Chapter 2

Portfolio Selection

The theory of option pricing is a theory of deterministic returns: we hedge our option with the underlying to eliminate risk, and our resulting risk-free portfolio then earns the risk free rate of interest. Banks make money from this hedging process; they sell something for a bit more than its worth and hedge away the risk to make a guaranteed profit.

But not everyone is hedging. Fund managers buy and sell assets (including derivatives) with the aim of beating the bank’s rate of return. In so doing they take risk.

2.1 Mean-variance Analysis: Single-period Model

Let us take into account a single period model.

2.1.1 Without a Riskfree Investment

Suppose there are \( n \) risky assets. The value today of the \( i \)th asset is \( S_i \) and its random return rate is \( R_i \) over the period considered. Here \( R_i \) is Normally distributed with mean \( \mu_i \) and standard deviation \( \sigma_i \). The correlation coefficient between these returns on the \( i \)th and \( j \)th assets is \( \rho_{ij} \) (with \( \rho_{ii} = 1 \)), that is

\[
\rho_{ij} = \frac{E[(R_i - \mu_i) (R_j - \mu_j)]}{\sigma_i \sigma_j}.
\]

In other words, the covariance matrix is

\[
(\rho_{ij} \sigma_i \sigma_j)_{n \times n} \overset{\Delta}{=} V = (v_{ij})_{n \times n}.
\]

Without loss of generality, we assume \( V \) is positive definite.

A portfolio: \( (w_1, w_2, \ldots, w_n) = w^T \) with

\[
\sum_{i=1}^{n} w_i = 1,
\]
where \( w_i \) denotes the proportion invested in the \( i \)th asset. Then the reward and risk of the portfolio can be described by the mean
\[
\sum_{i=1}^{n} w_i \mu_i
\]
and the variance
\[
\sigma_w^2 = \sum_{i=1}^{n} w_i w_j v_{ij} = w^T V w,
\]
respectively.

Markowitz’s problem (mean-variance analysis): we want
\[
\min_{w^T} \sigma_w^2
\]
subject to
\[
\sum_{i=1}^{n} w_i \mu_i = \bar{\mu}, \quad \sum_{i=1}^{n} w_i = 1,
\]
where \( \bar{\mu} > 0 \) is a given constant. (Clearly it is required that there exist some \( i \) and \( j \) such that \( \mu_i \neq \mu_j \).)

In the matrix form, the problem can be written as
\[
\min_{w^T} w^T V w,
\]
subject to
\[
w^T \mu = \bar{\mu}, \quad w^T e = 1,
\]
where \( \mu = (\mu_1, \mu_2, \ldots, \mu_n)^T \), and \( e = (1, 1, \ldots, 1)^T \).

To solve this problem, we introduce Lagrangian multipliers \( \lambda_1 \) and \( \lambda_2 \). It can be shown that the problem is equivalent to
\[
\min_{w^T, \lambda_1, \lambda_2} w^T V w + \lambda_1 (\bar{\mu} - w^T \mu) + \lambda_2 (1 - w^T e).
\]
Let \((\bar{w}, \lambda_1, \lambda_2)\) be the true solution. Then
\[
\begin{align*}
2V\bar{w} &= \lambda_1 \mu + \lambda_2 e \\
\bar{w}^T \mu &= \bar{\mu} \quad \text{(2.1)} \\
\bar{w}^T e &= 1 \quad \text{(2.2)}
\end{align*}
\]
2.1. MEAN-VARIANCE ANALYSIS: SINGLE-PERIOD MODEL

By (2.1), we have
\[ \overline{w} = \frac{1}{2} V^{-1} (\lambda_1 \mu + \lambda_2 e) = \frac{1}{2} V^{-1} (\mu, e) (\lambda_1, \lambda_2)^T. \] (2.4)

Note that (2.2) and (2.3) can be rewritten as
\[ (\mu, e)^T \overline{w} = (\overline{\mu}, 1)^T. \] (2.5)

Combination of (2.4) and (2.5) yields
\[ \frac{1}{2} (\mu, e)^T V^{-1} (\mu, e) (\lambda_1, \lambda_2)^T = (\overline{\mu}, 1)^T \]
or
\[ \frac{1}{2} (\lambda_1, \lambda_2)^T = A^{-1} (\overline{\mu}, 1)^T, \] (2.6)
where
\[ A = (\mu, e)^T V^{-1} (\mu, e). \] (2.7)

In terms of (2.4) and (2.6), we get
\[ \overline{w} = V^{-1} (\mu, e) A^{-1} (\overline{\mu}, 1)^T. \]

The corresponding variance
\[ \sigma^2 = \overline{w}^T V \overline{w} = (\overline{\mu}, 1) A^{-1} (\mu, e)^T V^{-1} V V^{-1} (\mu, e) A^{-1} (\overline{\mu}, 1)^T = (\overline{\mu}, 1) A^{-1} (\mu, e)^T V^{-1} (\mu, e) A^{-1} (\overline{\mu}, 1)^T = (\overline{\mu}, 1) A^{-1} (\mu, e)^T V^{-1} (\mu, e) A^{-1} (\overline{\mu}, 1)^T. \] (2.8)

Here the last inequality is due to (2.7).

