Chapter 1

Derivative Pricing

1.1 Basic Financial Derivatives: Forward Contracts and Options

A derivative is a financial instrument whose value depends on the values of other, more basic underlying variables such as stocks, indices, interest rate and so on. Typical examples of derivatives include forward contracts, futures, options, swaps, interest rate derivatives and so on. Futures and standard options are traded actively on many exchanges. Forward contracts, swaps, many different types of options are regularly traded by financial institutions, fund managers, and corporations in the over-the-counter market (OTC market).

In this section we introduce two kinds of basic derivative products: forward contracts and options.

1.1.1 Forward Contracts

A forward contract is an agreement between two parties to buy or sell an asset at a certain future time (called the expiry date or maturity) for a certain price (called delivery price). It can be contrasted with a spot contract, which is an agreement to buy or sell an asset today. One of the parties to the forward contract assumes a long position and agrees to buy the underlying asset at expiry for the delivery price. The other party assumes a short position and agrees to sell the asset at expiry for the delivery price. The payoff from a long position in a forward contract on one unit of an asset is

\[ S_T - K, \]

where \( K \) is the delivery price and \( S_T \) is the spot price of the asset at maturity of the contract. Similarly the payoff from a short position in a forward contract is \( K - S_T \). Observe that the payoff is linear with \( S_T \).

At the time the contract is entered into, it costs nothing to take either a long or a short position. This means that on the starting date the value of the forward contract to both sides is zero. A natural question:

\[ \text{how to choose the delivery price such that the value of the forward contract is zero when opening the contract?} \]  (1.1)
1.1.2 Options

The simplest financial option, a European vanilla call or put option, is a contract that gives its holder the right to buy or sell the underlying at a certain future time (expiry date) for a predetermined price (known as strike price). For the holder of the option, the contract is a right and not an obligation. The other party to the contract, who is known as the writer, does have a potential obligation.

The payoff of a European vanilla call option is

\[(S_T - K)^+,
\]

where \(K\) is the strike price and \(S_T\) is the spot price of the asset at maturity of the option. Similarly, the payoff a European vanilla put option is \((K - S_T)^+\). Note that the payoff of an option is nonlinear with \(S_T\).

If an option is allowed to be exercised at any time before expiry, the option is called an American style option. For example, an American put option has an early exercise payoff \(K - S_t\) if it is exercised at time \(t < T\), where \(S_t\) is the time \(t\) value of the underlying asset.

The terminal payoff of vanilla options only depends on the underlying asset price at maturity. There are some options, known as path-dependent options, which have a terminal payoff depending on the historic price. For example, a fixed strike arithmetic Asian call option has the payoff

\[
\left(\frac{1}{T} \int_0^T S_\tau d\tau - K\right)^+.
\]

Similarly, a floating strike lookback call has the payoff

\[
\left(S_T - \max_{\tau \in [0,T]} S_\tau\right)^+,
\]

while an up-out barrier call option has the payoff

\[
(S_T - K)^+ I_{\{S_\tau < H, \tau \in [0,T]\}},
\]

where \(H\) is the barrier level and \(I\) is the indicator function.

Since the option confers on its holder a right without obligation it must have some value at the time of opening the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed. So, there is a question:

\[
\text{how much would one pay to win the option?} \quad (1.2)
\]

1.2 No Arbitrage Principle

One of the fundamental concepts in derivatives pricing is the no-arbitrage principle, which can be loosely stated as ‘there is no such thing as a free lunch’. More formally, in financial term, there are never any opportunities to make an instantaneous risk-free profit. In fact, such opportunities may exist in a real market. But, they cannot last for a significant length of time before prices move to eliminate them because of the existence of arbitraguer in the
market. Throughout this notes, we always admit the no-arbitrage principle whose application will lead to some elegant modelling.

We often make use of two conclusions below derived from the no-arbitrage principle:

1) Let \( \Pi_1(t) \) and \( \Pi_2(t) \) be the value of two portfolios at time \( t \), respectively. Then
\[
\Pi_1(t) \leq \Pi_2(t) \text{ if } \Pi_1(T) \leq \Pi_2(T) \text{ a.s., } t < T.
\] (1.3)

Especially,
\[
\Pi_1(t) = \Pi_2(t) \text{ if } \Pi_1(T) = \Pi_2(T) \text{ a.s., } t < T.
\] (1.4)

2) All risk-free portfolios must earn the same return, i.e. riskless interest rate. Suppose \( \Pi \) is the value of a riskfree portfolio, and \( d\Pi \) is its price increment during a small period of time \( dt \). Then
\[
\frac{d\Pi}{\Pi} = r dt,
\] (1.5)
where \( r \) is the riskless interest rate.

Remark 1 When applying the no-arbitrage principle (for example, proving the above two conclusions), the assumption of short-selling is needed. Except for special claim, we suppose that short selling is allowed for any assets involved.

In what follows we attempt to derive the price of a forward contract by using the no-arbitrage principle.

### 1.2.1 Pricing Forward Contracts

Consider a forward contract whose delivery price is \( K \). Let \( S_t \) and \( V_t \) be the prices of the underlying asset and the long forward contract at time \( t \). The riskless interest rate \( r \) is a constant. In addition, we assume that the underlying asset has no storage costs and produces no income.

