n \) correlated samples from normal distributions where the coefficient of correlation between sample \( i \) and sample \( j \) is \( \rho_{ij} \). We first sample \( n \) independent variables \( x_i \) (1 \( \leq \) i \( \leq \) n), from univariate standardized normal distributions. The required samples are \( \epsilon_i \) (1 \( \leq \) i \( \leq \) n), where

\[
\epsilon_i = \sum_{k=1}^{i} \alpha_{ik} x_k.
\]

For \( \epsilon_i \) to have the correct variance and the correct correlation with the \( \epsilon_j \) (1 \( \leq \) j \( \leq \) n), we must have

\[
\sum_{k=1}^{i} \alpha_{ik}^2 = 1
\]

and, for all \( j \leq i \),

\[
\sum_{k=1}^{j} \alpha_{ik} \alpha_{jk} = \rho_{ij}.
\]

The first sample, \( \epsilon_1 \), is set equal to \( x_1 \). These equations for the \( \alpha \)'s can be solved so that \( \epsilon_2 \) is calculated from \( x_1 \) and \( x_2 \); \( \epsilon_3 \) is calculated from \( x_1 \), \( x_2 \) and \( x_3 \); and so on. The procedure is known as the Cholesky decomposition. For example, when \( n = 3 \),

\[
\begin{align*}
\epsilon_1 &= x_1 \\
\epsilon_2 &= \rho_{21} x_1 + \sqrt{1 - \rho_{21}^2} x_2 \\
\epsilon_3 &= \rho_{31} x_1 + \frac{\rho_{32} - \rho_{31} \rho_{21}}{\sqrt{1 - \rho_{21}^2}} x_2 + \sqrt{\frac{1 + 2 \rho_{32} \rho_{21} - \rho_{31}^2}{1 - \rho_{21}^2}} x_3
\end{align*}
\]