Let us denote
\[ A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \mu^T V^{-1} \mu & e^T V^{-1} e \\ e^T V^{-1} \mu & e^T V^{-1} e \end{pmatrix}. \]

It follows,
\[ A^{-1} = \frac{1}{(ac - b^2)} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}. \]

As a result, (2.8) gives
\[ \sigma^2 = \frac{a - 2b\overline{\mu} + \overline{\mu}^2}{ac - b^2}. \]

This is the relation between risk and reward of the optimal portfolio. In the \( \sigma - \mu \) plane, it is one branch of a hyperbola, which is called the frontier.

**Question:** What is an efficient frontier?
2.1.2 With a Riskfree Investment

Now let us allow a riskfree asset in our portfolio. Let \( S_0 \) be the value today of riskfree asset and \( r \) be the riskfree rate. We keep the previous notation and denote the portfolio by \((w_0, w^T)\), where

\[
w_0 + \sum_{i=1}^{n} w_i = 1.
\]

Consequently, the reward becomes

\[w_0 + \sum_{i=1}^{n} w_i \mu_i = \sum_{i=1}^{n} w_i (\mu_i - r) + r \leq w^T \mu' + r,
\]

where

\[
\mu' = (\mu_1 - r, \mu_2 - r, ..., \mu_n - r)^T.
\]

The risk (variance) is still

\[
\sigma_w^2 = \sum_{i=1}^{n} w_i w_j v_{ij} = w^T V w.
\]

Mean-variance analysis with a riskfree investment:

\[
\min_{w^T V w},
\]

subject to

\[
w^T \mu' = \mu - r = \mu'.
\]

Note that there is no constraint on \( w \).

Likewise, we can introduce a Lagrangian function

\[
\min_{w^T, \lambda} w^T V w + \lambda (\mu' - w^T \mu').
\]

Let \((\overline{w}, \lambda)\) be the true solution. Then

\[
\begin{align*}
2V \overline{w} &= \lambda \mu' \quad (2.9) \\
\overline{w}^T \mu' &= \overline{\mu}' \quad (2.10)
\end{align*}
\]

By (2.9), we have

\[
\overline{w} = \frac{\lambda}{2} V^{-1} \mu' \quad (2.11)
\]

It follows

\[
\frac{\lambda}{2} \mu^T V^{-1} \mu' = \overline{\mu}'
\]

or

\[
\frac{\lambda}{2} = \frac{\overline{\mu}'}{\mu'^T V^{-1} \mu'}
\]
Consequently,
\[ w = \frac{\mu'}{\mu'V^{-1}\mu'}V^{-1}\mu' \]
and
\[ \sigma^2 = w^TVw \]
\[ = \frac{\mu'}{\mu'^T V^{-1} \mu' \mu' V^{-1} \mu'} \frac{\mu'}{\mu'^T V^{-1} \mu'} \]
\[ = \frac{\mu'^2}{\mu'^T V^{-1} \mu'}. \]

That is,
\[ \sigma = \pm \frac{\mu'}{\sqrt{\mu'^T V^{-1} \mu'}} \]
\[ = \pm \frac{\mu - r}{\sqrt{\mu'^T V^{-1} \mu'}}. \]

In this case the frontier becomes two straight lines. The upward straight line \( \sigma = \frac{\mu - r}{\sqrt{\mu'^T V^{-1} \mu'}} \) is the efficient frontier (also called the Capital Market Line). And \( \frac{\mu - r}{\sigma} \) is called Sharp ratio, which is a measure for the portfolio’s performance.

### 2.2 Continuous-time Model with Utility Framework

#### 2.2.1 Optimal Investment

Suppose that there are only two assets available for investment: a riskless asset (bank account) and a risky asset (stock). Their prices, denoted by \( R_t \) and \( S_t \), respectively, evolve according to the following equations:

\[ dR_t = rR_t dt, \]
\[ dS_t = S_t [\mu dt + \sigma dW_t]. \]

where \( r > 0 \) is the constant riskless rate, \( \mu > 0 \) and \( \sigma > 0 \) are constants called the expected rate of return and the volatility, respectively, of the stock.