At time \( t \) we construct two portfolios:
- Portfolio A: a long forward contract + cash \( Ke^{-r(T-t)} \);
- Portfolio B: one share of underlying asset: \( S_t \).

At expiry date, both have the value of \( S_T \). At time \( t \), portfolio A and B have the values of \( V_t + Ke^{-r(T-t)} \) and \( S_t \), respectively.

We emphasize that the underlying asset discussed here is an investment asset (stock or gold, for example) for which short selling is allowed. Then we infer from the no-arbitrage principle that the two must have the same value at time \( t \), that is
\[
V_t + Ke^{-r(T-t)} = S_t
\]
or
\[
V_t = S_t - Ke^{-r(T-t)}.
\] (1.6)

Recall that the delivery price is chosen such that at the time when the contract is opened, the value of the contract to both long and short sides is zero. Let \( t = 0 \) be the time of opening the contract. Then we have
\[
S_0 - Ke^{-rT} = 0,
\]
namely,
\[ K = S_0 e^{rT}. \]

This answers Question 1.1.

Question: what happens if the underlying is oil?

1.2.2 Properties of Option Prices

Forward contract can be valued by the no-arbitrage principle. Unfortunately, because of the nonlinearity of the payoff of options, arbitrage arguments are not enough to obtain the price function of options. In fact, more assumptions are required to value options, which will be discussed in the subsequent sections.

The no-arbitrage principle can only result in some relationships between option prices and the underlying asset price, including (suppose the underlying pays no dividend):

1. \( C_t^E = C_t^A \). In other words, it is never optimal to exercise an American call option on a non-dividend-paying underlying asset before the expiration date.

2. Put-call Parity (European Options):
   \[ C_t^E - P_t^E = S_t - Ke^{-r(T-t)} \]

3. Upper and Lower Bound of Option Prices:
   \[ (S_t - Ke^{-r(T-t)})^+ \leq C_t^E = C_t^A \leq S_t \]
   \[ (Ke^{-r(T-t)} - S_t)^+ \leq P_t^E \leq Ke^{-r(T-t)}, \quad (K_t - S_t)^+ \leq P_t^A \leq K \]

Here \( C^E \) – European call; \( P^E \) – European put, \( C^A \) – American call, \( P^A \) – American put.

1.3 Cox-Ross-Rubinstein Model

1.3.1 Single-Period Model

Consider an option whose value, denoted by \( V_0 \) at current time \( t = 0 \), depends on the underlying asset price \( S_0 \). Let the expiration date of the option be \( T \). Assume that during the life of the option the underlying asset price \( S_0 \) can either move up to \( S_0u \) with probability \( p' \), or down to \( S_0d \) with probability \( 1 - p' \) (\( u > d, \ 0 < p' < 1 \)). Correspondingly, the payoff from the option will become either \( V_u \) (for up-movement in the underlying asset price) or \( V_d \) (for down-movement).

Exercise: Using the no-arbitrage principle, show that \( u > e^{rT} > d \).

Black-Scholes Analysis

We construct a portfolio that consists of a long position in the option and a short position in \( \Delta \) shares. At time \( t = 0 \), the portfolio has the value
\[ V_0 - \Delta S_0 \]
If there is an up movement in the underlying asset price, the value of the portfolio at \( t = T \) is
\[
V_u - \Delta S_0 u.
\]

If there is a down movement in the underlying asset price, the value becomes
\[
V_d - \Delta S_0 d.
\]

To make the portfolio riskfree, we let the two be equal, that is,
\[
V_u - \Delta S_0 u = V_d - \Delta S_0 d
\]
or
\[
\Delta = \frac{V_u - V_d}{S_0 (u - d)}. \tag{1.7}
\]

Again, by the no-arbitrage principle, a risk-free portfolio must earn the risk-free interest rate. As a result
\[
V_u - \Delta S_0 u = e^{rT}(V - \Delta S).
\]

Substituting (1.7) into the above formula, we get
\[
V_0 = e^{-rT} \left[ p V_u + (1 - p) V_d \right],
\]
where
\[
p = \frac{e^{rT} - d}{u - d}.
\]

This is the single-period binomial model.

**Risk Neutral Pricing**

Note that the objective probability \( p' \) does not appear in the binomial model. The probability \( p \) is called the risk-neutral probability that corresponds to an imaginary *risk neutral* world in which all assets earn the same risk-free rate. So, any asset price is the discounted expectation of its future price in the risk-neutral world.

It is important to emphasize that risk-neutral valuation (or the assumption that all investors are risk-neutral) is merely an artificial device for pricing derivatives. The derivative prices obtained are valid in all worlds.

**Option Replication**

Let us derive the binomial model from the point of view of option replication. The fair value of the option should be the cost to replicate the option’s payoff. After receiving the option premium \( V_0 \), the writer of the option would like to invest \( \Delta S_0 \) in stock and \( V_0 - \Delta S_0 \) in bond. Then, the payoff of this portfolio is
\[
\Delta S_0 u + e^{rT} (V_0 - \Delta S_0) \text{ for up movement}
\]
\[
\Delta S_0 d + e^{rT} (V_0 - \Delta S_0) \text{ for down movement}.
\]
The writer aims to replicate the option’s payoff by choosing $\Delta$. So,

$$
\Delta S_0 u + e^{rT} (V_0 - \Delta S_0) = V_u \\
\Delta S_0 d + e^{rT} (V_0 - \Delta S_0) = V_d.
$$

Solving the above system yields the binomial model again.