Now we consider an investment problem associated with the market. Assume that an investor has an initial wealth \( Z_0 \). Let \( Z_t \) be a self-financing process. At time \( t \), the investor holds \( Y_t \) and \( Z_t - Y_t \) in stock and bank respectively. Then,

\[ dZ_t = [rZ_t + (\mu - r)Y_t] dt + \sigma Y_t dW_t \]

with the initial wealth \( Z_0 \).
Assume the investor’s risk preference is described by a strictly increasing and concave utility function \( U : R^1 \to R^1 \). The investor’s problem is to choose an admissible strategy so as to maximize the expected utility of terminal wealth, that is,

\[
\sup_{Y_t} E_0^z [U(Z_T)].
\]

In the following we would like to choose the logarithm utility function \( U(Z) = \log(Z) \). In addition, we are confined within the solvency region

\[
\mathcal{S} = \{Z : Z > 0\}.
\]

We define the value function by

\[
\varphi(z, t) = \sup_{Y_t} E_t^Z [U(Z_T)] \mid Z_t > 0, \ t \in [0, T).
\]

It can be shown (see Oksendal (2003)) that \( \varphi \) satisfies the Hamilton-Jacobi-Bellman (HJB) equation

\[
\sup_y \left[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \varphi}{\partial z^2} + (rz + (\mu - r)y) \frac{\partial \varphi}{\partial z} \right] = 0
\]

(2.12)

Let \( y^* \) be the maximum. Then

\[
\sigma^2 y^* \frac{\partial^2 \varphi}{\partial z^2} + (\mu - r) \frac{\partial \varphi}{\partial z} = 0,
\]

namely

\[
y^* = -\frac{(\mu - r) \frac{\partial \varphi}{\partial z}}{\sigma^2 \frac{\partial^2 \varphi}{\partial z^2}}
\]

Substituting into (2.12), we get

\[
\frac{\partial \varphi}{\partial t} - \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \varphi}{\partial z^2} + rz \frac{\partial \varphi}{\partial z} = 0
\]

or

\[
\frac{\partial \varphi}{\partial t} - \frac{1}{2} \frac{(\mu - r)^2 \left( \frac{\partial \varphi}{\partial z} \right)^2}{\sigma^2 \frac{\partial^2 \varphi}{\partial z^2}} + rz \frac{\partial \varphi}{\partial z} = 0, \ z > 0, \ t \in (0, T]
\]

The terminal condition is

\[
\varphi(z, T) = \log z.
\]

It can be verified that the true solution to the above problem is

\[
\varphi(z, t) = \left( r + \frac{(\mu - r)^2}{2\sigma^2} \right) (T - t) + \log z.
\]

Correspondingly,

\[
y^* = \frac{\mu - r}{\sigma^2} z.
\]

This means that the optimal investment policy is to keep a constant fraction of the total wealth in the risky asset.

Remark 11  (a) The above result can be extended to the power utility function \( U(z) = \frac{z^\gamma}{\gamma} \), \( \gamma < 1, \gamma \neq 1 \). (b) We can similarly take into consideration the multi-assets case.
2.2.2 Optimal Investment and Consumption

If the consumption is allowed, then
\[
dZ_t = [rZ_t + (\mu - r) Y_t - C_t] \, dt + \sigma Y_t \, dW_t,
\]
where \( C_t \geq 0 \) is the consumption rate.

The investor’s problem is
\[
\sup_{Y_t, C_t} \mathbb{E}^Z \left[ \int_0^T e^{-\beta t} U(C_s) \, ds + e^{-\beta T} U(Z_T) \right].
\]

Denote the value function
\[
\varphi(Z_t, t) = \sup_{Y_t, C_t} \mathbb{E}^Z_t \left[ \int_t^T e^{-\beta (s-t)} U(C_s) \, ds + e^{-\beta (T-t)} U(Z_T) \right], \quad Z_t > 0, \ t \in [0, T).
\]

The resulting HJB equation becomes
\[
\sup_{y, C} \left[ \frac{\partial \varphi}{\partial t} + \frac{1}{2} \sigma^2 y^2 \frac{\partial^2 \varphi}{\partial z^2} + (rz + (\mu - r) y - C) \frac{\partial \varphi}{\partial z} - \beta \varphi + U(C) \right] = 0. \quad (2.13)
\]

The analytic solution to the above equation with boundary condition and the corresponding optimal investment/consumption strategy can be obtained. I leave it as an exercise.