### 1.3.2 Multi-Period Model

Let $T$ be expiration date, $[0, T]$ be the lifetime of the option. If $N$ is the number of discrete time points, we have time points $n\Delta t$, $n = 0, 1, ..., N$, with $\Delta t = \frac{T}{N}$. At time $t = 0$, the underlying asset price is known, denoted by $S_0$. At time $\Delta t$, there are two possible underlying asset prices, $S_0 u$ and $S_0 d$. Without loss of generality, we assume $ud = 1$.

At time $2\Delta t$, there are three possible underlying asset prices, $S_0 u^2$, $S_0$, and $S_0 d^2 = S_0 u^{-2}$; and so on. In general, at time $n\Delta t$, $n + 1$ underlying asset prices are considered. These are $S_0 u^{-n}$, $S_0 u^{-n+2}$, ..., $S_0 u^n$. A complete tree is then constructed. Let $V^n_j$ be the option price at time point $n\Delta t$ with underlying asset price $S_j = S_0 u^j$. Note that $S_j$ will jump either up to $S_{j+1}$ or down to $S_{j-1}$ at time $(n + 1)\Delta t$, and the value of the option at $(n + 1)\Delta t$ will become either $V^{n+1}_{j+1}$ or $V^{n+1}_{j-1}$. Since the length of time period is $\Delta t$, the discounting factor is $e^{-r\Delta t}$. Then, similar to the arguments in the single-period case, we have

$$
V^n_j = e^{-r\Delta t} \left[ p V^j_{j-1} + (1 - p) V^j_{j+1} \right], \quad j = -n, -n+2, ..., n, \quad n = 0, 1, ..., N - 1
$$

where

$$
p = \frac{e^{r\Delta t} - d}{u - d}.
$$

At expiry,

$$
V^N_j = \left\{ \begin{array}{ll} (S_0 u^j - K)^+ & \text{for call}, \\
(K - S_0 u^j)^+ & \text{for put}, \\
\end{array} \right. \quad j = -N, -N+2, ..., N.
$$

This is the multi-period binomial model.

**Remark 2** It is worthwhile pointing out that the single period BTM is valid for any European style derivatives, while the above multi-period algorithm only applies to the derivative that has a payoff depending only on the terminal price of the underlying asset.

### 1.4 Black-Scholes Model

#### 1.4.1 Black-Scholes Assumptions

We list the assumptions that we make for the Black-Scholes model.

1. The underlying asset price follows a geometric Brownian motion

$$
\frac{dS_t}{S_t} = \mu dt + \sigma dW_t
$$
where $\mu$ and $\sigma$ are the expected return rate and volatility of the underlying asset, $W_t$ is the Brownian motion.

2. There are no arbitrage opportunities. The absence of arbitrage opportunities means that all risk-free portfolios must earn the same risk-free return.

3. The underlying asset pays no dividends during the life of the option.

4. The risk-free interest rate $r$ and the asset volatility $\sigma$ are known constants over the life of the option.

5. Trading is done continuously. Short selling is permitted and the assets are divisible.

6. There are no transaction costs associated with hedging a position. Also no taxes.

### 1.4.2 Preliminary: Brownian Motion and Ito’s Lemma

Consider an Ito process:

$$dS_t = a(\cdot, t)dt + b(\cdot, t)dW_t,$$

(1.8)

where $W_t$ is a Brownian motion, and $a$ and $b$ are adaptive w.r.t. the filtration generated by $W_t$.

**Brownian motion**

Formally, $W_t$ is a Brownian motion if it has the following properties:

1. $W_0 = 0$, and $W_t$ is continuous in $t$.
2. The change $\Delta W$ in $[t, t+\Delta t]$ is a random variable, drawn from a normal distribution with zero mean and variance $\Delta t$, i.e.

   $$\Delta W = \varepsilon \sqrt{\Delta t},$$

where $\varepsilon$ is a random variable drawn from a standardized normal distribution which has zero mean, unit variance and a density function given by

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}, \ x \in (-\infty, \infty).$$

3. The values of $\Delta W$ for any two different intervals of time $\Delta t$ are independent.

**Ito Process and Ito Integral**

Let us go back to (1.8). Thanks to the properties of Brownian motion, we are able to simulate the sample path of $S_t$ in a given period $[0, T]$ by the following procedure: Let $\Delta t = \frac{T}{N}$, $t_n = n\Delta t$, $S_n = S_{t_n}$, $n = 0, 1, ..., N$,

$$S_{n+1} = S_n + a(\cdot, t_n)\Delta t + b(\cdot, t_n)\varepsilon\sqrt{\Delta t}.$$  

(1.9)

Here $\varepsilon$ should be taken independently for each time interval $[t_{n-1}, t_n]$.

A precise expression of (1.8) is

$$S_t = S_0 + \int_0^t a(\cdot, \tau)d\tau + \int_0^t b(\cdot, \tau)dW_\tau,$$
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where the first integral is the Lebesgue integral, and the second is the Ito integral.

For a rigorous definition of Ito integral, see Oksendal (2003). Here we only give a heuristic definition. For any partition $0 = t_1 < t_2 < \ldots < t_n = t$, the integral $\int_0^t f(\cdot, \tau)dW_\tau$ is the limit of

$$\sum_{i=0}^{n-1} f(\cdot, t_i) \left( W_{t_{i+1}} - W_{t_i} \right)$$

as $\max(t_{i+1} - t_i) \to 0$ ($f$ is adaptive w.r.t. the filtration generated by $W_t$). We emphasize that $f$ is taken at the left-hand side endpoint of the interval $(t_i, t_{i+1})$.

In contrast to the Riemann integral, $\int_0^t f(\tau)d\tau = \lim_{\max(t_{i+1} - t_i) \to 0} \sum_{i=0}^{n-1} f(\xi_i) (t_{i+1} - t_i)$, where $\xi_i \in [t_i, t_{i+1}]$.

The definition of Ito integral is in line with investment decision. Consider an investment strategy for which the trading takes place at discrete times $t_i, i = 0, ..., t_{n-1}$. Assume that the investor makes a decision to hold $\Delta t_i$ number of shares at $t_i$, based on all information up to time $t_i$. During $(t_i, t_{i+1})$, the number of shares remains constant. Then the profit/loss during $[t_i, t_{i+1}]$ should be $\Delta t_i (S_{t_{i+1}} - S_{t_i})$. The accumulative profit or loss during $[0, T]$ becomes

$$\sum_{i=0}^{n-1} \Delta t_i (S_{t_{i+1}} - S_{t_i}),$$

where we have ignored the riskfree return. If it is assumed that $dS = a(\cdot, t)dt + b(\cdot, t)dW_t$, the continuous-time limit is

$$\int_0^t \Delta \tau a(\cdot, \tau)d\tau + \int_0^t \Delta \tau b(\cdot, \tau)dW_\tau.$$

Ito’s Lemma

Ito’s Lemma is essentially the differential chain rule of a function involving random variable. First of all let us recall the ordinary differential chain rule of a function of deterministic variables. Let $V(S, t)$ be a function of two variables $S$ and $t$, where

$$dS = adt.$$ 

Then by Taylor series expansion,

$$dV(S_t, t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} adt$$

$$= \left[ \frac{\partial V}{\partial t} + a \frac{\partial V}{\partial S} \right] dt$$

Now let us come back to the stochastic process (1.8). Keep in mind that $dW = \phi \sqrt{dt}$ and $E(dW^2) = dt$ (in fact, $dW^2 = dt$ in some sense). So, formally we have

$$(dS_t)^2 = (adt + bdW)^2$$

$$= a^2 dt^2 + 2abdtdW + b^2 (dW)^2$$

$$= b^2 dt + \cdots.$$
As a result, when applying the Taylor series expansion to $V(S_t, t)$, we need to retain the second order term of $dS$. Thus,

$$dV(S_t, t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS_t)^2$$

$$= \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right] dt + \frac{\partial V}{\partial S} dS_t$$

$$= \left[ \frac{\partial V}{\partial t} + a \frac{\partial V}{\partial S} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} \right] dt + b \frac{\partial V}{\partial S} dW. \tag{1.10}$$

This is the Ito formula, the chain rule of stochastic calculus. Note that it is not a rigorous proof. We refer interested readers to Oksendal (2003) for rigorous proof of Ito’s formula.

A question: now that $dW \approx O(dt^{1/2})$, why don’t we omit the first order term of the right hand side in Eq. (1.10)?

### 1.4.3 Derivation of the Black-Scholes Equation

Let $V = V(S, t)$ be the value of an European option. To derive the model, we construct a portfolio of one long option position and a short position in some quantity $\Delta$, of the underlying.

$$\Pi = V - \Delta S.$$  

The increment of the value of the portfolio in one time-step is

$$d\Pi = dV - \Delta dS_t = (\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}) dt + \frac{\partial V}{\partial S} dS_t - \Delta dS_t.$$  

To eliminate the risk, we take

$$\Delta = \frac{\partial V}{\partial S}$$

and then

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$  

Since there is no random term, the portfolio is riskless. By the no-arbitrage principle, a riskless portfolio must earn a risk free return [see Eq. (1.5)]. So, we have

$$d\Pi = r\Pi dt = r(V - S \frac{\partial V}{\partial S}) dt.$$  

From the above two equalities, we obtain an equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \tag{1.11}$$

This is the well-known Black-Scholes equation. The solution domain is $D = \{(S, t) : S > 0, t \in [0, T]\}$. At expiry, we have

$$V(S, T) = \begin{cases} (S - K)^+, & \text{for call option}, \\ (K - S)^+, & \text{for put option.} \end{cases} \tag{1.12}$$
There is a unique solution to the model (1.11-1.12):

\[V(S,t) = \begin{cases} SN(d_1) - Ke^{-r(T-t)}N(d_2) & \text{for call option} \\ Ke^{-r(T-t)}N(-d_2) - SN(-d_1) & \text{for put option} \end{cases}\]

where

\[N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy, \quad d_1 = \frac{\log S_K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.
\]

Remark 3 The Black-Scholes equation is valid for any derivative that provides a terminal payoff \(f(S_T)\) depending only on the underlying asset price at maturity. So, the Black-Scholes equation is also valid for forward contracts.

Remark 4 In the Black-Scholes equation, \(S\) and \(t\) are independent.