2.3 Continuous-time Mean-variance Analysis

Assume the wealth process
\[
dZ_t = [rZ_t + (\mu - r) Y_t] \, dt + \sigma Y_t \, dW_t.
\]

The continuous-time mean-variance problem is
\[
\min_{Y_t} \text{Var} [Z_T]
\]
subject to \( E[Z_T] = a \). This is equivalent to
\[
\min_{Y_t} E \left[ Z_T^2 \right]
\]
subject to \( E[Z_T] = a \). By introducing a Lagrangian multiplier, we instead consider
\[
\min_{Y_t} E \left[ Z_T^2 \right] - 2\lambda (\text{E}[Z_T] - a)
\]
\[
= \min_{Y_t} \left[ (Z_T - \lambda)^2 \right] - \lambda^2 + 2\lambda a,
\]
where \( \lambda \) is a constant.

Now we define two value functions
\[
V(z, t) = \inf_{Y_t, \xi_t \in [t, T]} \mathbb{E}^Z_t [ (Z_T - \lambda)^2 ],
\]
and
\[ u(z, t) = \inf_{Y_t, s \in [t, T]} E_t^{Z_s=z} [Z_T^2]. \]

It is easy to see \( V(z, t) = u \left( z - \lambda e^{-r(T-t)}, t \right). \)

Let us find analytical representations of \( u \) and \( V. \) Note that \( u \) satisfies
\[ u_t - \left( \frac{(\mu - r)^2}{2\sigma^2} u^2_z + rz u_z \right) = 0, \quad \text{for } z > 0, \ t \in [0, T) \]
subject to
\[ u(z, T) = z^2. \]

Assume the solution takes the form \( u(z, t) = A(t) z^2. \) It follows
\[ A'(t) z^2 - \left( \frac{(\mu - r)^2}{4\sigma^2 A(t)} 4z^2 + 2rz^2 A(t) = 0 \right. \]
or
\[ A'(t) = \left( \frac{(\mu - r)^2}{\sigma^2} - 2r \right) A(t). \]

So \( A(t) = \exp \left[ - \left( \frac{(\mu - r)^2}{\sigma^2} - 2r \right) (T - t) \right]. \) Then
\[ V(z, t) = A(t) (z - \lambda e^{-r(T-t)})^2. \]

Next, we go back to (2.14) and find the optimal strategy. I leave this as an assignment.

### 2.4 Continuous-time Model: Probabilistic Approaches

Let us confine to the utility framework.

Wealth process:
\[ dZ_t = \left[ rZ_t + (\mu - r) Y_t \right] dt + \sigma Y_t dW_t. \]

We want to
\[ \max_{Y_t} E \left[ U(Z_T) \right], \]
give \( Z_0 = z. \) Without loss of generality, we only consider power utility, i.e.,
\[ U(Z_T) = \frac{1}{\gamma} Z_T^\gamma \text{ for } \gamma < 1, \ \gamma \neq 0. \]

#### 2.4.1 An Approach Based on Girsanov Transformation

For \( Z_t > 0, \) we can rewrite
\[ Y_t = \pi_t Z_t, \]
where \( \pi_t \) is the fraction in the risky asset. Then
\[ \frac{dZ_t}{Z_t} = \left[ r + (\mu - r) \pi_t \right] dt + \sigma \pi_t dW_t. \]
So, the portfolio selection problem becomes

$$\max_{\pi_t} E \left[ \frac{1}{\gamma} Z_T \right] ,$$

where

$$Z_T = e^{\exp \left[ \int_0^T \left( r + (\mu - r) \pi_s - \frac{\sigma^2 \pi_s^2}{2} \right) ds + \int_0^T \sigma \pi_s dW_s \right] } .$$

Note that

$$E \left[ \frac{1}{\gamma} Z_T \right] = \frac{1}{\gamma} e^{\exp \left[ \int_0^T \left( r + (\mu - r) \pi_s - \frac{\sigma^2 \pi_s^2}{2} \right) ds + \int_0^T \gamma \sigma \pi_s dW_s \right] } ,$$

where we have used the Girsanov transformation by taking

$$\overline{G}(t) = e^{\exp \left[ - \int_0^t \frac{\gamma^2 \sigma^2 \pi_s^2}{2} ds + \int_0^t \gamma \sigma \pi_s dW_s \right] }$$

and

$$\overline{F}(A) = \int_A \overline{G}(T) dP .$$

Now, it is enough to consider

$$\max_{\pi_s} \gamma \left( r + (\mu - r) \pi_s - (1 - \gamma) \frac{\sigma^2 \pi_s^2}{2} \right) .$$