1.4.4 Risk-Neutral Pricing and Monte-Carlo Simulation

There is a drawback in the derivation of the Black-Scholes equation in last section: we need a prior assumption that \(V = V(S,t)\). In addition, the Black-Scholes equation does not apply to path-dependent options that provide path-dependent payoff. Let us use another argument which can remove these restrictions.

Self-financing Process

Consider a market where only two basic assets are traded. One is a bond, whose price process is

\[dR_t = rR_t dt.\]

The other asset is a stock whose price process is governed by the geometric Brownian motion:

\[dS_t = \mu S_t dt + \sigma S_t dW_t.\]

Let us consider a self-financing process \(Z_t\) which means that there is no withdrawal or infusion of funds during the investment period. We denote by the amount \(\Delta_t S_t\) invested in the stock, where \(\Delta_t\) is an adapted process. The remaining amount \(Z_t - \Delta_t S_t\) is invested in the bond. The wealth process \(Z_t\) is

\[dZ_t = r (Z_t - \Delta_t S_t) dt + \Delta_t S_t dW_t = [rZ_t + (\mu - r) \Delta_t S_t] dt + \sigma \Delta_t S_t dW_t\] (1.13)

subject to an initial endowment \(Z_0 = z\).
Girsanov Theorem

Let $W_t$, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, F_t, P)$, and $\theta_t$, $0 \leq t \leq T$, be a process adapted to the filtration $F_t$ [where $F_t$ is generated by $W_t$]. For $0 \leq t \leq T$, define

$$\hat{W}_t = \int_0^t \theta_u du + W_t,$$

$$G(t) = \exp \left\{ - \int_0^t \theta_u dW_u - \frac{1}{2} \int_0^t \theta_u^2 du \right\}.$$

Assume

$$E \left[ \exp \left\{ \frac{1}{2} \int_0^T \theta_u^2 du \right\} \right] < \infty.$$

Define a new probability measure by

$$\hat{P}(A) = \int_A G(T) dP, \text{ for any } A \in F.$$

Then, under $\hat{P}$, the process $\hat{W}_t$, $0 \leq t \leq T$, is a Brownian motion.

Option Replication

For any financial derivative with a terminal payoff $V_T$, its fair value at time 0, denoted by $v$, should be the cost to perfectly replicate the payoff $V_T$ using a self-financing process. As a result, the option pricing problem is reduced to determining the replication cost $v$ (also the strategy $\Delta_t$) such that

$$Z_0 = v \text{ and } Z_T = V_T,$$

where $Z_t$ evolves according to (1.13) with some $\Delta_t$.

To do that, we consider the discounted asset prices:

$$d (e^{-rt} R_t) = 0$$

$$d (e^{-rt} S_t) = (\mu - r) e^{-rt} S_t dt + \sigma e^{-rt} S_t dW_t$$

$$= \sigma e^{-rt} S_t \left[ \frac{\mu - r}{\sigma} dt + dW_t \right]$$

$$= \sigma e^{-rt} S_t d\hat{W}_t,$$

where we have used the Girsanov transformation to turn to a new world (often referred to as the risk-neutral world associated with new measure $\hat{P}$) under which $\hat{W}_t$ is a new Brownian motion. It is worthwhile pointing out that in the risk-neutral world,

$$\frac{dS_t}{S_t} = r dt + \sigma d\hat{W}_t. \quad (1.14)$$

Clearly, both $e^{-rt} R_t$ and $e^{-rt} S_t$ are martingales under the measure $\hat{P}$ (referred to as martingale measure).
Consequently,
\[ d \left( e^{-rt} Z_t \right) = -re^{-rt} Z_t dt + e^{-rt} dZ_t = (\mu - r) \Delta_t e^{-rt} S_t dt + \sigma \Delta_t e^{-rt} S_t d\hat{W}_t = \sigma \Delta_t e^{-rt} S_t d\hat{W}_t, \]
which indicates that \( e^{-rt} Z_t \) is also a martingale \([martingale representation theorem, see Oksendal (2003)]\), i.e., \( e^{-rt} Z_t = \hat{E}_t \left[ e^{-rT} Z_T \right] \) or
\[
Z_t = \hat{E}_t \left[ e^{-r(T-t)} Z_T \right] = \hat{E}_t \left[ e^{-r(T-t)} V_T \right]. \tag{1.15}
\]
Here \( \hat{E}_t \) refers to the conditional expectation under the new measure.

**Remark 5** (1.15) is valid for any (European-style) derivatives whose terminal payoffs are allowed to depend on the historical prices of the underlying asset.

If the option is a vanilla call, then
\[ v = Z_0 = \hat{E}_0 \left[ e^{-rT} (S_T - K)^+ \right]. \]

In particular,
\[
Z_t = \hat{E}_t \left[ e^{-r(T-t)} (S_T - K)^+ \right] = \hat{E}_t \left[ e^{-r(T-t)} \left( S_t e^{(r-t)\frac{\sigma^2}{2}(T-t)+\sigma(W_T-W_t)} - K \right)^+ \right] = V(S_t, t). \tag{1.16}
\]

Note that \( V_t = V(S_t, t) \) is a martingale under the new measure \( \hat{P} \) and
\[
d \left( e^{-rt} V(S_t, t) \right) = e^{-rt} \left[ \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV \right) dt + \sigma S_t \frac{\partial V}{\partial S} d\hat{W}_t \right].
\]

We then deduce that the drift term must be 0. So the Black-Scholes equation follows.

**Remark 6** From (1.16), we can alternatively make use of the Feynman-Kac formula \([Oksendal (2003)]\) to obtain the Black-Scholes equation (a heuristic derivation is available based on the property of conditional expectation and Itô lemma).

**Remark 7** Note that in a real world (i.e. under the measure \( P \)),
\[
d V(S_t, t) = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \mu S_t \frac{\partial V}{\partial S} \right) dt + \sigma S_t \frac{\partial V}{\partial S} dW_t
\]
\[ = \left[ rV + (\mu - r) S_t \frac{\partial V}{\partial S} \right] dt + \sigma S_t \frac{\partial V}{\partial S} dW_t,
\]

where the Black-Scholes equation is used. Compared with (1.13), this yields \( \Delta_t = \frac{\partial V}{\partial S} \).

Question: how do we replicate an Asian option with the payoff \( \left( \frac{1}{T} \int_0^T S_t - K \right)^+ \)?
1.4. **BLACK-SCHOLES MODEL**

**Monte-Carlo Simulation**

In the risk-neutral world, all investors are risk-neutral, namely, the expected return on all securities is the risk-free rate of interest $r$. Thus, the present value of any cash flow in the world can be obtained by discounting its expected value at the risk-free rate.

Eq (1.15) is the theoretical basis of Monte-Carlo simulation for derivative pricing, where the pricing problem is reduced to computing the expectation of the terminal payoff in the risk-neutral world. We emphasize that the risk-neutral price process of the underlying asset is governed by (1.14).

The simulation can be carried out by the following procedure:

i) Simulate the price movement of the underlying asset in a risk-neutral world according to (1.14) (see the discrete scheme (1.9));

ii) Calculate the terminal payoff of the derivative for one path;

iii) Repeat i) and ii) for $N$ paths.

iv) Calculate the mean of the payoffs and discount it at the risk-free interest rate.

1.4.5 **Continuous-time Pricing Model for Asian Options**

Asian options have a payoff, denoted by $\Lambda(S_T, A_T)$, depending on the historical average of the underlying asset price, where

$$A_t = \begin{cases} \frac{1}{t} \int_0^t S_\tau d\tau, & \text{Asian arithmetic} \\ \exp\left(\frac{1}{t} \int_0^t \ln S_\tau d\tau\right), & \text{Asian geometric} \end{cases}$$

Let $V_t$ be the option value at time $t$. According to the risk neutral pricing principle,

$$V_t = e^{-r(T-t)} \widehat{E}_t [\Lambda(S_T, A_T)],$$

where $\widehat{E}_t$ is the expectation in the risk-neutral world and the corresponding risk-neutral process of stock price is

$$dS_t = rS_t dt + \sigma S_t d\widehat{W}_t.$$

Let us first show that we can write

$$V_t = V(S_t, A_t, t).$$

For illustration, we only consider the arithmetic Asian options:

$$\begin{align*}
V_t & = e^{-r(T-t)} \widehat{E}_t \left[ \Lambda \left( S_T, \frac{1}{T} \int_0^T S_\tau d\tau \right) \right] \\
& = e^{-r(T-t)} \widehat{E}_t \left[ \Lambda \left( S_T, \frac{1}{T} \int_0^t S_\tau d\tau + \frac{1}{T} \int_t^T S_\tau d\tau \right) \right] \\
& = e^{-r(T-t)} \widehat{E}_t \left[ \Lambda \left( S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\widehat{W}_T-\widehat{W}_t)}, \frac{1}{T} A_T + \frac{1}{T} S_t \int_t^T e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma(\widehat{W}_\tau-\widehat{W}_t)} d\tau \right) \right] \\
& = V(S_t, A_t, t)
\end{align*}$$

We then have
\[ d(e^{-rt}V_t) = d(e^{-rt}V(S_t, A_t, t)) \]
\[ = e^{-rt} \left[ \left( \frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right) dt + \sigma S \frac{\partial V}{\partial S} d\hat{W}_t \right] \]

In the risk-neutral world, \( e^{-rt}V_t \) is a martingale. It follows
\[ \frac{\partial V}{\partial t} + \frac{S - A}{t} \frac{\partial V}{\partial A} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \]  
(1.17)
The solution domain is \( S > 0, A > 0, t \in [0, T) \), and the terminal condition is
\[ V(S, A, T) = \Lambda(S, A) \]

1.5 Consistency of Discrete- and Continuous-time Models

To make the binomial process of the underlying asset price match the geometric Brownian motion, we need to choose \( u, d \) such that
\[ p'u + (1 - p')d = e^{\mu \Delta t} \]  
(1.18)
\[ p'u^2 + (1 - p')d^2 - e^{2\mu \Delta t} = \sigma^2 \Delta t. \]  
(1.19)
There are three unknowns \( u, d \) and \( p' \). Without loss of generality, we add one condition
\[ ud = 1. \]  
(1.20)
By neglecting the order of \( \Delta t \), we can solve the system of equations (1.18-1.20) to get
\[ u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}. \]