Clearly, the optimal \( \pi_s \), denoted by \( \pi^*_s \),

$$\pi^*_s = \frac{\mu - r}{(1 - \gamma) \sigma^2} .$$

The corresponding optimal value function is

$$E \left[ \frac{1}{\gamma} Z_T \right] = \frac{1}{\gamma} \frac{1}{\gamma} \overline{E} \left[ \exp \left[ \int_0^T \gamma \left( r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} \right) ds \right] \right]$$

$$= \frac{1}{\gamma} \frac{1}{\gamma} \exp \left[ \gamma \left( r + \frac{(\mu - r)^2}{2(1 - \gamma) \sigma^2} \right) T \right] .$$
2.4.2 Martingale Approach

For any strategy $Y_t$, we can think of the associated $Z_T$ as the payoff of a contingent claim. Then the initial wealth $z$ is its fair value, and

$$z = \hat{E}\left[e^{-rT}Z_T\right] = E\left[e^{-rT}\hat{G}_TZ_T\right].$$  \hspace{1cm} (2.15)

Here $\hat{E}$ is taken under the risk neutral world (i.e. martingale measure $\hat{P}$),

$$\hat{P}(A) = \int_A \hat{G}(T) dP$$

with $\hat{G}(t) = \exp\left[-\frac{(\mu-r)^2}{2\sigma^2} - \frac{\mu-r}{\sigma} W_t\right]$.

The idea is the following: we first find an optimal $Z_T$, i.e.,

$$\max_{Z_T} E\left[\frac{1}{\gamma}Z_T^\gamma\right];$$  \hspace{1cm} (2.16)

then we can replicate the optimal $Z_T$ to get the optimal strategy. Bear in mind that the problem (2.16) is subject to the constraint (2.15). We then introduce a Lagrangian multiplier $\lambda$ and instead take into consideration

$$\max_{Z_T} E\left[\frac{1}{\gamma}Z_T^\gamma - \lambda\left(E\left[e^{-rT}\hat{G}_TZ_T\right] - z\right)\right]$$

It suffices to consider

$$\max_{Z_T} \frac{1}{\gamma}Z_T^\gamma - \lambda e^{-rT}\hat{G}_TZ_T.$$

Clearly, the optimal $Z_T$ is

$$Z_{T,\lambda}^* = \left(\lambda e^{-rT}\hat{G}_T\right)^{1/(\gamma-1)}.$$  \hspace{1cm} (2.17)

Notice that $Z_{T,\lambda}$ is the solution to the problem (2.16) if

$$z = E\left[e^{-rT}\hat{G}_TZ_{T,\lambda}^*\right] = E\left[\lambda^{1/(\gamma-1)} \left(e^{-rT}\hat{G}_T\right)^{\gamma/(\gamma-1)}\right]$$

So

$$\lambda = z^{\gamma-1} e^{\gamma r} E\left[\hat{G}_T^{\gamma/(\gamma-1)}\right]^{-1}$$

$$= z^{\gamma-1} e^{\gamma r} E\left[\exp\left[\frac{\gamma}{\gamma-1} \left(-\frac{(\mu-r)^2}{2\sigma^2}\frac{T}{\sigma} - \frac{(\mu-r)}{\sigma} W_T\right)\right]\right]^{-1}$$

$$= z^{\gamma-1} e^{\gamma r} \exp\left[\frac{\gamma}{\gamma-1} \left(-\frac{(\mu-r)^2}{2\sigma^2}\frac{T}{\sigma} + \frac{\gamma}{\gamma-1} \frac{(\mu-r)^2}{2\sigma^2}\frac{T}{\sigma}\right)\right]^{-1}$$

$$= z^{\gamma-1} \exp\left[\gamma \left(f - \frac{1}{\gamma} \frac{(\mu-r)^2}{2\sigma^2}\right) T\right].$$
Substitute to (2.17), we then obtain

\[
Z^*_T = z \left( \exp \left[ \left( -r (1 - \gamma) - \frac{\gamma (\mu - r)^2}{2\sigma^2} \right) T - \frac{(\mu - r)^2 T}{2\sigma^2} - \frac{(\mu - r)}{\sigma} W_T \right] \right)^{1/(\gamma-1)}
\]

\[
= z \exp \left[ \left( r + \frac{(\mu - r)^2}{2 (1 - \gamma)^2 \sigma^2} \right) T + \frac{1}{1 - \gamma} \frac{(\mu - r)}{\sigma} W_T \right].
\]

A question: what is the optimal strategy implied by the above expression?

2.5 Advanced topics

Portfolio selection with portfolio constraints, transaction costs, and/or non-concave utility.