1.5.1 First Order Consistency

We take the European vanilla option as an example. The binomial tree method can be rewritten as
\[ V(S, t - \Delta t) = e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)]. \]
Here, for the convenience of presentation, we take the current time to be \( t - \Delta t \). Assuming sufficient smoothness of the \( V(S, t) \), we perform the Taylor series expansion of the binomial scheme at \( (S, t) \) as follows
\[ 0 = -V(S, t - \Delta t) + e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)] \]
\[ = -V(S, t) + \frac{\partial V}{\partial t} \Delta t + O(\Delta t^2) \]
\[ + e^{-r\Delta t}V(S, t) + \frac{\partial^2 V}{\partial S^2}S e^{-r\Delta t}[p(u - 1) + (1 - p)(d - 1)] \]
\[ + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} e^{-r\Delta t}[p(u - 1)^2 + (1 - p)(d - 1)^2] \]
\[ + \frac{1}{6} \frac{\partial^3 V}{\partial S^3} S^3 e^{-r\Delta t}[p(u - 1)^3 + (1 - p)(d - 1)^3] + O(\Delta t^3) \]
Observe that
\[ e^{-r\Delta t}[p(u - 1) + (1 - p)(d - 1)] = r\Delta t + O(\Delta t^2), \]
\[ e^{-r\Delta t}[p(u - 1)^2 + (1 - p)(d - 1)^2] = \sigma^2\Delta t + O(\Delta t^2), \]
\[ e^{-r\Delta t}[p(u - 1)^3 + (1 - p)(d - 1)^3] = O(\Delta t^3). \]

We then get
\[
0 = -V(S, t - \Delta t) + e^{-r\Delta t}[pV(Su, t) + (1 - p)V(Sd, t)]
\]
\[
= [-rV(S, t) + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}]\Delta t + O(\Delta t^2)
\]
or
\[
-rV(S, t) + \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2} = O(\Delta t).
\]

This implies the consistency of two models.

### 1.5.2 Equivalence of Discrete-Time Model and an Explicit Difference Scheme

We claim the discrete time binomial model is equivalent to an explicit difference scheme for the continuous-time model.

Using the transformations \( u(x, t) = V(S, t) \), \( S = e^x \), (1.11-1.12) become the following constant-coefficient PDE problem
\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + (r - \frac{\sigma^2}{2}) \frac{\partial u}{\partial x} - ru & = 0 \\
\phi(x) & \in (-\infty, \infty), \quad \phi(x)^+ \text{ in } (-\infty, \infty),
\end{align*}
\]

where \( \phi(x) = e^x - K \) (call option) or \( \phi(x) = K - e^x \) (put option).

We now present the explicit difference scheme for (1.21). Given mesh size \( \Delta x, \Delta t > 0 \), \( N\Delta t = T \), let \( Q = \{(j\Delta x, n\Delta t) : 0 \leq n \leq N, j \in Z\} \) stand for the lattice. \( U^n_j \) represents the value of numerical approximation at \((j\Delta x, n\Delta t)\) and \( \varphi_j = \phi(j\Delta x) \). Taking the explicit difference for time and the conventional difference discretization for space, we have

\[
\frac{U_{j+1}^n - U_j^n}{\Delta t} + \frac{\sigma^2 U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2} + (r - \frac{\sigma^2}{2}) \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} - rU_j^n = 0
\]
or
\[
U_j^n = \frac{1}{1 + r\Delta t} \left( (1 - \frac{\sigma^2\Delta t}{\Delta x^2})U_j^{n+1} + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} + (r - \frac{\sigma^2}{2})\frac{\Delta x}{2\sigma^2})U_{j+1}^{n+1} + \frac{\sigma^2\Delta t}{\Delta x^2}(\frac{1}{2} - (r - \frac{\sigma^2}{2})\frac{\Delta x}{2\sigma^2})U_{j-1}^{n+1} \right),
\]

which is denoted by
\[
U_j^n = \frac{1}{1 + r\Delta t} \left[ (1 - \alpha)U_{j+1}^{n+1} + \alpha(aU_{j+1}^{n+1} + (1 - a)U_{j-1}^{n+1}) \right],
\]

(1.22)
where
\[ \alpha = \sigma^2 \frac{\Delta t}{\Delta x^2}, \quad a = \frac{1}{2} + \left( r - \frac{\sigma^2}{2} \right) \frac{\Delta x}{2\sigma^2}. \]

By putting \( \alpha = 1 \) in (1.22), namely \( \sigma^2 \Delta t/\Delta x^2 = 1 \), we get
\[ U^n_j = \frac{1}{1 + r \Delta t} \left[ a U^{n+1}_{j+1} + (1 - a) U^{n+1}_{j-1} \right]. \tag{1.23} \]

The final values are given as follows:
\[ U^N_j = \varphi_j^+, \quad j \in \mathbb{Z}. \]

Recall the binomial tree method can be described as follows by adopting the same lattice:
\[ V^n_j = e^{-r \Delta t} \left[ p V^{n+1}_{j+1} + (1 - p) V^{n+1}_{j-1} \right], \quad j = n, n-2, \ldots, -n, \tag{1.24} \]
\[ V^N_j = \varphi_j^+, \quad j = N, N-2, \ldots, -N \tag{1.25} \]

In view of
\[ e^{r \Delta t} = 1 + r \Delta t + O(\Delta t^2) \]
and
\[ p = \frac{1}{2} \left( 1 + \frac{\sqrt{\Delta t}}{\sigma} (r - \frac{\sigma^2}{2}) \right) + O(\Delta t), \]

### 1.6 American Options and Early Exercise

American options are contracts that may be exercised early, prior to expiry. These options are contrasted with European options for which exercise is only permitted at expiry. Most traded stock and futures options are American style, while most index options are European.

#### 1.6.1 Continuous-time Model

We now consider the pricing model for American options. Here we take into account a put as an example. Let \( V = V(S, t) \) be the option value. At expiry, we still have
\[ V(S, T) = (K - S)^+. \tag{1.26} \]

The early exercise feature gives the constraint
\[ V(S, t) \geq K - S. \tag{1.27} \]

As before, we construct a portfolio of one long American option position and a short position in some quantity \( \Delta \), of the underlying.
\[ \Pi = V - \Delta S. \]

With the choice \( \Delta = \frac{\partial V}{\partial S} \), the value of this portfolio changes by the amount
\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt \]
In the Black-Scholes argument for European options, we set this expression equal to riskless return, in order to preclude arbitrage. However, when the option in the portfolio is of American style, all we can say is that we can earn no more than the risk-free rate on our portfolio, that is, 

\[ d\Pi \leq r\Pi dt = r(V - S \frac{\partial V}{\partial S}) dt. \]

The reason is the holder of the option controls the early exercise feature. If he/she fails to optimally exercise the option, the change of the portfolio value would be less than riskless return (bear in mind that the holder always holds the option during \([t, t + dt])\). Thus we arrive at an inequality

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0. \] (1.28)

Remark 8 For American options, the long/short position is asymmetrical. The holder of an American option is given more rights, as well as more headaches: when should he exercise? Whereas the writer of the option can do no more than sit back and enjoy the view. The writer of the American option can make more than the risk-free rate if the holder does not exercise optimally. A question: what happens if the portfolio is composed of a long position in some quantity of the underlying and one short American option?

It is clear that (1.26)-(1.28) are insufficient to form a model because solution is not unique. We need to exploit more information. Note that if \(V(S, t) > K - S\), which implies that the option should not be exercised at the moment, then the equality holds in the inequality (1.28), namely

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \] if \(V > K - S\).

If \(V(S, t) = K - S\), of course we still have the inequality, that is

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0 \] if \(V = K - S\).

The above two formulas imply that at least one holds in equality between (1.27-1.28). So we arrive at a complete model:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \leq 0, \ V \geq K - S \]

\[ \left[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV \right] [V - (K - S)] = 0, \ (S, t) \in D \]

\[ V(S, T) = (K - S)^+ \]

It can be shown that there exists a unique solution to the model.
A succinct expression of the above model is

\[
\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV, V - (K - S) \right\} = 0, \quad (S, t) \in D
\]

\[ V(S, T) = (K - S)^+ \quad (1.29) \]

For American call options, we similarly have

\[
\min \left\{ -\frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV, V - (S - K) \right\} = 0, \quad (S, t) \in D
\]

\[ V(S, T) = (S - K)^+ \]

We claim the price function of European call option \( C(S, t) \) just satisfies the above model. Indeed, \( C(S, t) > S - K \) for \( t < T \) and \( C(S, t) \) clearly satisfies the Black-Scholes equation. So \( C(S, t) \) must be the (unique) solution to the American option pricing model. The result \( C(S, t) > S - K \) implies that the option should never be exercised before expiry.

**Remark 9** From the view point of probabilistic approach, we have (for an American put)

\[
V(S, t) = \max_{t'} \mathbb{E}\left[ e^{-r(t'-t)}(K - S_{t'})^+ | S_t = S \right],
\]

where \( t' \) is a stopping time. Intuitively \( t'(.) \) can be thought of as a strategy to exercise the option and the option’s value corresponds to the optimal exercise strategy. It can be shown that (1.30) is equivalent to the above variational inequality [see Oksendal (2003)].

**Remark 10** It turns out that the solution of (1.29) \( V(S, t) \notin C^2 \) in \( S \). So, \( \frac{\partial^2 V}{\partial S^2} \) in (1.29) is not in the classical sense.

### 1.6.2 Discrete-time Binomial Model

Let \( T \) be the expiration date, \([0, N]\) be the lifetime of the option. If \( N \) is the number of discrete time points, we have time points \( n\Delta t, n = 0, 1, ..., N \), with \( \Delta t = T/N \). Let \( V^n_j \) be the option price at time point \( n\Delta t \) with underlying asset price \( S_j \). Suppose the underlying asset price \( S_j \) will move either up to \( S_{j+1} = S_ju \) or down to \( S_{j-1} = S_jd \) after the next time step.

Similar to the arguments in the continuous time case, we are able to derive the binomial tree model:

\[
\begin{cases}
V^n_j = \max \left\{ e^{-r\Delta t}[pV^n_{j+1} + (1-p)V^n_{j-1}], \varphi_j \right\}, \\
\text{for } j = -n, -n + 2, ..., n \\
V^n_N = \varphi^+_j, \quad \text{for } j = -N, -N + 2, ..., N
\end{cases}
\]

where \( \varphi_j = K - S_0u^j \),

\[
p = \frac{e^{r\Delta t} - d}{u - d},
\]

\[
u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}.
\]
1.7 Advanced Topics

1. Pricing exotic options like barrier options, Asian option, lookback options;
3. Interest rate models: short rate models, HJM models, LIBOR models.
5. Efficient numerical methods.